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# On bivariate Hermite interpolation and the limit of certain bivariate Lagrange projectors

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**Abstract.** We give a new poised bivariate Hermite scheme and a formula for the interpolation polynomial. We show that the Hermite interpolation polynomial is the limit of bivariate Lagrange interpolation polynomials at Bos configurations on circles.

**1. Introduction.** Let $\mathcal{P}(\mathbb{R}^2)$ be the vector space of all polynomials (with real coefficients) on $\mathbb{R}^2$, and $\mathcal{P}_d(\mathbb{R}^2)$ the subspace consisting of all polynomials of degree at most $d$. The vector space $\mathcal{P}(\mathbb{R}^2)$ is endowed with the norm

$$\|P\|_\infty = \max_{j+k\le d} |c_{jk}| \quad \text{for} \quad P(\mathbf{x}) = \sum_{j+k\le d} c_{jk} x^j y^k.$$

Suppose that $E \subset \mathbb{R}^2$ is some set. Then the polynomials in $\mathcal{P}_d(\mathbb{R}^2)$, when restricted to $E$, form a certain vector space, say $\mathcal{P}_d(E)$. When $E = \mathbb{R}^2$ or $E$ contains some open disk, $\mathcal{P}_d(E)$ is identical with $\mathcal{P}_d(\mathbb{R}^2)$. The dimension of $\mathcal{P}_d(E)$ is denoted by $m_d(E)$. It is well-known that $m_d = m_d(\mathbb{R}^2) = \binom{d+2}{2}$. If $E$ is a circle in $\mathbb{R}^2$, then $m_d(E) = 2d+1$.

A subset $X = \{\mathbf{x}_1, \dots, \mathbf{x}_{m_d(E)}\}$ of $E$ that consists of $m_d(E)$ distinct points is said to be *unisolvent* for $\mathcal{P}_d(E)$ (or degree $d$ on $E$) if, for every function $f$ defined on $X$, there exists a unique $P \in \mathcal{P}_d(E)$ such that $f(\mathbf{x}) = P(\mathbf{x})$ for all $\mathbf{x} \in X$. This polynomial is called the *Lagrange interpolation polynomial* of $f$ at $X$ and is denoted by $\mathbf{L}[X; f]$. In studying Lagrange interpolation, it is useful to introduce the (generalized) Vandermonde determinant. Choose a basis $\mathcal{B} = \{p_1, \dots, p_{m_d(E)}\}$ for $\mathcal{P}_d(E)$. Then

$$\mathrm{VDM}(\mathcal{B}; X) = \det\,[p_i(\mathbf{x}_j)]_{1\le i,j\le m_d(E)}$$

is called the *Vandermonde determinant*. Here $j$ is the row index of the matrix. It is well-known that $X$ is unisolvent if and only if $\mathrm{VDM}(\mathcal{B}; X) \neq 0$. Of course, the condition is independent of the choice of the basis $\mathcal{B}$. We can

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use the Vandermonde determinant to write the formula for the Lagrange interpolation polynomial,

(1.1)
$$\mathbf{L}[X;f](\mathbf{x}) = \sum_{i=1}^{m_d(E)} f(\mathbf{x}_i)\frac{\mathrm{VDM}(\mathcal{B};X[\mathbf{x}_i \leftarrow \mathbf{x}])}{\mathrm{VDM}(\mathcal{B};X)} = \sum_{p\in\mathcal{B}} \frac{\mathrm{VDM}(\mathcal{B}[p\leftarrow f];X)}{\mathrm{VDM}(\mathcal{B};X)}p,$$

where $X[\mathbf{x}_i \leftarrow \mathbf{x}]$ means that we substitute $\mathbf{x}$ for $\mathbf{x}_i$ in $X$ and likewise for $\mathcal{B}[p \leftarrow f]$.

We consider the problem of Hermite interpolation by a polynomial of two variables. More precisely, the problem is to find a polynomial which matches, on a set of distinct points in $\mathbb{R}^2$, the values of a function and its partial derivatives. We deal with the case where the number of interpolation conditions is equal to the dimension of $\mathcal{P}_d(\mathbb{R}^2)$. If the interpolation problem has a unique solution, then we say that the problem is *poised*. Unlike univariate Hermite interpolation, bivariate Hermite interpolation is not always poised. Moreover, it is difficult to check whether a particular bivariate Hermite problem is poised.

We now state a general problem. Associated with a homogeneous polynomial $P(\mathbf{x}) = \sum_{j+k=d} c_{jk}x^jy^k$, $\mathbf{x} = (x, y)$, we define a homogeneous differential operator $P(D)$ by

$$P(D)f = \sum_{j+k=d} c_{jk}\frac{\partial^d f}{\partial x^j \partial y^k}.$$

In the case when $P(\mathbf{x}) = c$, we set $P(D)f = cf$.

Problem 1. *Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ be $m$ distinct points in $\mathbb{R}^2$. Let $n_1, \dots, n_m$ and $d$ be positive integers such that $n_1 + \dots + n_m = m_d(\mathbb{R}^2)$. Find homogeneous differential operators $P_{jk}(D)$ for $1 \le j \le m$ and $k = 0, \dots, n_j - 1$ for which the interpolation problem*

$$P_{jk}(D)f(\mathbf{a}_j) = f_{jk}, \qquad 1 \le j \le m,\, 0 \le k \le n_j - 1,$$

*has a unique solution in $\mathcal{P}_d(\mathbb{R}^2)$ for any given data $\{f_{jk}\}$.*

In this paper, we give a solution of Problem 1 in which the homogeneous differential operators are the real and imaginary parts of complex differential operators of the form $\partial^d/\partial\mathbf{x}^d$ with $\mathbf{x} = x + yi$. Theorem 2.1 gives a poised Hermite scheme. A formula for the Hermite interpolation polynomial is given in Theorem 2.2.

Roughly speaking, a univariate Hermite interpolation is the result of the collapsing of points in a univariate Lagrange interpolation. However, in several variables, the problem of determining the limit of Lagrange interpolations is not easy.

PROBLEM 2. *Suppose that the points of a unisolvent set $X$ for $\mathcal{P}_d(\mathbb{R}^2)$ converge to some limit points. Determine the limit of the Lagrange interpolation polynomial at $X$ of a sufficiently smooth function $f$.*

In [1], the authors gave a condition implying that multivariate Lagrange projectors tends to a Taylor projector. However, it is difficult to check the Bloom–Calvi conditions. In [8], the authors treated the case when $X$ is a natural lattice. They gave a natural geometric condition on $X$ that ensures that the corresponding Lagrange interpolation polynomial (of fixed degree) of a sufficient smooth function converges to a Taylor polynomial. In a recent work, based on a beautiful result of Bos and Calvi, Calvi and Phung [9] proved that the limit of Lagrange projectors at Bos configurations on irreducible algebraic curves in $\mathbb{R}^2$ are the Hermite projectors introduced by Bos and Calvi [5, 6]. Note that in [9] the authors fixed the curves and let Bos configurations move along them to Taylorian points.

In this paper, we collect interpolation points from disjoint circles to get a unisolvent set $X$. Then we fix the centers of the circles and let the radii tend to 0. We prove in Theorem 4.1 that the Lagrange interpolation polynomial converges to the Hermite interpolation polynomial given in Theorem 2.1. Theorem 4.2 is a slight modification of Theorem 4.1 in which the radius of the circle containing the largest number of interpolation points remains constant. We show that the corresponding Lagrange interpolation tends to the mixed Lagrange/Hermite interpolation constructed in Theorem 2.6.

The tools for proving the convergence theorems are given in Section 3 in which we study the limit of Vandermonde determinants on circles when the radii tend to 0. Note that there are many ways to collapse points in $\mathbb{R}^n$, $n \geq 2$. In general, each way will give a Hermite interpolation scheme. For a recent account of the theory of Hermite interpolation, we refer the reader to [11, 13] and the references therein.

**2. A new Hermite scheme in $\mathbb{R}^2$.** As usual, to $\mathbf{x} = (x, y) \in \mathbb{R}^2$, we associate the complex number $x + yi$ which we still denote by $\mathbf{x}$. It is understood that $\mathbf{x}$ is a complex number when we write $\mathbf{x}^k$ for $k \in \mathbb{N}$. The Euclidean norm of $\mathbf{x}$ is defined by $|\mathbf{x}| = \sqrt{x^2 + y^2}$. The derivatives in the complex setting are defined as usual,

$$\frac{\partial}{\partial \mathbf{x}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{\mathbf{x}}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If $f$ and $g$ are sufficiently differentiable real-valued functions in a neighborhood of $\mathbf{a} \in \mathbb{R}^2$, we have the law $\overline{\left(\frac{\partial^k f(\mathbf{a})}{\partial \mathbf{x}^k}\right)} = \frac{\partial^k f(\mathbf{a})}{\partial \overline{\mathbf{x}}^k}$ and the Leibniz formula

$$\frac{\partial^k (fg)(\mathbf{a})}{\partial \mathbf{x}^k} = \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^{k-j} f(\mathbf{a})}{\partial \mathbf{x}^{k-j}} \cdot \frac{\partial^j g(\mathbf{a})}{\partial \mathbf{x}^j}. \tag{2.1}$$

Moreover, since for each $k \geq 1$,

$$\frac{\partial^k f(\mathbf{a})}{\partial \mathbf{x}^k} = \frac{1}{2^k}\Bigg(\sum_{0\leq j\leq k,\, j \text{ even}} (-1)^{j/2}\binom{k}{j}\frac{\partial f^k(\mathbf{a})}{\partial x^{k-j}\partial y^j} - i \sum_{0\leq j\leq k,\, j \text{ odd}} (-1)^{(j-1)/2}\binom{k}{j}\frac{\partial f^k(\mathbf{a})}{\partial x^{k-j}\partial y^j}\Bigg),$$

the relation $\frac{\partial^k f(\mathbf{a})}{\partial \mathbf{x}^k} = a + bi$ is equivalent to the following two relations in the real setting:

$$\sum_{0\leq j\leq k,\, j \text{ even}} (-1)^{j/2}\binom{k}{j}\frac{\partial f^k(\mathbf{a})}{\partial x^{k-j}\partial y^j} = 2^k a,$$

$$\sum_{0\leq j\leq k,\, j \text{ odd}} (-1)^{(j-1)/2}\binom{k}{j}\frac{\partial f^k(\mathbf{a})}{\partial x^{k-j}\partial y^j} = -2^k b.$$

The following theorem gives a poised Hermite scheme.

THEOREM 2.1. *Let $d \geq 2$ be a positive integer and $m = [d/2] + 1$. Let $s_k = d - 2k + 2$ for $k = 1, \ldots, m$ and let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be $m$ distinct points in $\mathbb{R}^2$. Then the Hermite scheme*

$$\frac{\partial^j H(\mathbf{a}_k)}{\partial \mathbf{x}^j} = f_{jk}, \qquad \forall 1 \leq k \leq m,\, 0 \leq j \leq s_k, \tag{2.2}$$

*is poised for $\mathcal{P}_d(\mathbb{R}^2)$. Here the $\{f_{jk}\}$ are any given data.*

*Proof.* We first remark that (2.2) gives $2s_k + 1 = 2d - 4k + 5$ interpolation conditions for $k = 1, \ldots, m$. Hence the total number of interpolation conditions is equal to

$$\sum_{k=1}^{m}(2d - 4k + 5) = 2md - 2m(m+1) + 5m = \binom{d+2}{2} = m_d(\mathbb{R}^2).$$

To prove the Hermite scheme is poised, it suffices to check that if $H \in \mathcal{P}_d(\mathbb{R}^2)$ and

$$\frac{\partial^j H(\mathbf{a}_k)}{\partial \mathbf{x}^j} = 0, \qquad \forall 1 \leq k \leq m,\, 0 \leq j \leq s_k, \tag{2.3}$$

then $H = 0$. Since $H$ is a polynomial of degree at most $d$, it is equal to its Taylor expansion at $\mathbf{a}_1$ in the complex form (see for instance [12, p. 74]):

$$H(\mathbf{x}) = \sum_{j+k\leq d} \frac{1}{j!k!}\frac{\partial^{j+k} H(\mathbf{a}_1)}{\partial \mathbf{x}^j \partial \overline{\mathbf{x}}^k}(\mathbf{x} - \mathbf{a}_1)^j(\overline{\mathbf{x}} - \overline{\mathbf{a}}_1)^k. \tag{2.4}$$

From (2.3) we have

$$\frac{\partial^j H(\mathbf{a}_1)}{\partial \mathbf{x}^j} = \frac{\partial^j H(\mathbf{a}_1)}{\partial \overline{\mathbf{x}}^j} = 0, \qquad \forall j = 0, \ldots, s_1.$$

Since $s_1 = d$, we conclude from (2.4) that

$$H(\mathbf{x}) = (\mathbf{x}-\mathbf{a}_1)(\overline{\mathbf{x}}-\overline{\mathbf{a}}_1)\sum_{\substack{j,k>0\\ j+k\le d}}\frac{1}{j!k!}\frac{\partial^{j+k}H(\mathbf{a}_1)}{\partial\mathbf{x}^j\partial\overline{\mathbf{x}}^k}(\mathbf{x}-\mathbf{a}_1)^{j-1}(\overline{\mathbf{x}}-\overline{\mathbf{a}}_1)^{k-1}$$
$$= |\mathbf{x}-\mathbf{a}_1|^2 H_1(\mathbf{x}),$$

where $H_1 \in \mathcal{P}_{d-2}(\mathbb{R}^2)$. Applying the Leibniz formula for $H_1(\mathbf{x}) = \frac{H(\mathbf{x})}{|\mathbf{x}-\mathbf{a}_1|^2}$ and using (2.3) again, we obtain

$$\frac{\partial^j H_1(\mathbf{a}_k)}{\partial\mathbf{x}^j} = 0, \qquad \forall 2\le k\le m,\ 0\le j\le s_k.$$

By similar arguments, we have $H_1(\mathbf{x}) = |\mathbf{x}-\mathbf{a}_2|^2 H_2(\mathbf{x})$ with $H_2 \in \mathcal{P}_{d-4}(\mathbb{R}^2)$. We continue in this fashion to obtain

$$H(\mathbf{x}) = \prod_{k=1}^{m}|\mathbf{x}-\mathbf{a}_k|^2 H_m(\mathbf{x}), \qquad H_m \in \mathcal{P}(\mathbb{R}^2).$$

It follows from the last relation that $H = 0$. Conversely, suppose that $H \neq 0$. Then the degree of the polynomial on the right hand side is at least $2m > d$. This contradics the fact that $\deg H \le d$, and the proof is complete. ∎

We shall give a formula for the Hermite interpolation polynomial. Let $f$ be a real-valued function of class $C^k$ in a neighborhood of $\mathbf{a}\in\mathbb{R}^2$. Define

$$(2.5)\qquad H^k_{\mathbf{a}}(f)(\mathbf{x}) = f(\mathbf{a}) + \sum_{j=1}^{k}\frac{1}{j!}\Big(\frac{\partial^j f(\mathbf{a})}{\partial\mathbf{x}^j}(\mathbf{x}-\mathbf{a})^j + \frac{\partial^j f(\mathbf{a})}{\partial\overline{\mathbf{x}}^j}(\overline{\mathbf{x}}-\overline{\mathbf{a}})^j\Big)$$
$$= f(\mathbf{a}) + 2\,\mathrm{Re}\sum_{j=1}^{k}\frac{1}{j!}\frac{\partial^j f(\mathbf{a})}{\partial\mathbf{x}^j}(\mathbf{x}-\mathbf{a})^j.$$

It is easily seen that $H^k_{\mathbf{a}}(f)$ is a polynomial in $\mathcal{P}_k(\mathbb{R}^2)$. Moreover,

$$(2.6)\qquad \frac{\partial^j H^k_{\mathbf{a}}(f)(\mathbf{a})}{\partial\mathbf{x}^j} = \frac{\partial^j f(\mathbf{a})}{\partial\mathbf{x}^j}, \qquad \forall 0\le j\le k.$$

If $k = 0$, then the empty sum in (2.5) is taken to be 0. In this case, we have $H^0_{\mathbf{a}}(f)(\mathbf{x}) = f(\mathbf{a})$.

THEOREM 2.2. *With the assumptions of Theorem* 2.1*, the Hermite interpolation polynomial of a function $f$ of class $C^{s_k}$ in a neighborhood of $\mathbf{a}_k$, $k = 1,\ldots,m$, is given by the formula*

$$H = \sum_{k=1}^{m} T_k$$

*where*

$$T_1(\mathbf{x}) = H^{s_1}_{\mathbf{a}_1}(f)(\mathbf{x}), \qquad T_k(\mathbf{x}) = \prod_{j=1}^{k-1} |\mathbf{x}-\mathbf{a}_j|^2 H^{s_k}_{\mathbf{a}_k}\Big(\frac{f - T_1 - \cdots - T_{k-1}}{\prod_{j=1}^{k-1}|\cdot - \mathbf{a}_j|^2}\Big)(\mathbf{x}),$$

*for* $k = 2, \dots, m$.

*Proof.* We see at once that $H \in \mathcal{P}_d(\mathbb{R}^2)$. We need to verify that

$$\frac{\partial^j H(\mathbf{a}_k)}{\partial \mathbf{x}^j} = \frac{\partial^j f(\mathbf{a}_k)}{\partial \mathbf{x}^j}, \qquad \forall 1 \le k \le m,\ 0 \le j \le s_k. \tag{2.7}$$

For simplicity of notation, set $q_k(\mathbf{x}) = |\mathbf{x}-\mathbf{a}_k|^2$. As $q_k(\mathbf{x}) = (\mathbf{x}-\mathbf{a}_k)(\overline{\mathbf{x}}-\overline{\mathbf{a}}_k)$, we have

$$\frac{\partial^j q_k(\mathbf{a}_k)}{\partial \mathbf{x}^j} = \frac{\partial^j q_k(\mathbf{a}_k)}{\partial \overline{\mathbf{x}}^j} = 0, \qquad \forall j = 0, 1, \dots,\ k = 1, \dots, m. \tag{2.8}$$

Fix $k \in \{1, \dots, m\}$. Then for every $n > k$, $T_n$ contains the factor $q_k$. Therefore, relation (2.8) and the Leibniz formula imply that

$$\frac{\partial^j T_n(\mathbf{a}_k)}{\partial \mathbf{x}^j} = 0, \qquad \forall 0 \le j \le s_k,\ k < n \le m. \tag{2.9}$$

To shorten notation, we write $Q_{k-1}$ for $q_1 \cdots q_{k-1}$. Now applying the Leibniz formula again and the interpolation property of $H^{s_k}_{\mathbf{a}_k}(\cdot)$ in (2.6) we find that, for $0 \le j \le s_k$,

$$\begin{aligned}
\frac{\partial^j T_k(\mathbf{a}_k)}{\partial \mathbf{x}^j} &= \frac{\partial^j \big(Q_{k-1} H^{s_k}_{\mathbf{a}_k}\big(\frac{f-T_1-\cdots-T_{k-1}}{Q_{k-1}}\big)\big)}{\partial \mathbf{x}^j}(\mathbf{a}_k) \\
&= \sum_{l=0}^{j} \binom{j}{l} \frac{\partial^{j-l} Q_{k-1}}{\partial \mathbf{x}^{j-l}}(\mathbf{a}_k) \cdot \frac{\partial^l \big(H^{s_k}_{\mathbf{a}_k}\big(\frac{f-T_1-\cdots-T_{k-1}}{Q_{k-1}}\big)\big)}{\partial \mathbf{x}^l}(\mathbf{a}_k) \\
&= \sum_{l=0}^{j} \binom{j}{l} \frac{\partial^{j-l} Q_{k-1}}{\partial \mathbf{x}^{j-l}}(\mathbf{a}_k) \cdot \frac{\partial^l \big(\frac{f-T_1-\cdots-T_{k-1}}{Q_{k-1}}\big)}{\partial \mathbf{x}^l}(\mathbf{a}_k) \\
&= \frac{\partial^j \big(Q_{k-1} \frac{f-T_1-\cdots-T_{k-1}}{Q_{k-1}}\big)}{\partial \mathbf{x}^j}(\mathbf{a}_k) = \frac{\partial^j f(\mathbf{a}_k)}{\partial \mathbf{x}^j} - \sum_{l=1}^{k-1} \frac{\partial^j T_l(\mathbf{a}_k)}{\partial \mathbf{x}^j}.
\end{aligned}$$

Combining the last relation with (2.9), we obtain (2.7). ∎

Definition 2.3. The Hermite interpolation polynomial of a function $f$ in Theorems 2.2 is denoted briefly by $\mathbf{H}[(A, S); f]$, where $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and $S = \{s_1, \dots, s_m\}$.

Example 2.4. Take $d = 2$. Then $m = 2$, $s_1 = 2$ and $s_2 = 0$. The interpolation conditions for $\mathbf{H}[(\{\mathbf{a}_1, \mathbf{a}_2\}, \{2, 0\}); f]$ at $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ are given by

$$f \mapsto f(\mathbf{a}_1),\ f \mapsto \frac{\partial}{\partial x} f(\mathbf{a}_1),\ f \mapsto \frac{\partial}{\partial y} f(\mathbf{a}_1),\ f \mapsto \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f(\mathbf{a}_1),$$
$$f \mapsto \frac{\partial^2}{\partial x \partial y} f(\mathbf{a}_1),\ f \mapsto f(\mathbf{a}_2).$$

According to Theorem 2.2, the Hermite interpolation polynomial is given by $\mathbf{H}[(\{\mathbf{a}_1, \mathbf{a}_2\}, \{2, 0\}); f] = T_1 + T_2$, where

$$T_1(\mathbf{x}) = f(\mathbf{a}_1) + \frac{\partial f(\mathbf{a}_1)}{\partial x}(x - b_1) + \frac{\partial f(\mathbf{a}_1)}{\partial y}(y - c_1) + \frac{\partial^2 f(\mathbf{a}_1)}{\partial x \partial y}(x - b_1)(y - c_1)$$
$$+ \frac{1}{4}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x^2} - \frac{\partial^2 f(\mathbf{a}_1)}{\partial y^2}\right)((x - b_1)^2 - (y - c_1)^2), \qquad \mathbf{a}_1 = (b_1, c_1),$$

and

$$T_2(\mathbf{x}) = \frac{f(\mathbf{a}_2) - T_1(\mathbf{a}_2)}{|\mathbf{a}_2 - \mathbf{a}_1|^2} |\mathbf{x} - \mathbf{a}_1|^2.$$

REMARK 2.5. There is another way to obtain the conditions for the factorization in Theorem 2.1. Let $H \in \mathcal{P}_d(\mathbb{R}^2)$ and $q(\mathbf{x}) = |\mathbf{x} - \mathbf{a}|^2$ with $\mathbf{a} = (a, b)$. We have $q(\mathbf{x}) = [(y - b) + i(x - a)][(y - b) - i(x - a)]$. By [14, Lemma 2.5], $(y-b)+i(x-a)$ is a factor of $H$ if $\widetilde{H}(x) := H(x, b-i(x-a)) = 0$ for all $x \in \mathbb{R}$. Since $\widetilde{H}$ is a univariate polynomial of degree at most $d$, $H = 0$ is equivalent to

$$\frac{d^k \widetilde{H}}{dx^k}(a) = 0, \qquad k = 0, 1, \ldots, d. \tag{2.10}$$

Since $\frac{d\widetilde{H}(x)}{dx} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) H(x, b - i(x - a))$, relation (2.10) gives

$$\frac{\partial^k H}{\partial \mathbf{x}^k}(\mathbf{a}) = 0, \qquad k = 0, 1, \ldots, d. \tag{2.11}$$

Similarly, $(y - b) - i(x - a)$ is a factor of $H$ if

$$\frac{\partial^k H}{\partial \overline{\mathbf{x}}^k}(\mathbf{a}) = 0, \qquad k = 0, 1, \ldots, d, \tag{2.12}$$

which is equivalent to (2.11). Hence $H$ is a multiple of $q$ if (2.11) holds.

A well-known result shows that the set $\mathcal{P}_d(\mathbb{R}^2)$ restricted to the circle $C(0, r) = \{\mathbf{x} : |\mathbf{x}| = r\}$ forms a $(2d + 1)$-dimensional vector space whose basis can be taken to be

$$\mathcal{B} = \{1, \mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x}), \ldots, \mathrm{Re}(\mathbf{x}^d), \mathrm{Im}(\mathbf{x}^d)\}. \tag{2.13}$$

Let $X$ be a set of $2d+1$ distinct points on $C(0, r)$. For a function $f$ defined on $X$, we set

$$\mathbb{L}[X; f](\mathbf{x}) = \sum_{p \in \mathcal{B}} \frac{\mathrm{VDM}(\mathcal{B}[p \leftarrow f]; X)}{\mathrm{VDM}(\mathcal{B}; X)} p(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2. \tag{2.14}$$

By Lemma 3.3 below, $\mathrm{VDM}(\mathcal{B}; X) \neq 0$. Therefore $\mathbb{L}[X; f]$ is well-defined and belongs to $\mathcal{P}_d(\mathbb{R}^2)$. When we restrict the polynomial $\mathbb{L}[X; f](\mathbf{x})$ to $C(0, r)$, it is identical with the Lagrange interpolation polynomial of $f$ at $X$ on $C(0, r)$. In other words,

$$\mathbb{L}[X; f](\mathbf{a}) = f(\mathbf{a}), \quad \forall \mathbf{a} \in X. \tag{2.15}$$

As usual, when a Hermite scheme is constructed by using the factorization method, it can combine with another interpolation scheme (of Hermite type) to make up a new interpolation process. Here we only exhibit the simplest case when the additional interpolation process is Lagrange interpolation.

Theorem 2.6. *Let $d \geq 2$ be a positive integer and $m = [d/2] + 1$. Let $s_k = d - 2k + 2$ for $k = 2, \ldots, m$ and let $A' = \{\mathbf{a}_2, \ldots, \mathbf{a}_m\}$ be $m - 1$ distinct points in $\mathbb{R}^2$. Let $B$ be a set of $2d + 1$ distinct points on the circle $C(\mathbf{a}_1, r_1)$ with $C(\mathbf{a}_1, r_1) \cap A' = \emptyset$. Then, for any sufficiently smooth function $f$, there exists a unique $H \in \mathcal{P}_d(\mathbb{R}^2)$ such that*

$$H(\mathbf{a}) = f(\mathbf{a}), \quad \forall \mathbf{a} \in B,$$

*and*

$$\frac{\partial^j H(\mathbf{a}_k)}{\partial \mathbf{x}^j} = \frac{\partial^j f(\mathbf{a}_k)}{\partial \mathbf{x}^j}, \quad \forall 2 \leq k \leq m, \, 0 \leq j \leq s_k,$$

*Furthermore,*

$$H = \sum_{k=1}^{m} \widetilde{T}_k \tag{2.16}$$

*where*

$$\widetilde{T}_1(\mathbf{x}) = \mathbb{L}[B; f](\mathbf{x}), \qquad \widetilde{T}_k(\mathbf{x}) = q(\mathbf{x}) \prod_{j=2}^{k-1} q_j(\mathbf{x}) H_{\mathbf{a}_k}^{s_k}\Big(\frac{f - \widetilde{T}_1 - \cdots - \widetilde{T}_{k-1}}{q \prod_{j=2}^{k-1} q_j}\Big)(\mathbf{x}),$$

*$k = 2, \ldots, m$. Here $q(\mathbf{x}) = |\mathbf{x} - \mathbf{a}_1|^2 - r_1^2$ and $q_k(\mathbf{x}) = |\mathbf{x} - \mathbf{a}_k|^2$ for $k = 2, \ldots, m$.*

*Proof.* To verify the first assertion, it suffices to prove that if $H \in \mathcal{P}_d(\mathbb{R}^2)$ vanishes on $B$ and satisfies

$$\frac{\partial^j H(\mathbf{a}_k)}{\partial \mathbf{x}^j} = 0, \quad \forall 2 \leq k \leq m, \, 0 \leq j \leq s_k, \tag{2.17}$$

then $H$ is identically zero. By Lemma 3.3 below, $B$ is unisolvent for the space $\mathcal{P}_d(C(\mathbf{a}_1, r_1))$. The assumption shows that $H$ must vanish on $C(\mathbf{a}_1, r_1)$. It follows that $H$ is a multiple of $q$. This enables us to write $H = qH_1$ with $H_1 \in \mathcal{P}_{d-2}(\mathbb{R}^2)$. Using the Leibniz formula for $H/q$ and relation (2.17) we obtain

$$\frac{\partial^j H_1(\mathbf{a}_k)}{\partial \mathbf{x}^j} = 0, \quad \forall 2 \leq k \leq m, \, 0 \leq j \leq s_k, \tag{2.18}$$

Analysis similar to that in the proof of Theorem 2.1 shows that $H_1 = 0$. Hence $H = 0$. To prove the formula for the interpolation polynomial, we make use of a reasoning inspired from the proof of Theorem 2.2. We only need to show that $H$ given in (2.16) satisfies the interpolation conditions. Since $\widetilde{T}_k$ is a multiple of $q$ for all $k = 2, \dots, m$, we have $\widetilde{T}_k(\mathbf{a}) = 0$ for all $\mathbf{a} \in B$ and $2 \le k \le m$. It follows that

$$H(\mathbf{a}) = \widetilde{T}_1(\mathbf{a}) = \mathbb{L}[B; f](\mathbf{a}) = f(\mathbf{a}), \qquad \forall \mathbf{a} \in B,$$

where we use (2.15) to obtain the last relation. The rest of the proof runs as before. The details are left to the reader. ∎

DEFINITION 2.7. The Hermite interpolation polynomial of a function $f$ in Theorem 2.6 is denoted briefly by $\mathbf{H}[B, (A', S'); f]$, where $S' = \{s_2, \dots, s_m\}$.

**3. Vandermonde determinants on circles.** The aim of this section is to study the asymptotic behavior of Vandermonde determinants at points lying on a circle when the radius goes to 0. We only work with circles centered at the origin. But all results still hold for arbitrary centers. We recall the basis $\mathcal{B}$ for $\mathcal{P}_d(C(0,r))$,

$$\mathcal{B} = \{1, \mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x}), \dots, \mathrm{Re}(\mathbf{x}^d), \mathrm{Im}(\mathbf{x}^d)\}.$$

In the trigonometric form, by setting $\mathbf{x} = e^{i\theta}$, we get a basis for the space of all trigonometric polynomials of degree at most $d$,

$$\mathcal{T} = \{1, \cos\theta, \sin\theta, \dots, \cos d\theta, \sin d\theta\}. \tag{3.1}$$

LEMMA 3.1. *Let $P$ be a polynomial of degree at most $d$ in $\mathbb{R}^2$. If we write*

$$P(\mathbf{x}) = c_{r,0} + \sum_{k=1}^{d} c_{r,k}\,\mathrm{Re}(\mathbf{x}^k) + \sum_{k=1}^{d} d_{r,k}\,\mathrm{Im}(\mathbf{x}^k), \qquad \mathbf{x} \in C(0,r), \tag{3.2}$$

*then* $\lim_{r\to 0} c_{r,0} = P(0)$ *and for* $1 \le k \le d$,

$$\lim_{r\to 0} c_{r,k} = \frac{1}{k!}\left(\frac{\partial^k P(0)}{\partial \mathbf{x}^k} + \frac{\partial^k P(0)}{\partial \overline{\mathbf{x}}^k}\right), \qquad \lim_{r\to 0} d_{r,k} = \frac{i}{k!}\left(\frac{\partial^k P(0)}{\partial \mathbf{x}^k} - \frac{\partial^k P(0)}{\partial \overline{\mathbf{x}}^k}\right). \tag{3.3}$$

*Proof.* In the polar coordinate $\mathbf{x} = (r\cos\theta, r\sin\theta)$, $\theta \in [0, 2\pi)$, equation (3.2) becomes

$$\begin{aligned} &P(r\cos\theta, r\sin\theta) \\ &\quad = c_{r,0} + \sum_{k=1}^{d} c_{r,k} r^k \cos k\theta + \sum_{k=1}^{d} d_{r,k} r^k \sin k\theta, \qquad \theta \in [0, 2\pi). \end{aligned} \tag{3.4}$$

On the other hand, since $P \in \mathcal{P}_d(\mathbb{R}^2)$, we have

$$P(\mathbf{x}) = \sum_{j+k\le d} \frac{1}{j!k!} \frac{\partial^{j+k} P(0)}{\partial \mathbf{x}^j \partial \overline{\mathbf{x}}^k} \mathbf{x}^j \overline{\mathbf{x}}^k.$$

Substituting $\mathbf{x} = r(\cos\theta + i\sin\theta)$ we obtain

$$(3.5)\qquad P(r\cos\theta, r\sin\theta) = \sum_{j+k\le d} \frac{1}{j!k!}\frac{\partial^{j+k}P(0)}{\partial\mathbf{x}^j\partial\overline{\mathbf{x}}^k} r^{j+k}\big(\cos(j-k)\theta + i\sin(j-k)\theta\big).$$

Comparing the coefficients of the trigonometric polynomials on the right hand sides of (3.4) and (3.5) we have

$$c_{r,0} = P(0) + rg_0(r), \qquad c_{r,j} = \frac{1}{j!}\bigg(\frac{\partial^j P(0)}{\partial\mathbf{x}^j} + \frac{\partial^j P(0)}{\partial\overline{\mathbf{x}}^j}\bigg) + rg_j(r),$$

and

$$d_{r,j} = \frac{i}{j!}\bigg(\frac{\partial^j P(0)}{\partial\mathbf{x}^j} - \frac{\partial^j P(0)}{\partial\overline{\mathbf{x}}^j}\bigg) + rh_j(r), \qquad 1 \le j \le d,$$

where $g_j(r), h_j(r)$ are polynomials in $r$. In the formulas for $c_{r,j}$ and $d_{r,j}$ we let $r \to 0$ and get the desired equations. ∎

DEFINITION 3.2. Let $\delta > 0$. A set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{2d+1}\} \subset C(\mathbf{a}, r)$ with $\mathbf{x}_k = \mathbf{a} + re^{i\theta_k}$ for $k = 1, \ldots, 2d+1$ is said to be *$\delta$-separate* if

$$|e^{i\theta_k} - e^{i\theta_j}| \ge \delta, \qquad \forall k \ne j.$$

LEMMA 3.3. *Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{2d+1}\}$ be $2d+1$ distinct points on $C(0,r)$ with $\mathbf{x}_k = re^{i\theta_k}$. Then*

$$\mathrm{VDM}(\mathcal{B}; X) = (-2i)^{-d} e^{-di(\theta_1+\cdots+\theta_{2d+1})} r^{d(d+1)} \prod_{1\le j<k\le 2d+1} (e^{i\theta_k} - e^{i\theta_j}).$$

*In particular, $X$ is unisolvent for $\mathcal{P}_d(C(0,r))$. Moreover, if $X$ is $\delta$-separate for $\delta > 0$, then*

$$|\mathrm{VDM}(\mathcal{B}; X)| \ge 2^{-d} r^{d(d+1)} \delta^{d(2d+1)}.$$

*Proof.* It is easily seen that

$$(3.6)\qquad \mathrm{VDM}(\mathcal{B}; X) = r^{d(d+1)}\mathrm{VDM}(\mathcal{T}; \Theta),$$

where $\Theta = \{\theta_1, \ldots, \theta_{2d+1}\}$. Here $\mathrm{VDM}(\mathcal{T}; \Theta) = \det[h(\theta)]_{h\in\mathcal{T},\theta\in\Theta}$ is the (generalized) Vandermonde determinant. It follows from the computation in [10, pp. 30–31] that

$$(3.7)\qquad \mathrm{VDM}(\mathcal{T}; \Theta) = (-2i)^{-d} e^{-di(\theta_1+\cdots+\theta_{2d+1})} \prod_{1\le j<k\le 2d+1} (e^{i\theta_k} - e^{i\theta_j}).$$

Hence the first assertion is proved. Next, it is easily seen that $\mathrm{VDM}(\mathcal{B}; X) \ne 0$. Hence $X$ is unisolvent for $\mathcal{P}_d(C(0,r))$. The last assertion is a consequence of the fact that $|e^{i\theta_k} - e^{i\theta_j}| \ge \delta$ for all $k \ne j$,

$$(3.8)\qquad |\mathrm{VDM}(\mathcal{T}; \Theta)| \ge 2^{-d}\delta^{d(2d+1)}. \ ∎$$

LEMMA 3.4. *Let $\delta > 0$ and $\{r_n\}$ a sequence of positive numbers tending to 0. For each $n$, let $X_n$ be a set of $2d+1$ points on $C(0, r_n)$ such that*

*$X_n$ is $\delta$-separate. Then for any function $f$ of class $C^d$ in a neighborhood of $0 \in \mathbb{R}^2$ we have*

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{B}[1 \leftarrow f]; X_n)}{\mathrm{VDM}(\mathcal{B}; X_n)} = f(0),$$

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow f]; X_n)}{\mathrm{VDM}(\mathcal{B}; X_n)} = \frac{1}{k!}\left(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\right),$$

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{B}[\mathrm{Im}(\mathbf{x}^k) \leftarrow f]; X_n)}{\mathrm{VDM}(\mathcal{B}; X_n)} = \frac{i}{k!}\left(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} - \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\right), \qquad 1 \le k \le d,$$

*where $\mathcal{B}[p \leftarrow f]$ means that we substitute $f$ for $p$ in $\mathcal{B}$.*

*Proof.* We will denote by $\mathbf{T}_0^d(f)$ the Taylor expansion of $f$ at 0 up to order $d$. By the Taylor theorem, we can write

$$f(\mathbf{x}) = \mathbf{T}_0^d(f)(\mathbf{x}) + |\mathbf{x}|^d \varphi(\mathbf{x}), \tag{3.9}$$

where $\varphi(\mathbf{x}) \to 0$ as $\mathbf{x} \to 0$. The polynomial $\mathbf{T}_0^d(f)$ restricted to $C(0,r)$ is a linear combination of polynomials in $\mathcal{B}$, that is,

$$\mathbf{T}_0^d(f)(\mathbf{x}) = c_{r,0} + \sum_{k=1}^{d} c_{r,k}\,\mathrm{Re}(\mathbf{x}^k) + \sum_{k=1}^{d} d_{r,k}\,\mathrm{Im}(\mathbf{x}^k), \qquad \mathbf{x} \in C(0,r). \tag{3.10}$$

We first deal with $\mathrm{Re}(\mathbf{x}^k)$ for $k = 1, \dots, d$. Looking at (3.9), (3.10) and using the column operation rule for the Vandermonde determinant

$$\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow f]; X_n)$$

we get

$$\begin{aligned}\mathrm{VDM}&(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow f]; X_n)\\ &= \mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow c_{r_n,k}\,\mathrm{Re}(\mathbf{x}^k) + |\mathbf{x}|^d\varphi(\mathbf{x})]; X_n).\end{aligned}$$

Writing the right hand side in the trigonometric form as in the proof of Lemma 3.3, we obtain

$$\begin{aligned}\mathrm{VDM}&(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow f]; X_n)\\ &= r_n^{d(d+1)}\mathrm{VDM}(\mathcal{T}[\cos k\theta \leftarrow c_{r_n,k}\cos k\theta + r_n^{d-k}\varphi(r_n e^{i\theta})]; \varTheta_n)\\ &= r_n^{d(d+1)}\big(c_{r_n,k}\mathrm{VDM}(\mathcal{T}; \varTheta_n) + \mathrm{VDM}(\mathcal{T}[\cos k\theta \leftarrow r_n^{d-k}\varphi(r_n e^{i\theta})]; \varTheta_n)\big),\end{aligned}$$

where $\varTheta_n$ is the set of the arguments of $X_n$. From (3.6) we conclude that

$$\begin{aligned}&\frac{\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow f]; X_n)}{\mathrm{VDM}(\mathcal{B}; X_n)}\\ &\qquad = c_{r_n,k} + \frac{\mathrm{VDM}(\mathcal{T}[\cos k\theta \leftarrow r_n^{d-k}\varphi(r_n e^{i\theta})]; \varTheta_n)}{\mathrm{VDM}(\mathcal{T}; \varTheta_n)}.\end{aligned} \tag{3.11}$$

By Lemma 3.1, we have

$$\lim_{n\to\infty} c_{r_n,k} = \frac{1}{k!}\left(\frac{\partial^k \mathbf{T}_0^d(f)(0)}{\partial \mathbf{x}^k} + \frac{\partial^k \mathbf{T}_0^d(f)(0)}{\partial \overline{\mathbf{x}}^k}\right) = \frac{1}{k!}\left(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\right). \tag{3.12}$$

On the other hand, since $\varphi(r_n e^{i\theta}) \to 0$ as $n \to \infty$ uniformly in $\theta$, we get

$$\lim_{n\to\infty} \mathrm{VDM}(\mathcal{T}[\cos k\theta \leftarrow r_n^{d-k}\varphi(r_n e^{i\theta})]; \Theta_n) = 0. \tag{3.13}$$

Hence the fraction on the right hand side of (3.11) tends to 0 as $n \to \infty$ since $|\mathrm{VDM}(\mathcal{T}; \Theta_n)|$ is bounded from below by $2^{-d}\delta^{d(2d+1)}$ due to (3.8). It follows that

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(z^k) \leftarrow f]; X_n)}{\mathrm{VDM}(\mathcal{B}; X_n)} = \frac{1}{k!}\left(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\right), \qquad 1 \le k \le d. \tag{3.14}$$

The proofs of the remaining equations are similar. The details are left to the reader. ■

Remark 3.5. The condition that $X_n$ is $\delta$-separate cannot be removed. For simplicity, we work with three points. Set $\mathcal{B} = \{1, \mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x})\}$ and $X^r = \{re^{i\theta_{r,1}}, re^{i\theta_{r,2}}, re^{i\theta_{r,3}}\}$. The computation in Lemma 3.3 gives

$$\begin{aligned}\mathrm{VDM}(\mathcal{B}; X^r) &= (-2i)^{-1} r^2 e^{-i(\theta_{r,1}+\theta_{r,2}+\theta_{r,3})} \prod_{1\le j<k\le 3} (e^{i\theta_{r,k}} - e^{i\theta_{r,j}}) \\ &= 4r^2 \prod_{1\le j<k\le 3} \sin\frac{\theta_{r,k}-\theta_{r,j}}{2}.\end{aligned} \tag{3.15}$$

Take $f(\mathbf{x}) = y^{4/3}$. Clearly, $f \in C^1(\mathbb{R}^2)$. Now, for $k = 1, 2, 3$ choose $\theta_{r,k} = k^3 r^d$ with $d \ge 1$. Then

$$\begin{aligned}\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}) \leftarrow f]; X^r) &= \begin{vmatrix} 1 & (r\sin(r^d))^{4/3} & r\sin(r^d) \\ 1 & (r\sin(8r^d))^{4/3} & r\sin(8r^d) \\ 1 & (r\sin(27r^d))^{4/3} & r\sin(27r^d) \end{vmatrix} \\ &\sim r^{\frac{7}{3}(d+1)} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 16 & 8 \\ 1 & 81 & 27 \end{vmatrix} = -170 r^{\frac{7}{3}(d+1)}.\end{aligned}$$

where we use the Taylor expansion $\sin x = x + o(x)$ in the second relation.

On the other hand, from (3.15) we have

$$\mathrm{VDM}(\mathcal{B}; X^r) = 4r^2 \prod_{1\le j<k\le 3} \sin \frac{(k^3-j^3)r^d}{2} \sim 1729 r^{3d+2}.$$

Consequently,

$$\frac{\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}) \leftarrow f]; X^r)}{\mathrm{VDM}(\mathcal{B}; X^r)} \sim \frac{-170 r^{\frac{7}{3}(d+1)}}{1729 r^{3d+2}} = \frac{-170}{1729 r^{\frac{2}{3}d-1/3}}.$$

The last fraction tends to $-\infty$ as $r \to 0$.

Corollary 3.6. *Under the same assumptions of Lemma* 3.4*, we have*

$$\lim_{n\to\infty} \mathbb{L}[X_n, f](\mathbf{x}) = H_0^d(f)(\mathbf{x}),$$

*where* $\mathbb{L}[X_n; f]$ *is defined in* (2.14)*.*

*Proof.* By Lemma 3.4 and the formula for $\mathbb{L}[X_n, f](\mathbf{x})$, we have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{L}[X_n, f](\mathbf{x}) &= f(0) + \sum_{k=1}^{d}\Bigg[\frac{1}{k!}\bigg(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\bigg)\mathrm{Re}(\mathbf{x}^k) \\ &\quad + \frac{i}{k!}\bigg(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} - \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\bigg)\mathrm{Im}(\mathbf{x}^k)\Bigg] \\ &= f(0) + \sum_{k=1}^{d}\frac{1}{k!}\bigg(\frac{\partial^k f(0)}{\partial \mathbf{x}^k}\mathbf{x}^k + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\overline{\mathbf{x}}^k\bigg) = H_0^d(f)(\mathbf{x}),\end{aligned}$$

which completes the proof. ∎

Suppose $K$ is a compact neighborhood of $\mathbf{a} \in \mathbb{R}^2$. We denote by $C^d(K)$ the space of functions of class $C^d$ in neighborhoods of $K$. Let us define a norm on $C^d(K)$ by

$$\|f\|_{K,d} = \max_{j+k\le d} \sup_{\mathbf{x}\in K} \left|\frac{\partial^{j+k} f(\mathbf{x})}{\partial x^j \partial y^k}\right|, \qquad f \in C^d(K).$$

Definition 3.7. Let $\{f_n\}$ be a sequence of functions of class $C^d$ in a compact neighborhood $K$ of $\mathbf{a}$. We say that $\{f_n\}$ is *d-regular* at $\mathbf{a}$ if for every $\varepsilon_1 > 0$, there exist $\varepsilon_2 > 0$ and $n_0 \in \mathbb{N}$ such that

$$|\varphi_n(\mathbf{x})| < \varepsilon_1, \qquad \forall \mathbf{x} \in D(\mathbf{a}, \varepsilon_2),\ n > n_0,$$

where $\varphi_n$ is defined by

$$f_n(\mathbf{x}) = \mathbf{T}_{\mathbf{a}}^d(f_n)(\mathbf{x}) + |\mathbf{x}-\mathbf{a}|^d \varphi_n(\mathbf{x}).$$

Theorem 3.8. *Let* $\delta > 0$ *and* $\{r_n\}$ *a sequence of positive numbers tending to* 0*. For each* $n$*, let* $X_n$ *be a* $\delta$*-separate set of* $2d+1$ *points on* $C(\mathbf{a}, r_n)$*. If* $K$ *is a compact neighborhood of* $\mathbf{a}$*,* $\{f_n\} \subset C^d(K)$ *is d-regular at* $\mathbf{a}$ *and* $f_n \to f$ *in* $C^d(K)$*, then*

$$\lim_{n\to\infty} \mathbb{L}[X_n; f_n](\mathbf{x}) = H_{\mathbf{a}}^d(f)(\mathbf{x}),$$

*where* $\mathbb{L}[X_n; f_n]$ *is defined in* (2.14) *in which*

$$\mathcal{B} = \{1, \mathrm{Re}(\mathbf{x}-\mathbf{a}), \mathrm{Im}(\mathbf{x}-\mathbf{a}), \ldots, \mathrm{Re}((\mathbf{x}-\mathbf{a})^d), \mathrm{Im}((\mathbf{x}-\mathbf{a})^d)\}.$$

*Proof.* There is no loss of generality in assuming $\mathbf{a} = 0$. It suffices to prove that the fraction $\mathrm{VDM}(\mathcal{B}[p \leftarrow f_n]; X_n)/\mathrm{VDM}(\mathcal{B}; X_n)$ has the limit given in Lemma 3.4 for every $p \in \mathcal{B}$. We need to check that

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow f_n]; X_n)}{\mathrm{VDM}(\mathcal{B}; X_n)} = \frac{1}{k!}\Big(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\Big), \qquad 1 \le k \le d.$$

As in the proof of Lemma 3.4, we write $f_n$ as the sum of its Taylor expansion and an error,

$$f_n(\mathbf{x}) = \mathbf{T}_0^d(f_n)(\mathbf{x}) + |\mathbf{x}|^d \varphi_n(\mathbf{x}). \tag{3.16}$$

We also set

$$\begin{aligned} &\mathbf{T}_0^d(f_n)(\mathbf{x}) \\ &\quad = c_{r_n,0} + \sum_{k=1}^d c_{r_n,k} \mathrm{Re}(\mathbf{x}^k) + \sum_{k=1}^d d_{r_n,k} \mathrm{Im}(\mathbf{x}^k), \qquad \mathbf{x} \in C(0, r_n). \end{aligned} \tag{3.17}$$

By the arguments in the proof of Lemma 3.1, we have

$$\begin{aligned} c_{r_n,k} &= \frac{1}{k!}\Big(\frac{\partial^k \mathbf{T}_0^d(f_n)(0)}{\partial \mathbf{x}^k} + \frac{\partial^k \mathbf{T}_0^d(f_n)(0)}{\partial \overline{\mathbf{x}}^k}\Big) + r_n g_{n,k}(r_n) \\ &= \frac{1}{k!}\Big(\frac{\partial^k f_n(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f_n(0)}{\partial \overline{\mathbf{x}}^k}\Big) + r_n g_{n,k}(r_n), \end{aligned}$$

where $g_{n,k}(r_n)$ is a polynomial in $r_n$. Moreover, the coefficients of $g_{n,k}(r_n)$ are linear combinations of partial derivatives of $f_n$ at 0 of order up to $d$, and hence these coefficients are uniformly bounded in $n$. It follows that $r_n g_{n,k}(r_n) \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} c_{r_n,k} = \lim_{n\to\infty} \frac{1}{k!}\Big(\frac{\partial^k f_n(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f_n(0)}{\partial \overline{\mathbf{x}}^k}\Big) = \frac{1}{k!}\Big(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\Big).$$

Since $\{f_n\}$ is $d$-regular at 0, $\varphi_n(r_n e^{i\theta}) \to 0$ as $n \to \infty$ uniformly in $\theta$. A passage to the limit similar to the proof of Lemma 3.4 implies that

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{B}[\mathrm{Re}(\mathbf{x}^k) \leftarrow f_n]; X_n)}{\mathrm{VDM}(\mathcal{B}; X_n)} = \lim_{n\to\infty} c_{r_n,k} = \frac{1}{k!}\Big(\frac{\partial^k f(0)}{\partial \mathbf{x}^k} + \frac{\partial^k f(0)}{\partial \overline{\mathbf{x}}^k}\Big).$$

The proof is complete. ∎

**4. Lagrange interpolation at Bos configuration on circles.** Let $d \ge 2$ and $m = [d/2] + 1$. We set $s_k = d - 2k + 2$ for $k = 1, \ldots, m$. Let $C_1, \ldots, C_m$ be pairwise disjoint circles in $\mathbb{R}^2$. On $C_k$, we take a set $X_k$ of $2s_k + 1$ distinct points. In [4, Theorem 3.3], Bos pointed out that $X = \bigcup_{k=1}^m X_k$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$. Such a configuration of points will be called a *Bos configuration on circles*. Note that Bos considers two cases, when $d$ is even and odd. But our setting includes both of them.

THEOREM 4.1. *Let $\delta > 0$ and $d \geq 2$ be a positive integer. Define $m = [d/2] + 1$ and $S = \{s_1, \ldots, s_m\}$ with $s_k = d - 2k + 2$ for $k = 1, \ldots, m$. Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be $m$ distinct points in $\mathbb{R}^2$. To each point $\mathbf{a}_k$ is associated a sequence $\{C_k^n\}$ of circles $C_k^n = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}_k| = r_{k,n}\}$ such that $\lim_{n\to\infty} r_{k,n} = 0$. For $1 \leq k \leq m$ and $n \geq 1$, let $X_k^n$ be a $\delta$-separate set of $2s_k + 1$ points on $C_k^n$. Set $X^n = \bigcup_{k=1}^m X_k^n$. Then for any function $f$ of class $C^{s_k}$ in neighborhoods of the $\mathbf{a}_k$'s, we have*

$$\lim_{n\to\infty} \mathbf{L}[X^n; f] = \mathbf{H}[(A, S); f].$$

In Figure 1, we show a Bos configuration formed by 21=11+7+3 points on three pairwise disjoint circles. It is unisolvent for $\mathcal{P}_5(\mathbb{R}^2)$. Fix the centers and let the radii of the circles tend to 0. By Theorem 4.1, the Lagrange operator converges to the Hermite operator $\mathbf{H}[(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}, \{5, 3, 1\}); \cdot]$.

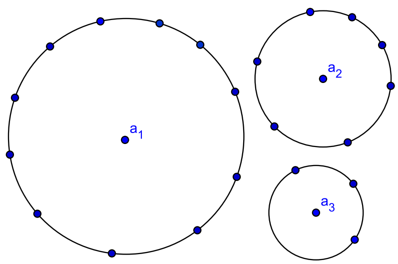


Fig. 1. An example of a Bos configuration

In the following result, we fix the radius of the first circle which contains the largest number of points. We let points on this circle move along it and get an analogous result.

THEOREM 4.2. *Let $\delta > 0$ and $d \geq 2$ be a positive integer. Define $m = [d/2] + 1$ and $S' = \{s_2, \ldots, s_m\}$ with $s_k = d - 2k + 2$ for $k = 2, \ldots, m$. Let $A' = \{\mathbf{a}_2, \ldots, \mathbf{a}_m\}$ be $m$ distinct points in $\mathbb{R}^2$. To each point $\mathbf{a}_k$ with $k = 2, \ldots, m$ is associated to a sequence $\{C_k^n\}$ of circles with $C_k^n = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}_k| = r_{k,n}\}$ such that $\lim_{n\to\infty} r_{k,n} = 0$. For $2 \leq k \leq m$ and $n \geq 1$, let $X_k^n$ be a $\delta$-separate set of $2s_k + 1$ points on $C_k^n$. Let $C(\mathbf{a}_1, r_1)$ be a fixed circle which is disjoint from $A'$. Let $X_1^n$ and $B$ be sets of $2d + 1$ distinct points on $C(\mathbf{a}_1, r_1)$ such that $X_1^n \to B$ as $n \to \infty$. Set $X^n = \bigcup_{k=1}^m X_k^n$. Then, for any continuous function $f$ on $C(\mathbf{a}_1, r_1)$ and of class $C^{s_k}$ in neighborhoods of the $\mathbf{a}_k$'s for $k = 2, \ldots, m$, we have*

$$\lim_{n\to\infty} \mathbf{L}[X^n; f] = \mathbf{H}[B, (A', S'); f]$$

*where the right hand side is defined in Definition* 2.7.

The assumption $X_1^n \to B$ as $n \to \infty$ is understood as follows. Assume that $X_1^n = \{\mathbf{b}_1^n, \dots, \mathbf{b}_{2d+1}^n\}$ and $B = \{\mathbf{b}_1, \dots, \mathbf{b}_{2d+1}\}$. Then $\lim_{n\to\infty} X_1^n = B$ if

$$\lim_{n\to\infty} \mathbf{b}_k^n = \mathbf{b}_k, \quad \forall k = 1, \dots, 2d+1.$$

We first need some auxiliary results.

LEMMA 4.3. *Let $K$ be a compact neighborhood of $\mathbf{a} \in \mathbb{R}^2$. Let $\{g_n\}$ be a sequence in $C^{d+1}(K)$ that converges to $g$ in $C^{d+1}(K)$. If $f \in C^d(K)$, then the sequence $\{fg_n\}$ is $d$-regular at $\mathbf{a}$.*

*Proof.* We can assume that $\mathbf{a} = 0$ and $K = \{\mathbf{x} : |\mathbf{x}| \le r\}$. We write

$$f(\mathbf{x}) = \mathbf{T}_0^d(f)(\mathbf{x}) + |\mathbf{x}|^d \varphi(\mathbf{x}), \quad g_n(\mathbf{x}) = \mathbf{T}_0^d(g_n)(\mathbf{x}) + |\mathbf{x}|^d \psi_n(\mathbf{x}). \tag{4.1}$$

We have $\lim_{\mathbf{x}\to 0} \varphi(\mathbf{x}) = 0$. Since $g_n$ converges to $g$ on $C^{d+1}(K)$, the sequence $\{D^\alpha g_n\}$ is uniformly bounded on $K$ for all $|\alpha| \le d+1$ and we can find $M_1 > 0$ such that $\psi_n(\mathbf{x}) \le M_1|\mathbf{x}|$, $\mathbf{x} \in K$. Here, to shorten the notation, we write $D^\alpha f$ for $\frac{\partial^{\alpha_1+\alpha_2} f}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$ with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$. From (4.1) we get

$$\begin{aligned} f(\mathbf{x})g_n(\mathbf{x}) &= \big(\mathbf{T}_0^d(f)(\mathbf{x}) + |\mathbf{x}|^d\varphi(\mathbf{x})\big)\big(\mathbf{T}_0^d(g_n)(\mathbf{x}) + |\mathbf{x}|^d\psi_n(\mathbf{x})\big) \\ &= \Big(\sum_{|\beta|\le d} \frac{1}{\beta!} D^\beta(f)(0)\mathbf{x}^\beta + |\mathbf{x}|^d\varphi(\mathbf{x})\Big)\Big(\sum_{|\gamma|\le d} \frac{1}{\gamma!} D^\gamma(g_n)(0)\mathbf{x}^\gamma + |\mathbf{x}|^d\psi_n(\mathbf{x})\Big). \end{aligned}$$

On the other hand, using the Leibniz formula for partial derivatives, we obtain

$$\begin{aligned} \mathbf{T}_0^d(fg_n)(\mathbf{x}) &= \sum_{|\alpha|=0}^{d} \frac{1}{\alpha!} D^\alpha(fg_n)(0)\mathbf{x}^\alpha \\ &= \sum_{|\alpha|=0}^{d} \frac{\mathbf{x}^\alpha}{\alpha!} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^\beta(f)(0) D^\gamma(g_n)(0) \\ &= \sum_{|\beta|+|\gamma|\le d} \Big(\frac{1}{\beta!} D^\beta(f)(0)\mathbf{x}^\beta\Big)\Big(\frac{1}{\gamma!} D^\gamma(g_n)(0)\mathbf{x}^\gamma\Big). \end{aligned}$$

Consequently,

$$\begin{aligned} f(\mathbf{x})g_n(\mathbf{x}) - \mathbf{T}_0^d(fg_n)(\mathbf{x}) &= \sum_{\substack{|\beta|+|\gamma|\ge d+1 \\ |\beta|,|\gamma|\le d}} \frac{1}{\beta!\gamma!} D^\beta(f)(0) D^\gamma(g_n)(0)\mathbf{x}^{\beta+\gamma} \\ &\quad + |\mathbf{x}|^d\varphi(\mathbf{x})\mathbf{T}_0^d(g_n)(\mathbf{x}) + |\mathbf{x}|^d\psi_n(\mathbf{x})\mathbf{T}_0^d(f)(\mathbf{x}) + |\mathbf{x}|^{2d}\varphi(\mathbf{x})\psi_n(\mathbf{x}). \end{aligned} \tag{4.2}$$

Since $\mathbf{T}_0^d(g_n)(\mathbf{x})$ is uniformly bounded on $K$ and $\psi_n(\mathbf{x}) \le M_1|\mathbf{x}|$ for all $\mathbf{x} \in K$, the right hand side of (4.2) can be written in the form $|\mathbf{x}|^d\eta_n(\mathbf{x})$ where $\{\eta_n(\mathbf{x})\}$ satisfies the condition: For all $\varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$

such that

$$|\eta_n(\mathbf{x})| < \varepsilon_1, \qquad \forall |\mathbf{x}| < \varepsilon_2,\, n \geq 1.$$

This proves the lemma. ∎

A simple argument gives the following result.

LEMMA 4.4. *Let $K$ be a compact set in the plane that does not contain $\mathbf{a}_1, \ldots, \mathbf{a}_k$. Set $q_j(\mathbf{x}) = |\mathbf{x} - \mathbf{a}_j|^2$, $1 \leq j \leq k$. For each $j$, let $\{r_{j,n}\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_{j,n} = 0$. Then*

$$\frac{1}{(q_1 - r_{1,n}^2)\cdots(q_k - r_{k,n}^2)} \to \frac{1}{q_1\cdots q_k} \quad \text{in } C^m(K) \text{ for all } m \geq 0.$$

*Moreover if $P_n, P \in \mathcal{P}_d(\mathbb{R}^2)$ are such that $P_n \to P$ as $n \to \infty$, then*

$$\frac{P_n}{(q_1 - r_{1,n}^2)\cdots(q_k - r_{k,n}^2)} \to \frac{P}{q_1\cdots q_k} \quad \text{in } C^m(K) \text{ for all } m \geq 0.$$

Combining Lemmas 4.3 and 4.4 we obtain the following result.

COROLLARY 4.5. *Under the assumptions of Lemma 4.4 where $K$ is a compact neighborhood of $\mathbf{a}$, if $f \in C^d(K)$, then the sequence*

$$\left\{\frac{f}{(q_1 - r_{1,n}^2)\cdots(q_k - r_{k,n}^2)}\right\}$$

*is d-regular $\mathbf{a}$. Moreover, if $P_n, P \in \mathcal{P}_m(\mathbb{R}^2)$ are such that $P_n$ tends to $P$ as $n \to \infty$, then so does the sequence $\left\{\frac{P_n}{(q_1 - r_{1,n}^2)\cdots(q_k - r_{k,n}^2)}\right\}$.*

*Proof of Theorem 4.1.* We first recall a formula for the Lagrange interpolation polynomial at the Bos configuration. It can be found in [9, proof of Theorem 6.3]. The polynomial $L^n = \mathbf{L}[X^n; f]$ is given by

(4.3)
$$L^n = \sum_{k=1}^{m} L_k^n \text{ with } L_k^n = \prod_{j=1}^{k-1}(q_j - r_{j,n}^2)\mathbb{L}\left[X_k^n; \frac{f - \sum_{j=1}^{k-1} L_j^n}{\prod_{j=1}^{k-1}(q_j - r_{j,n}^2)}\right],\; 1 \leq k \leq m,$$

where $q_k(\mathbf{x}) = |\mathbf{x} - \mathbf{a}_k|^2$, the empty product in the definition of $L_k^n$ is taken as 1 and the empty sum is taken as 0. Note that when $n$ is sufficiently large, the $m$ circles $C_k^n$, $k = 1, \ldots, m$, are pairwise disjoint. Hence $X^n$ is unisolvent for $\mathcal{P}_d(\mathbb{R}^2)$. For the convenience of the reader, we recall the proof of (4.3). On $X_1^n$, we have $L_k^n = 0$ for all $k \geq 2$ since it contains the factor $q_1 - r_{1,n}^2$. Hence $L^n = L_1^n = f$ on $X_1^n$ by (2.15). Similarly, on $X_j^n$, for $j \geq 2$, we have

$L_k^n = 0$ for all $k > j$. Hence, on $X_j^n$, we can write

$$(4.4)\qquad L^n = \sum_{k=1}^{j} L_k^n = L_j^n + \sum_{k=1}^{j-1} L_k^n$$
$$= \prod_{k=1}^{j-1}(q_k - r_{k,n}^2)\left(\frac{f - \sum_{k=1}^{j-1} L_k^n}{\prod_{k=1}^{j-1}(q_k - r_{k,n}^2)}\right) + \sum_{k=1}^{j-1} L_k^n = f.$$

We will prove that $\lim_{n\to\infty} L_k^n = T_k$, where $T_k$ is defined in Theorem 2.2. Indeed, if $k = 1$, then using Corollary 3.6, we have

$$\lim_{n\to\infty} L_1^n = \lim_{n\to\infty} \mathbb{L}[X_1^n; f] = H_{\mathbf{a}_1}^{s_1}(f) = T_1.$$

Assume the assertion holds up to $k < m$; we will prove it for $k + 1$. By Corollary 4.5, the sequence

$$\left\{\frac{f - L_1^n - \cdots - L_{k-1}^n}{\prod_{j=1}^{k-1}(q_j - r_{j,n}^2)}\right\}$$

is $s_{k+1}$-regular at $\mathbf{a}_{k+1}$. Furthermore, when $r$ is small enough, this sequence converges in $C^{s_{k+1}}(\bar{D}(\mathbf{a}_{k+1}, r))$ to

$$\frac{f - T_1 - \cdots - T_{k-1}}{\prod_{j=1}^{k-1} q_k}$$

as $n \to \infty$. Hence Theorem 3.8 implies that $\lim_{n\to\infty} L_{k+1}^n = T_{k+1}$. Consequently,

$$\lim_{n\to\infty} L^n = \sum_{k=1}^{m} T_k = \mathbf{H}[(A, S); f]. \quad \blacksquare$$

*Proof of Theorem 4.2.* The proof is a simple adaptation of that of Theorem 4.1. It is sufficient to show that

$$\lim_{n\to\infty} L_k^n = \widetilde{T}_k, \qquad k = 1, \ldots, m,$$

where $L_k^n$ and $\widetilde{T}_k$ are defined in (4.3) and Theorem 2.6 respectively. We first prove the claim for $k = 1$. For this purpose, we see at once that

$$L_1^n = \mathbb{L}[X_1^n; f] = \sum_{p\in\mathcal{B}} \frac{\mathrm{VDM}(\mathcal{B}[p \leftarrow f]; X_1^n)}{\mathrm{VDM}(\mathcal{B}; X_1^n)} p(\mathbf{x}),$$

where $\mathcal{B}$ is defined in (2.13). Since $\lim_{n\to\infty} X_1^n = B$ and $\mathrm{VDM}(\mathcal{B}; B) \neq 0$, and since $f$ is continuous on $C(\mathbf{a}_1, r_1)$, we have

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{B}[p \leftarrow f]; X_1^n)}{\mathrm{VDM}(\mathcal{B}; X_1^n)} = \frac{\mathrm{VDM}(\mathcal{B}[p \leftarrow f]; B)}{\mathrm{VDM}(\mathcal{B}; B)}, \qquad \forall p \in \mathcal{B}.$$

It follows that

$$\lim_{n\to\infty} L_1^n = \sum_{p\in\mathcal{B}} \frac{\mathrm{VDM}(\mathcal{B}[p \leftarrow f]; B)}{\mathrm{VDM}(\mathcal{B}; B)} p = \mathbb{L}[B; f] = \widetilde{T}_1.$$

A passage to the limit similar to the above implies that $\lim_{n\to\infty} L_k^n = \widetilde{T}_k$ for all $k = 2, \ldots, m$, and the proof is complete. ■

In Theorem 4.1 we consider the case when the Bos configurations lie on $m$ circles whose centers are distinct. Now we turn to the problem of finding the limit of the Lagrange interpolation at Bos configurations when the interpolation points are taken on concentric circles. In this case, it is to be expected that the limit is the Taylor polynomial.

Let $\mathbf{a} \in \mathbb{R}^2$. For $d \geq 2$, we define $m = [d/2] + 1$ and $s_k = d - 2k + 2$ for $k = 1, \ldots, m$. Fix $m$ pairwise distinct radii $r_1, \ldots, r_m$. Let $\{\rho_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \rho_n = 0$. For $1 \leq k \leq m$ and $n \geq 1$, let $X_k^n$ be a set of $2s_k + 1$ points on $C_k^n = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| = r_k\rho_n\}$. Set $X^n = \bigcup_{k=1}^m X_k^n$. Then $X^n$ is a unisolvent set for $\mathcal{P}_d(\mathbb{R}^2)$ (see Bos [4]).

Proposition 4.6. *Suppose $X_k^n$ is $\delta$-separate for every $n, k$. Let $f$ be a function of class $C^{m_d - 1}$ in a neighborhood of $\mathbf{a}$. Then*

$$\lim_{n\to\infty} \mathbf{L}[X^n; f] = \mathbf{T}_{\mathbf{a}}^d(f).$$

*Proof.* Without loss of generality, we assume that $\mathbf{a} = 0$. By [1, Theorem 3.3], it suffices to check that for all monomial functions $f_\alpha(\mathbf{x}) = x^{\alpha_1} y^{\alpha_2}$ with $|\alpha| = \alpha_1 + \alpha_2 = d + 1$, we have

$$\lim_{n\to\infty} \mathbf{L}[X^n; f_\alpha] = 0. \tag{4.5}$$

We follow [4, p. 276] in setting

$$\mathcal{B}_k = \{1, \mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x}), \ldots, \mathrm{Re}(\mathbf{x}^{s_k}), \mathrm{Im}(\mathbf{x}^{s_k})\},$$

and

$$\mathcal{D}_1^n = \mathcal{B}_1, \quad \mathcal{D}_j^n = \Big(\prod_{k=1}^{j-1} (|\mathbf{x}|^2 - (r_k\rho_n)^2)\Big)\mathcal{B}_j, \quad j = 2, \ldots, m.$$

Then $\mathcal{D}^n = \bigcup_{j=1}^m \mathcal{D}_j^n$ is a basis for $\mathcal{P}_d(\mathbb{R}^2)$. From the arguments in [4, pp. 279–280], we have

$$\begin{aligned}
&\mathrm{VDM}(\mathcal{D}^n; X^n)\\
&\quad = \prod_{k=1}^m (r_k\rho_n)^{s_k(s_k+1)} \Big\{\prod_{j=1}^{k-1} ((r_k\rho_n)^2 - (r_j\rho_n)^2)\Big\}^{2s_k+1} \mathrm{VDM}(\mathcal{T}_k; \Theta_k^n)\\
&\quad = M\rho_n^{\sigma_d} \prod_{k=1}^m \mathrm{VDM}(\mathcal{T}_k; \Theta_k^n),
\end{aligned}$$

where $\mathcal{T}_k = \{1, \cos\theta, \sin\theta, \ldots, \cos(s_k\theta), \sin(s_k\theta)\}$ and $\Theta_k^n$ is the set of arguments of $X_k^n$. The constant $M$ depends only on the $r_k$'s, and

$$\sigma_d = \sum_{k=1}^m \big(s_k(s_k + 1) + 2(k - 1)(2s_k + 1)\big).$$

Since $X_k^n$ is $\delta$-separate, the proof of Lemma 3.3 implies that there exists $M_k > 0$ depending only on $s_k$ and $\delta$ such that $|\mathrm{VDM}(\mathcal{T}_k; \Theta_k^n)| \geq M_k$. Hence

$$|\mathrm{VDM}(\mathcal{D}^n; X^n)| \geq M\Big(\prod_{k=1}^{m} M_k\Big)\rho_n^{\sigma_d}. \tag{4.6}$$

On the other hand, for each $p \in \mathcal{D}^n$, we can write

$$p(X^n) = \rho_n^{\deg p} Y_p,$$

and since $f_\alpha$ is a monomial function of degree $d+1$, we have

$$f_\alpha(X^n) = \rho_n^{d+1} Y,$$

where $Y, Y_p \in \mathbb{R}^{d(d+1)/2}$ whose entries are linear combinations of polynomials of the $r_k$'s (with certain coefficients) multiplied by multivariate trigonometric polynomials evaluated at $\bigcup_{k=1}^m \Theta_k^n$. Consequently,

$$\mathrm{VDM}(\mathcal{D}^n[p \leftarrow f_\alpha]; X^n) = \rho_n^{\sigma_d + d + 1 - \deg p} \det(V_p), \tag{4.7}$$

where $V_p$ is the matrix whose column vectors are the $Y_q$'s for all $q \in \mathcal{D}^n$ except for $Y_p$ which is replaced by $Y$. There exists a constant $M' > 0$ (depending only on the $r_k$ and $d$) such that $|\det(V_p)| < M'$ for all $p \in \mathcal{D}^n$. It follows that

$$|\mathrm{VDM}(\mathcal{D}^n[p \leftarrow f_\alpha]; X^n)| \leq M' \rho_n^{\sigma_d + d + 1 - \deg p}. \tag{4.8}$$

Combining (4.6) and (4.8) we obtain

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathcal{D}^n[p \leftarrow f_\alpha]; X^n)}{\mathrm{VDM}(\mathcal{D}^n; X^n)} = 0, \qquad p \in \mathcal{D}^n.$$

The formula of $\mathbf{L}[X^n; f_\alpha]$ in (1.1) implies (4.5). ∎

Example 4.7. In Proposition 4.6, every point of $X^n$ tends to $\mathbf{a}$ with the same speed as $n \to \infty$. This example shows that when the speeds are different, the Lagrange projector may not converge. Let $r_1, r_2$ be different positive numbers. Take $\mathbf{a}_1 = (r_1, 0)$, $\mathbf{a}_2 = (-r_1, 0)$ on $C(0, r_1)$ and $\mathbf{a}_3 = (0, r_2)$ on $C(0, r_2)$. Then $X = \{\mathbf{a}_i : i = 1, 2, 3\}$ is unisolvent for $\mathcal{P}_1(\mathbb{R}^2)$. Let $f(\mathbf{x}) = x^2$. It is easy to check that

$$\mathbf{L}[X; f](\mathbf{x}) = (r_1)^2 - \frac{(r_1)^2}{r_2} y, \qquad \mathbf{x} = (x, y).$$

If we take $r_1 = \delta$, $r_2 = \delta^3$, then $\mathbf{L}[X; f](\mathbf{x}) = \delta^2 - y/\delta$. Thus $\mathbf{L}[X; f]$ does not converge as $\delta \to 0$ although $X$ tends to 0.

Remark 4.8. The results in this paper are very two-dimensional being based on complex derivatives. One may ask whether analogous results hold for a higher-dimensional space using Clifford algebras, especially the quaternions.

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