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Normality criteria for families of zero-free meromorphic functions

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Abstract. Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, let n, k and m be positive integers with $n \geq m+1$, and let $a \neq 0$ and b be finite complex numbers. If for each $f \in \mathcal{F}$, $f^m + a(f^{(k)})^n - b$ has at most nk zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D. The examples show that the result is sharp.

1. Introduction. Let D be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of functions meromorphic in D. Then $\mathcal{F}$ is said to be *normal* in D, in the sense of Montel, if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D to a meromorphic function or the constant ∞ (see [4, 9, 12]).

In 1959, Hayman [3] proved that if f is a transcendental meromorphic function in $\mathbb{C}$, then $f' + af^n$ assumes every finite value infinitely often for a positive integer $n \geq 5$ and a nonzero finite complex number a. Mues [7] showed that this is false for $n = 3, 4$ by some counter-examples. Corresponding to the above result, Ye [13] for $n \geq 3$ and Fang and Zalcman [2] for $n \geq 2$ studied a similar problem where $f' + af^n$ is replaced by $f + a(f')^n$. Moreover, Fang and Zalcman [2] gave a related normal family analogue. Later on, Xu, Wu and Liao [10] considered the case of higher derivatives and proposed a conjecture. Recently, Li [5] studied this conjecture and proved the following result.

Theorem A. *Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, let $n \geq 2$ and k be positive integers, and let $a \neq 0$ and b be finite complex numbers. If for each $f \in \mathcal{F}$, $f + a(f^{(k)})^n - b$ has at most nk zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.*

In this paper, we generalize Theorem A by replacing $f + a(f^{(k)})^n - b$ by $f^m + a(f^{(k)})^n - b$ and prove the following result.

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Theorem 1. *Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, let n, k and m be positive integers with $n \geq m+1$, and let $a \neq 0$ and b be finite complex numbers. If for each $f \in \mathcal{F}$, $f^m + a(f^{(k)})^n - b$ has at most nk zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.*

Example 1. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_j : j = 1, 2, \ldots\}$, where $f_j(z) = e^{jz}$, and let n, k and m be positive integers. Then, for each $f_j \in \mathcal{F}$, we have $f_j \neq 0$ and $f_j^m + (f_j^{(k)})^m = e^{mjz}(1 + j^{mk}) \neq 0$ in D. But $\mathcal{F}$ fails to be normal in D. This shows that the condition $n \geq m+1$ in Theorem 1 is necessary.

Example 2. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_j : j = 1, 2, \ldots\}$, where $f_j(z) = jz^k$, and let n, k and m be positive integers with $n \geq m+1$. Then, for each $f_j \in \mathcal{F}$, we have $f_j^m + (f_j^{(k)})^n = j^m(z^{mk} + j^{n-m}k!^n) \neq 0$ in D. But $\mathcal{F}$ fails to be normal in D. This shows that the condition of zero-freeness in Theorem 1 cannot been removed.

Example 3. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_j : j = 1, 2, \ldots\}$, where $f_j(z) = 1/(jz)$, and let n, k and m be positive integers with $n \geq m+1$. Then, for each $f_j \in \mathcal{F}$,

$$f_j^m + (f_j^{(k)})^n = \frac{j^{-m}z^{n(k+1)-m} + j^{-n}(-1)^{nk}k!^n}{z^{n(k+1)}}$$

has at least $nk+1$ zeros in D, ignoring multiplicities. But $\mathcal{F}$ fails to be normal in D. This shows that the condition in Theorem 1 that $f^m + a(f^{(k)})^n - b$ has at most nk zeros in D, ignoring multiplicities, is the best possible.

2. Some lemmas. Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. We shall use standard notation of Nevanlinna theory (see e.g. [4, 12]), and denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure.

Lemma 1 (see [8, 14]). *Let $\alpha \in \mathbb{R}$ satisfy $-1 < \alpha < \infty$, and let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D. Then, if $\mathcal{F}$ is not normal at some point $z_0 \in D$, there exist*

(i) *points $z_j \in D$, $z_j \to z_0$,*
(ii) *functions $f_j \in \mathcal{F}$, and*
(iii) *positive numbers $\rho_j \to 0$*

such that

$$\frac{f_j(z_j + \rho_j\zeta)}{\rho_j^\alpha} = g_j(\zeta) \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant zero-free meromorphic function on $\mathbb{C}$ of order at most 2. In particular, if g is an entire function, then g is of order at most 1.

LEMMA 2 (see [12]). *Let f be a transcendental meromorphic function in $\mathbb{C}$. Then*

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=\infty.$$

LEMMA 3 (see [11]). *Let k be a positive integer, and let f be a transcendental meromorphic function in $\mathbb{C}$. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f^{(k)}).$$

LEMMA 4. *Let f be a zero-free transcendental meromorphic function in $\mathbb{C}$, let $n\geq 2$, k and m be positive integers, and let $a\neq 0$ be a finite complex number. Then $f^m+a(f^{(k)})^n$ has infinitely many zeros.*

Proof. Suppose that $f^m+a(f^{(k)})^n$ has finitely many zeros. Then by Lemma 2,

$$N\left(r,\frac{1}{f^m+a(f^{(k)})^n}\right)=O(\log r)=S(r,f^{(k)}). \tag{2.1}$$

On the other hand, it follows from the first and second fundamental theorem and Lemma 3 that

$$\begin{aligned}
m\left(r,\frac{1}{a(f^{(k)})^n}\right)&\leq m\left(r,\frac{f^m}{a(f^{(k)})^n}\right)+m\left(r,\frac{1}{f^m}\right)\\
&\leq m\left(r,\frac{f^m}{a(f^{(k)})^n}+1\right)+m\left(r,\frac{1}{(f^{(k)})^m}\right)+m\left(r,\frac{(f^{(k)})^m}{f^m}\right)+O(1)\\
&\leq m\left(r,\frac{f^m+a(f^{(k)})^n}{a(f^{(k)})^n}\right)+m\left(r,\frac{1}{(f^{(k)})^m}\right)+S(r,f^{(k)})\\
&\leq T\left(r,\frac{a(f^{(k)})^n}{f^m+a(f^{(k)})^n}\right)-N\left(r,\frac{f^m+a(f^{(k)})^n}{a(f^{(k)})^n}\right)\\
&\quad+m\left(r,\frac{1}{(f^{(k)})^m}\right)+S(r,f^{(k)})\\
&\leq \overline{N}\left(r,\frac{a(f^{(k)})^n}{f^m+a(f^{(k)})^n}\right)+\overline{N}\left(r,\frac{f^m+a(f^{(k)})^n}{a(f^{(k)})^n}\right)\\
&\quad+\overline{N}\left(r,\frac{1}{a(f^{(k)})^n/(f^m+a(f^{(k)})^n)-1}\right)+m\left(r,\frac{1}{(f^{(k)})^m}\right)\\
&\quad-N\left(r,\frac{f^m+a(f^{(k)})^n}{a(f^{(k)})^n}\right)+S(r,f^{(k)}).
\end{aligned} \tag{2.2}$$

A simple calculation shows that

$$\overline{N}\left(r,\frac{a(f^{(k)})^n}{f^m+a(f^{(k)})^n}\right)\leq\overline{N}\left(r,\frac{1}{f^m+a(f^{(k)})^n}\right), \tag{2.3}$$

$$(2.4)\quad \overline{N}\left(r, \frac{f^m + a(f^{(k)})^n}{a(f^{(k)})^n}\right) \le \overline{N}\left(r, \frac{1}{f^{(k)}}\right),$$

$$(2.5)\quad \overline{N}\left(r, \frac{1}{a(f^{(k)})^n/(f^m + a(f^{(k)})^n) - 1}\right) \le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f),$$

$$(2.6)\quad N\left(r, \frac{f^m + a(f^{(k)})^n}{a(f^{(k)})^n}\right) \ge N\left(r, \frac{1}{(f^{(k)})^n}\right) - N\left(r, \frac{1}{f^m + a(f^{(k)})^n}\right).$$

Now by (2.1)–(2.6), Lemma 2 and the first fundamental theorem we obtain

$$\begin{aligned} nT(r, f^{(k)}) &\le \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}(r, f) + m\left(r, \frac{1}{(f^{(k)})^m}\right) + S(r, f^{(k)}) \\ &\le \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}(r, f^{(k)}) + mm\left(r, \frac{1}{f^{(k)}}\right) + S(r, f^{(k)}) \\ &\le mN\left(r, \frac{1}{f^{(k)}}\right) + \frac{1}{k+1}N(r, f^{(k)}) + mm\left(r, \frac{1}{f^{(k)}}\right) + S(r, f^{(k)}) \\ &\le mT\left(r, \frac{1}{f^{(k)}}\right) + \frac{1}{k+1}T(r, f^{(k)}) + S(r, f^{(k)}) \\ &\le mT(r, f^{(k)}) + \frac{1}{k+1}T(r, f^{(k)}) + S(r, f^{(k)}). \end{aligned}$$

Noting $n \ge m + 1$, from this we get $T(r, f^{(k)}) = S(r, f^{(k)})$, a contradiction. Therefore $f^m + a(f^{(k)})^n$ has infinitely many zeros.

This completes the proof of Lemma 4.

Using the idea of [1], we obtain the following important lemma.

Lemma 5. *Let n, k and m be positive integers with $n \ge m+1$, let $a \ne 0$ and b be finite complex numbers, and let f be a nonconstant zero-free rational function. Then $f^m + a(f^{(k)})^n - b$ has at least $nk + 1$ distinct zeros in $\mathbb{C}$.*

Proof. Since f is a nonconstant zero-free rational function, f is not a polynomial, i.e., f has at least one finite pole. Thus we can write

$$(2.7)\qquad f(z) = \frac{C_1}{\prod_{i=1}^{q}(z + z_i)^{p_i}},$$

$$(2.8)\qquad f^m(z) = \frac{C_2}{\prod_{i=1}^{q}(z + z_i)^{mp_i}},$$

where C_1 and C_2 $(= C_1^m)$ are nonzero constants, q and p_i are positive integers, the z_i (when $1 \le i \le q$) are distinct complex numbers, $p = \sum_{i=1}^{q} p_i$. By induction, we deduce from (2.7) that

$$(2.9)\qquad f^{(k)}(z) = \frac{P(z)}{\prod_{i=1}^{q}(z + z_i)^{p_i+k}},$$

where $P(z)$ is a polynomial of degree $(q-1)k$. Further, by (2.7)–(2.9), we get

$$f^m(z)+a(f^{(k)}(z))^n=\frac{C_2\prod_{i=1}^{q}(z+z_i)^{(n-m)p_i+nk}+aP^n(z)}{\prod_{i=1}^{q}(z+z_i)^{n(p_i+k)}},$$

and so, by simple calculation, $f^m(z)+a(f^{(k)}(z))^n-b$ has at least one zero in $\mathbb{C}$. Thus we can write

$$f^m(z)+a(f^{(k)}(z))^n-b=\frac{C_3\prod_{i=1}^{s}(z+\omega_i)^{l_i}}{\prod_{i=1}^{q}(z+z_i)^{n(p_i+k)}}, \tag{2.10}$$

where C_3 is a nonzero constant, s and l_i are positive integers, the z_i (when $1\le i\le q$) and ω_i (when $1\le i\le s$) are distinct complex numbers. From (2.8)–(2.10), we have

$$C_2\prod_{i=1}^{q}(z+z_i)^{(n-m)p_i+nk}+aP^n(z)=b\prod_{i=1}^{q}(z+z_i)^{n(p_i+k)}+C_3\prod_{i=1}^{s}(z+\omega_i)^{l_i}. \tag{2.11}$$

We now consider two cases.

Case 1: $b=0$. Then by (2.11) it follows that $\sum_{i=1}^{q}[(n-m)p_i+nk]=\sum_{i=1}^{s}l_i$, $C_2=C_3$,

$$\prod_{i=1}^{q}(1+z_it)^{(n-m)p_i+nk}-\prod_{i=1}^{s}(1+\omega_it)^{l_i}=t^{(n-m)p+nk}Q(t), \tag{2.12}$$

where $Q(t)=-(a/C_2)t^{n(q-1)k}P^n(1/t)$ is a polynomial of degree less than $n(q-1)k$. From (2.12), we get

$$\frac{\prod_{i=1}^{q}(1+z_it)^{(n-m)p_i+nk}}{\prod_{i=1}^{s}(1+\omega_it)^{l_i}}=1+\frac{t^{(n-m)p+nk}Q(t)}{\prod_{i=1}^{s}(1+\omega_it)^{l_i}}=1+O(t^{(n-m)p+nk}) \tag{2.13}$$

as $t\to 0$. Thus by taking logarithmic derivatives of both sides of (2.13), it follows that

$$\sum_{i=1}^{q}\frac{[(n-m)p_i+nk]z_i}{1+z_it}-\sum_{i=1}^{s}\frac{l_i\omega_i}{1+\omega_it}=O(t^{(n-m)p+nk-1}) \tag{2.14}$$

as $t\to 0$. If we compare the coefficients of t^j, $j=0,1,\dots,(n-m)p+nk-2$, in (2.14), we obtain

$$\sum_{i=1}^{q}[(n-m)p_i+nk]z_i^j-\sum_{i=1}^{s}l_i\omega_i^j=0, \tag{2.15}$$

$j=1,\dots,(n-m)p+nk-1$. Let $z_{q+i}=\omega_i$ when $1\le i\le s$. Noting that $\sum_{i=1}^{q}[(n-m)p_i+nk]=\sum_{i=1}^{s}l_i$ and using (2.15), we deduce that the system of linear equations

$$\sum_{i=1}^{q+s} z_i^j x_i = 0, \tag{2.16}$$

where $0 \le j \le (n-m)p + nk - 1$, has a nonzero solution

$$(x_1, \dots, x_{q+1}, \dots, x_{q+s}) = ((n-m)p_1 + nk, \dots, -l_1, \dots, -l_s).$$

If $(n-m)p + nk \ge q + s$, then the determinant $\det(z_i^j)_{(q+s)\times(q+s)}$ of the coefficients of the system of equations (2.15), where $0 \le j \le q + s - 1$, is equal to zero, by Cramer's rule (see e.g. [6]). However, the z_i are distinct complex numbers when $1 \le i \le q+s$, and the determinant is a Vandermonde determinant, so it cannot be zero (again see [6]), which is a contradiction.

Hence we conclude that $(n-m)p + nk < q + s$. It follows from this and the two inequalities $n \ge m + 1$ and $p = \sum_{i=1}^q p_i \ge q$ that $s \ge nk + 1$.

Case 2: $b \ne 0$. Let

$$b\prod_{i=1}^q (z+z_i)^{n(p_i+k)} - C_2 \prod_{i=1}^q (z+z_i)^{(n-m)p_i+nk} = b\prod_{i=1}^q (z+z_i)^{(n-m)p_i+nk} \prod_{i=1}^l (z+\alpha_i)^{m_i}, \tag{2.17}$$

where the z_i (when $1 \le i \le q$) and α_i (when $1 \le i \le l$) are distinct complex numbers, and $\sum_{i=1}^l m_i = mp$. Then from (2.11) and (2.17) we get

$$b\prod_{i=1}^q (z+z_i)^{(n-m)p_i+nk} \prod_{i=1}^l (z+\alpha_i)^{m_i} + C_3 \prod_{i=1}^s (z+\omega_i)^{l_i} = aP^n(z). \tag{2.18}$$

We see by (2.17)–(2.18) that $\sum_{i=1}^q [(n-m)p_i + nk] + \sum_{i=1}^l m_i = np + nkq = \sum_{i=1}^s l_i$ and $b = -C_3$. Thus by (2.18), we get

$$\prod_{i=1}^q (1+z_i t)^{(n-m)p_i+nk} \prod_{i=1}^l (1+\alpha_i t)^{m_i} - \prod_{i=1}^s (1+\omega_i t)^{l_i} = t^{np+nk} Q_1(t), \tag{2.19}$$

where $Q_1(t) = \frac{a}{b} t^{n(q-1)k} P^n(1/t)$ is a polynomial of degree less than $n(q-1)k$. From (2.19), we get

$$\frac{\prod_{i=1}^q (1+z_i t)^{(n-m)p_i+nk} \prod_{i=1}^l (1+\alpha_i t)^{m_i}}{\prod_{i=1}^s (1+\omega_i t)^{l_i}} = 1 + O(t^{n(p+k)}) \tag{2.20}$$

as $t \to 0$. Thus taking logarithmic derivatives of both sides of (2.20) shows that

$$\sum_{i=1}^q \frac{[(n-m)p_i + nk] z_i}{1+z_i t} + \sum_{i=1}^l \frac{m_i \alpha_i}{1+\alpha_i t} - \sum_{i=1}^s \frac{l_i \omega_i}{1+\omega_i t} = O(t^{n(p+k)-1}) \tag{2.21}$$

as $t \to 0$.

Set $S = \{\alpha_1, \ldots, \alpha_l\} \cap \{\omega_1, \ldots, \omega_s\}$. We consider two subcases.

CASE 2.1: $S = \emptyset$. Let $z_{q+i} = \alpha_i$ when $1 \le i \le l$, and

$$N_i = \begin{cases} (n-m)p_i + nk, & 1 \le i \le q, \\ m_{i-q}, & q+1 \le i \le q+l. \end{cases}$$

Then (2.21) can be rewritten as

$$\sum_{i=1}^{q+l} \frac{N_i z_i}{1 + z_i t} - \sum_{i=1}^{s} \frac{l_i \omega_i}{1 + \omega_i t} = O(t^{n(p+k)-1})$$

as $t \to 0$. Using the same argument as in Case 1, we get $s \ge nk + 1$.

CASE 2.2: $S \ne \emptyset$. Without loss of generality, we can assume that $S = \{\alpha_1, \ldots, \alpha_M\}$. Then $\alpha_i = \omega_i$ when $1 \le i \le M$. Let $M_1 = l - M$. Again we discuss two subcases.

CASE 2.2.1: $M_1 \ge 1$. Let $\omega_{s+i} = \alpha_{M+i}$ when $1 \le i \le M_1$. If $M < s$, then we set

$$L_i = \begin{cases} l_i - m_i, & 1 \le i \le M, \\ l_i, & M+1 \le i \le s, \\ -m_{M-s+i}, & s+1 \le i \le s+M_1. \end{cases}$$

If $M = s$, then we set

$$L_i = \begin{cases} l_i - m_i, & 1 \le i \le M = s, \\ -m_{M-s+i}, & s+1 \le i \le s+M_1. \end{cases}$$

CASE 2.2.2: $M_1 = 0$. If $M < s$, then we set

$$L_i = \begin{cases} l_i - m_i, & 1 \le i \le M, \\ l_i, & M+1 \le i \le s. \end{cases}$$

If $M = s$, then we set $L_i = l_i - m_i$ when $1 \le i \le M = s = l$.

In both Case 2.2.1 and Case 2.2.2, (2.21) can be rewritten as

$$\sum_{i=1}^{q} \frac{[(n-m)p_i + nk] z_i}{1 + z_i t} - \sum_{i=1}^{s+M_1} \frac{L_i \omega_i}{1 + \omega_i t} = O(t^{n(p+k)-1})$$

as $t \to 0$, where $0 \le M_1 \le l - 1$. Using the same argument as in Case 1, we get $s \ge nk + 1$.

This completes the proof of Lemma 5.

LEMMA 6 (see [1]). *Let k be a positive integer, let $b \ne 0$ be a finite complex number, and let f be a nonconstant zero-free rational function. Then $f^{(k)} - b$ has at least $k+1$ distinct zeros in $\mathbb{C}$.*

3. Proof of Theorem 1. Suppose that $\mathcal{F}$ is not normal in D. Then there exists at least one $z_0 \in D$ such that $\mathcal{F}$ is not normal at the point z_0. We consider two cases.

Case 1: $b=0$. Then from Lemma 1 we can find

(i) points $z_j \in D$, $z_j \to z_0$,
(ii) functions $f_j \in \mathcal{F}$, and
(iii) positive numbers $\rho_j \to 0$,

such that

$$\frac{f_j(z_j+\rho_j\zeta)}{\rho_j^{nk/(n-m)}} = g_j(\zeta) \to g(\zeta) \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant zero-free meromorphic function on $\mathbb{C}$ of order at most 2. In particular, if g is an entire function, then g is of order at most 1. From (3.1), we deduce that

$$\begin{aligned} &g_j^m(\zeta)+a(g_j^{(k)}(\zeta))^n \\ &= \rho_j^{-mnk/(n-m)}[f_j^m(z_j+\rho_j\zeta)+a(f_j^{(k)}(z_j+\rho_j\zeta))^n] \to g^m(\zeta)+a(g^{(k)}(\zeta))^n \end{aligned} \tag{3.2}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g.

We claim that $g^m(\zeta)+a(g^{(k)}(\zeta))^n$ has at most nk distinct zeros.

Suppose that $g^m(\zeta)+a(g^{(k)}(\zeta))^n$ has at least $nk+1$ distinct zeros ζ_i, $1 \le i \le nk+1$. First we show $g^m(\zeta)+a(g^{(k)}(\zeta))^n \not\equiv 0$. If $g^m(\zeta)+a(g^{(k)}(\zeta))^n \equiv 0$, then by $n \ge m+1$ we know that g is an entire function. Since g is nonconstant zero-free and of order at most 1, it follows that $g(\zeta)=e^{c_1\zeta+c_2}$, where $c_1 \ne 0$ and c_2 are constants. Thus

$$g^m(\zeta)+a(g^{(k)}(\zeta))^n = e^{m(c_1\zeta+c_2)} + ac_1^{kn}e^{n(c_1\zeta+c_2)} \equiv 0,$$

which is impossible because $n \ge m+1$. Therefore, $g^m(\zeta)+a(g^{(k)}(\zeta))^n \not\equiv 0$. Now by (3.2) and the Hurwitz theorem, there exist $\zeta_{j,i}$, $i=1,\dots,nk+1$, $\zeta_{j,i} \to \zeta_i$, such that, for j sufficiently large,

$$f_j^m(z_j+\rho_j\zeta_{j,i})+a(f_j^{(k)}(z_j+\rho_j\zeta_{j,i}))^n = 0.$$

But $f_j^m(z)+a(f_j^{(k)}(z))^n$ has at most nk distinct zeros in D, and $z_j+\rho_j\zeta_{j,i} \to z_0$, which is a contradiction. Hence $g^m(\zeta)+a(g^{(k)}(\zeta))^n$ has at most nk distinct zeros.

However, from Lemmas 4 and 5, we see that there do not exist nonconstant meromorphic functions that have the above properties. This contradiction shows that $\mathcal{F}$ is normal in D.

Case 2: $b \ne 0$. Then from Lemma 1 we can once more find

(i) points $z_j \in D$, $z_j \to z_0$,
(ii) functions $f_j \in \mathcal{F}$, and
(iii) positive numbers $\rho_j \to 0$,

such that

$$\frac{f_j(z_j+\rho_j\zeta)}{\rho_j^k} = g_j(\zeta) \to g(\zeta) \tag{3.3}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant zero-free meromorphic function on $\mathbb{C}$ of order at most 2. From (3.3), we deduce that

$$\begin{aligned}(3.4)\quad & \rho_j^{mk} g_j^m(\zeta) + a(g_j^{(k)}(\zeta))^n - b \\ &= f_j^m(z_j+\rho_j\zeta) + a(f_j^{(k)}(z_j+\rho_j\zeta))^n - b \to a(g^{(k)}(\zeta))^n - b\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g.

We claim that $a(g^{(k)}(\zeta))^n - b$ has at most nk distinct zeros.

Suppose that $a(g^{(k)}(\zeta))^n - b$ has at least $nk+1$ distinct zeros ζ_i, $1 \le i \le nk+1$. Clearly, $a(g^{(k)}(\zeta))^n \not\equiv b$, for otherwise g would be a nonconstant polynomial of degree k, which contradicts the fact that $g \neq 0$. Then by (3.4) and Hurwitz's theorem, there exist $\zeta_{j,i}$, $i = 1, \dots, nk+1$, $\zeta_{j,i} \to \zeta_i$, such that, for j sufficiently large, $f_j^m(z_j+\rho_j\zeta_{j,i}) + a(f_j^{(k)}(z_j+\rho_j\zeta_{j,i}))^n = b$. However $f_j^m(z) + a(f_j^{(k)}(z))^n - b$ has at most nk distinct zeros in D, and $z_j+\rho_j\zeta_{j,i} \to z_0$, which is a contradiction. Hence $a(g^{(k)}(\zeta))^n - b$ has at most nk distinct zeros.

Let $c_1, \dots, c_n$ be distinct roots of $\omega^n - b/a = 0$. Then

$$a(g^{(k)}(\zeta))^n - b = a\prod_{i=1}^{n}[g^{(k)}(\zeta) - c_i]. \tag{3.5}$$

Now if g is a rational function, then it follows by (3.5) and Lemma 6 that $a(g^{(k)}(\zeta))^n - b$ has at least $n(k+1)$ distinct zeros, which contradicts that $a(g^{(k)}(\zeta))^n - b$ has at most nk distinct zeros. And if g is a transcendental meromorphic function, then noting that $n \ge m+1$, from Nevanlinna's second fundamental theorem we deduce

$$\begin{aligned}T(r,g^{(k)}) &\le \overline{N}(r,g^{(k)}) + \sum_{i=1}^{n}\overline{N}\left(r,\frac{1}{g^{(k)}-c_i}\right) + S(r,g^{(k)}) \\ &= \overline{N}(r,g^{(k)}) + \overline{N}\left(r,\frac{1}{a(g^{(k)}(\zeta))^n - b}\right) + S(r,g^{(k)}) \\ &\le \frac{1}{k+1}N(r,g^{(k)}) + S(r,g^{(k)}) \\ &\le \frac{1}{k+1}T(r,g^{(k)}) + S(r,g^{(k)}),\end{aligned}$$

which implies that $T(r,g^{(k)}) = S(r,g^{(k)})$, a contradiction. Hence $\mathcal{F}$ is normal at z_0.

The proof of Theorem 1 is complete.

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