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Some characterizations of the class $\mathcal{E}_m(\Omega)$ and applications

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Abstract. We give some characterizations of the class $\mathcal{E}_m(\Omega)$ and use them to establish a lower estimate for the log canonical threshold of plurisubharmonic functions in this class.

1. Introduction. The complex Monge–Ampère operator has a central role in pluripotential theory and has been extensively studied for many years. This operator was used to obtain many important results of pluripotential theory in $\mathbb{C}^n$, $n > 1$. An example of such application is the proof of quasi-continuity of plurisubharmonic functions, yielding the pluripolarity of negligible sets. In [BT1] Bedford and Taylor have shown that this operator is well defined on the class of locally bounded plurisubharmonic functions with range in the class of nonnegative measures. Recently, to extend the domain of definition of this operator to plurisubharmonic functions which may or not be locally bounded, Cegrell [C1, C2] has introduced and investigated the classes $\mathcal{E}_0(\Omega)$, $\mathcal{F}(\Omega)$ and $\mathcal{E}(\Omega)$ on which the complex Monge–Ampère operator is well defined. He has developed pluripotential theory on these classes. To extend the class of plurisubharmonic functions and to study a class of complex differential operators more general than the Monge–Ampère operator, in [B1] and [DK2], the authors introduced m-subharmonic functions and studied the complex Hessian operator. They were also interested in the complex Hessian equations in $\mathbb{C}^n$ and on compact Kähler manifolds. In order to continue the study of the complex Hessian operator for m-subharmonic functions which are not locally bounded, in a recent preprint [Lu], Chinh Hoang Lu introduced the Cegrell classes $\mathcal{E}^0_m(\Omega)$, $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ associated to m-subharmonic functions, and proved that the complex Hessian operator is well defined on these classes. Thus it is of interest to obtain a characterization of these classes analytically and geometrically.

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In Section 3 we show the local property and give an analytic characterization of the class $\mathcal{E}_m(\Omega)$. At the beginning of Section 4, we prove the following:

THEOREM 4.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{E}_m(\Omega) \cap \mathrm{PSH}^-(\Omega)$. Then for all $c > 0$, the upper level sets*

$$E(u,c) = \{z \in \Omega : \nu(u,z) \geq c\}$$

are analytic subsets in Ω of dimension $\leq n - m$.

Next, by relying on a recent result of Demailly and Pham [DP] we give a lower bound for the log canonical threshold of plurisubharmonic functions in the class $\mathcal{E}_m(\Omega)$. This is the following theorem.

THEOREM 4.5. *Let $u \in \mathrm{PSH}(\Omega) \cap \mathcal{E}_m(\Omega)$, $1 \leq m \leq n-1$ and $0 \in \Omega$. Then*

$$c_u(0) \geq \sum_{j=1}^{m} \frac{e_{j-1}(u)}{e_j(u)},$$

where $e_0(u) = 1$.

Note that in the above theorem $c_u(0)$ and $e_j(u)$ denote, respectively, the log canonical threshold and the intersection number of the plurisubharmonic function u, whose definitions are given in Section 4. In the case $m = n$, from the above theorem we get the result of Demailly and Pham.

Finally, using a result in [FS] we will prove the same lower bound for the log canonical threshold of plurisubharmonic functions which are bounded outside a closed subset of small Hausdorff measure:

THEOREM 4.6. *Let Ω be an open subset in $\mathbb{C}^n$, $0 \in \Omega$ and $E \subset \Omega$ be a closed subset in Ω with $\mathcal{H}_{2(n-m)+2}(E) = 0$, where $1 \leq m \leq n-1$. Assume that $u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega \setminus E)$. Then*

$$c_u(0) \geq \sum_{j=1}^{m} \frac{e_{j-1}(u)}{e_j(u)},$$

where $e_0(u) = 1$.

The paper is organized as follows. In Section 2 we recall the definitions and results concerning m-subharmonic functions, which were introduced and investigated intensively in recent years by many authors (see [B1], [DK2]). We also recall the Cegrell classes of m-subharmonic functions: $\mathcal{E}^0_m(\Omega)$, $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ introduced and studied in [Lu]. At the same time, we deal with the Lelong numbers associated to a closed positive current T and to a plurisubharmonic function φ on an open set $\Omega \subset \mathbb{C}^n$. Results of Siu and Demailly on the analyticity of upper level sets of the Lelong numbers associated to a closed positive current T are recalled in that section. Section 3 is devoted to the proof of the local property and of an analytic characterization of $\mathcal{E}_m(\Omega)$. Next, in Section 4 we give a geometrical characterization of $\mathcal{E}_m(\Omega)$, and by relying on the result of Demailly and Pham [DP], we give a lower estimate for the log canonical threshold of plurisubharmonic functions in the class $\mathcal{E}_m(\Omega)$ and of plurisubharmonic functions which are bounded outside a subset of small Hausdorff measure.

2. Preliminaries. Some elements of pluripotential theory that will be used throughout the paper can be found in [BT1], [K1], [K2], [K3], while elements of the theory of m-subharmonic functions and the complex Hessian operator can be found in [B1], [DK2], [SA]. Now we recall the definition of the class of m-subharmonic functions introduced by Błocki [B1] and the classes $\mathcal{E}_m^0(\Omega)$ and $\mathcal{F}_m(\Omega)$ introduced and investigated by Chinh Hoang Lu [Lu]. Let Ω be an open subset in $\mathbb{C}^n$. We denote by $\mathrm{PSH}(\Omega)$ the set of plurisubharmonic functions on Ω, while $\mathrm{PSH}^-(\Omega)$ denotes the set of negative plurisubharmonic functions on Ω. By $\beta = dd^c|z|^2$ we denote the canonical Kähler form on $\mathbb{C}^n$ with the volume form $dV_n = \frac{1}{n!}\beta^n$ where $d = \partial + \overline{\partial}$ and $d^c = \frac{\partial - \overline{\partial}}{4i}$, hence $dd^c = \frac{i}{2}\partial\overline{\partial}$.

2.1. First, we recall the class of m-subharmonic functions introduced and investigated in [B1]. For $1 \le m \le n$, we define

$$\widehat{\Gamma}_m = \{\eta \in \mathbb{C}_{(1,1)} : \eta \wedge \beta^{n-1} \ge 0, \dots, \eta^m \wedge \beta^{n-m} \ge 0\},$$

where $\mathbb{C}_{(1,1)}$ denotes the space of $(1,1)$-forms with constant coefficients.

Definition 2.1. Let u be a subharmonic function on an open subset $\Omega \subset \mathbb{C}^n$. Then u is said to be an *m-subharmonic* function on Ω if for every $\eta_1, \dots, \eta_{m-1}$ in $\widehat{\Gamma}_m$ the inequality

$$dd^c u \wedge \eta_1 \wedge \cdots \wedge \eta_{m-1} \wedge \beta^{n-m} \ge 0$$

holds in the sense of currents.

By $\mathrm{SH}_m(\Omega)$ we denote the set of all m-subharmonic functions on Ω, while $\mathrm{SH}_m^-(\Omega)$ denotes the set of negative m-subharmonic functions on Ω. Now assume that Ω is an open set in $\mathbb{C}^n$ and $u \in C^2(\Omega)$. Then from [B1, Proposition 3.1] (see also [SA, Definition 1.2]) we note that u is m-subharmonic on Ω if and only if $(dd^c u)^k \wedge \beta^{n-k} \ge 0$ for $k = 1, \dots, m$.

Now as in [B1] and [DK2], we define the complex Hessian operator of locally bounded m-subharmonic functions.

Definition 2.2. Assume that $u_1, \dots, u_p \in \mathrm{SH}_m(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Then the *complex Hessian operator $H_m(u_1, \dots, u_p)$* is defined inductively by

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m} = dd^c(u_p dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m}).$$

In [B1] and [DK2] it is proved that $H_m(u_1, \ldots, u_p)$ is a closed positive current of bidegree $(n-m+p, n-m+p)$ and this operator is continuous under decreasing sequences of locally bounded m-subharmonic functions. In particular, when $u = u_1 = \cdots = u_m \in \mathrm{SH}_m(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, the Borel measure $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ is well defined and is called the *complex Hessian* of u.

EXAMPLE 2.3. By using an example due to Sadullaev and Abullaev [SA] we show that there exists a function which is m-subharmonic but not $(m+1)$-subharmonic. Let $\Omega \subset \mathbb{C}^n$ be a domain and $0 \notin \Omega$. Consider the Riesz kernel

$$K_m(z) = -\frac{1}{|z|^{2(n/m-1)}}, \qquad 1 \le m < n.$$

We note that $K_m \in C^2(\Omega)$. As in [SA] we have

$$(dd^c K_m)^k \wedge \beta^{n-k} = n(n/m-1)^k(1-k/m)|z|^{-2kn/m}\beta^n.$$

Thus $(dd^c K_m)^k \wedge \beta^{n-k} \ge 0$ for all $k = 1, \ldots, m$, and hence $K_m \in \mathrm{SH}_m(\Omega)$. However, $(dd^c K_m)^{m+1} \wedge \beta^{n-m-1} < 0$ and so $K_m \notin \mathrm{SH}_{m+1}(\Omega)$.

2.2. Now we recall the definition of m-maximal subharmonic functions introduced and investigated in [B1].

DEFINITION 2.4. An m-subharmonic function $u \in \mathrm{SH}_m(\Omega)$ is called *m-maximal* if for every $K \Subset \Omega$ and every $v \in \mathrm{SH}_m(\Omega)$, if $v \le u$ on $\Omega \setminus K$ then $v \le u$ on Ω.

We denote by $\mathrm{MSH}_m(\Omega)$ the set of m-maximal functions on Ω. Theorem 3.6 in [B1] states that a locally bounded m-subharmonic function u on a bounded domain $\Omega \subset \mathbb{C}^n$ belongs to $\mathrm{MSH}_m(\Omega)$ if and only if it solves the homogeneous Hessian equation $H_m(u) = (dd^c u)^m \wedge \beta^{n-m} = 0$.

2.3. Next, we recall the classes $\mathcal{E}^0_m(\Omega)$, $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ introduced and investigated in [Lu]. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Set

$$\mathcal{E}^0_m = \mathcal{E}^0_m(\Omega) = \Big\{u \in \mathrm{SH}^-_m(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial\Omega} u(z) = 0, \int_\Omega H_m(u) < \infty\Big\},$$

$$\mathcal{F}_m = \mathcal{F}_m(\Omega) = \Big\{u \in \mathrm{SH}^-_m(\Omega) : \exists\, \mathcal{E}^0_m \ni u_j \searrow u, \sup_j \int_\Omega H_m(u_j) < \infty\Big\},$$

$$\mathcal{E}_m = \mathcal{E}_m(\Omega) = \Big\{u \in \mathrm{SH}^-_m(\Omega) : \forall z_0 \in \Omega, \exists \text{ a neighborhood } \omega \ni z_0, \text{ and } \exists\, \mathcal{E}^0_m \ni u_j \searrow u \text{ on } \omega, \sup_j \int_\Omega H_m(u_j) < \infty\Big\}.$$

In the case $m = n$ these classes coincide with, respectively, the classes $\mathcal{E}^0(\Omega)$, $\mathcal{F}(\Omega)$ and $\mathcal{E}(\Omega)$ introduced and investigated earlier by Cegrell [C2].

From [Lu, Theorem 3.14] it follows that if $u \in \mathcal{E}_m(\Omega)$, the complex Hessian $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ is well defined and is a Radon measure on Ω. On the other hand, by [Lu, Remark 3.6] the following description of $\mathcal{E}_m(\Omega)$ may be given:

$$\mathcal{E}_m = \mathcal{E}_m(\Omega) = \{u \in \mathrm{SH}_m^-(\Omega) : \forall U \Subset \Omega,\ \exists v \in \mathcal{F}_m(\Omega), v = u \text{ on } U\}.$$

2.4. We recall the definition of the pluricomplex Green function. Let Ω be an open subset in $\mathbb{C}^n$, and let a be a point in Ω. The *pluricomplex Green function* with pole at a, denoted by g_a, is defined by

$$g_a(z) = \sup\{u(z) : u \in \mathrm{PSH}^-(\Omega), u(z) \le \log\|z-a\| + c_u \text{ for } z \text{ near } a\}.$$

It is well known that if Ω is bounded and $\overline{\mathbb{B}}(a,r) \subset \Omega \subset \mathbb{B}(a,R)$ then [K1, Proposition 6.1.1] implies that

$$\log(\|z-a\|/R) \le g_a(z) \le \log(\|z-a\|/r)$$

for $z \in \Omega$, and $z \mapsto g_a(z)$ is a negative plurisubharmonic function with a logarithmic pole at a. In the case when Ω is a bounded hyperconvex domain we have $\lim_{z\to\partial\Omega} g_a(z) = 0$. At the same time, by a result of Demailly [D1], the Monge–Ampère measure $(dd^c g_a)^n$ is well defined and $(dd^c g_a)^n = \delta_a$, where δ_a is the Dirac measure at a. On the other hand, it is not difficult to see that $g_a \in \mathcal{F}(\Omega)$.

2.5. Now we recall the definition of Lelong numbers associated to a closed positive current T and Lelong numbers of a plurisubharmonic function introduced and investigated in [D2] and [D3]. Let $\Omega \subset \mathbb{C}^n$ be an open set and T be a closed positive current of bidimension (p,p) on Ω. Assume that φ is a plurisubharmonic function bounded near the boundary $\partial\Omega$ of Ω. Then as in [D3] the measure $T \wedge (dd^c\varphi)^p$ is well defined on Ω. The *Lelong number of T with respect to the weight φ* is denoted by $\nu(T,\varphi)$ and defined by

$$\nu(T,\varphi) = \int_{\{\varphi=-\infty\}} T \wedge (dd^c\varphi)^p = \lim_{r\to-\infty} \int_{\{\varphi<r\}} T \wedge (dd^c\varphi)^p.$$

If $a \in \Omega$ and we take $\varphi_a(z) = \log\|z-a\|$ then we get the definition of the *Lelong number of T at a* which we denote by $\nu(T,a)$. Thus

$$\nu(T,a) = \int_{\{a\}} T \wedge (dd^c\varphi_a)^p = \lim_{r\to 0} \int_{\{\|z-a\|<r\}} T \wedge (dd^c\varphi_a)^p.$$

By 2.4 and by using the comparison theorems for Lelong numbers in [D3] we note that $\nu(T,a)$ can also be defined by

$$\nu(T,a) = \int_{\{a\}} T \wedge (dd^c g_a)^p = \lim_{r\to 0} \int_{\{\|z-a\|<r\}} T \wedge (dd^c g_a)^p.$$

Now assume that $a \in \Omega$ and $\varphi \in \mathrm{PSH}^-(\Omega)$. If we take $T = dd^c\varphi$ then we have the definition of the Lelong number of φ at a. Namely, with the notation φ_a above, the *Lelong number of* φ *at* a, denoted by $\nu(\varphi, a)$, is defined by

$$\nu(\varphi, a) = \int_{\{a\}} dd^c\varphi \wedge (dd^c\varphi_a)^{n-1} = \lim_{r\to 0} \int_{\{\|z-a\|<r\}} dd^c\varphi \wedge (dd^c\varphi_a)^{n-1}.$$

As above, we can also define $\nu(\varphi, a)$ by

$$\nu(\varphi, a) = \int_{\{a\}} dd^c\varphi \wedge (dd^c g_a)^{n-1} = \lim_{r\to 0} \int_{\{\|z-a\|<r\}} dd^c\varphi \wedge (dd^c g_a)^{n-1}.$$

A celebrated result of Siu [Si] for upper level sets of the Lelong numbers of plurisubharmonic functions, later generalized by Demailly [D3] to the Lelong numbers of closed positive currents, says that if T is a closed positive current of bidimension (p, p) on an open set $\Omega \subset \mathbb{C}^n$ then for all $c > 0$ the upper level sets $E_c(T) = \{x \in \Omega : \nu(T, x) \geq c\}$ are analytic subsets of Ω of dimension $\leq p$.

2.6. Throughout the paper we write $A \lesssim B$ if there exists a constant C such that $A \leq CB$.

3. The local property and an analytic characterization for the class $\mathcal{E}_m(\Omega)$. In this section we show that to belong to the class $\mathcal{E}_m(\Omega)$ is a local property. Relying on this result we give an analytic characterization for this class.

First we need the following.

Lemma 3.1. *Let $u, v \in \mathrm{SH}_m^-(\Omega) \cap L^\infty(\Omega)$ with $u \leq v$ on Ω and $T = dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_{m-1} \wedge \beta^{n-m}$ with $\varphi_j \in \mathrm{SH}_m^-(\Omega) \cap L^\infty(\Omega)$, $j = 1, \ldots, m-1$. Then for every $p \geq 0$ we have*

$$\int_{\Omega'} (-u)^p dd^c v \wedge T \leq c \int_{\Omega''} (-u)^p (dd^c u + |u|\beta) \wedge T,$$

where $\Omega' \Subset \Omega'' \Subset \Omega$ and c is a constant depending on Ω', Ω'', Ω and p.

Proof. Repeat the argument for [LPH, Lemma 3.2]. ∎

We also need the following result on subextension for the class $\mathcal{F}_m(\Omega)$.

Lemma 3.2. *Assume that $\Omega \Subset \widetilde{\Omega}$ and $u \in \mathcal{F}_m(\Omega)$. Then there exists a $\widetilde{u} \in \mathcal{F}_m(\widetilde{\Omega})$ such that $\widetilde{u} \leq u$ on Ω.*

Proof. We split the proof into three steps.

Step 1. We prove that if $v \in \mathcal{C}(\widetilde{\Omega})$, $v \leq 0$, $\operatorname{supp} v \Subset \widetilde{\Omega}$ then $\widetilde{v} := \sup\{w \in \mathrm{SH}_m^-(\widetilde{\Omega}) : w \leq v \text{ on } \widetilde{\Omega}\} \in \mathcal{E}_m^0(\widetilde{\Omega}) \cap \mathcal{C}(\widetilde{\Omega})$ and $(dd^c\widetilde{v})^m \wedge \beta^{n-m} = 0$ on $\{\widetilde{v} < v\}$. Indeed, let $\varphi \in \mathcal{E}_m^0(\widetilde{\Omega}) \cap \mathcal{C}(\widetilde{\Omega})$ with $\varphi \leq \inf_{\widetilde{\Omega}} v$ on $\operatorname{supp} v$. Since $\varphi \leq \widetilde{v}$ we get $\widetilde{v} \in \mathcal{E}_m^0(\widetilde{\Omega})$. Moreover, by [B1, Proposition 3.2] we have

$\widetilde{v} \in \mathcal{C}(\widetilde{\Omega})$. Let $w \in \mathrm{SH}_m^-(\{\widetilde{v} < v\})$ be such that $w \le \widetilde{v}$ outside a compact subset K of $\{\widetilde{v} < v\}$. Define

$$w_1 = \begin{cases} \max(w, \widetilde{v}) & \text{on } \{\widetilde{v} < v\}, \\ \widetilde{v} & \text{on } \widetilde{\Omega} \setminus \{\widetilde{v} < v\}. \end{cases}$$

Since $\widetilde{v}$ and v are continuous, $\varepsilon = -\sup_K(\widetilde{v} - v) > 0$. Choose $\delta \in (0,1)$ such that $-\delta \inf_{\widetilde{\Omega}} \widetilde{v} < \varepsilon$. Then $(1-\delta)\widetilde{v} \le \widetilde{v} + \varepsilon \le v$ on K. Hence, $(1-\delta)\widetilde{v} + \delta w_1 \le v$ on $\widetilde{\Omega}$, and we get $(1-\delta)\widetilde{v} + \delta w_1 \le \widetilde{v}$. Thus, $w \le \widetilde{v}$ on $\{\widetilde{v} < v\}$. Therefore, $\widetilde{v}$ is m-maximal in $\{\widetilde{v} < v\}$. By [B1] we get $(dd^c\widetilde{v})^m \wedge \beta^{n-m} = 0$ on $\{\widetilde{v} < v\}$.

Step 2. Next, we prove that if $u \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ then there exists $\widetilde{u} \in \mathcal{E}_m^0(\widetilde{\Omega})$ for which $(dd^c\widetilde{u})^m \wedge \beta^{n-m} = 0$ on $(\widetilde{\Omega}\setminus\Omega) \cup (\{\widetilde{u} < u\} \cap \Omega)$ and $(dd^c\widetilde{u})^m \wedge \beta^{n-m} \le (dd^c u)^m \wedge \beta^{n-m}$ on $\{\widetilde{u} = u\} \cap \Omega$. Indeed, set

$$v = \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \widetilde{\Omega} \setminus \Omega. \end{cases}$$

It is easy to see that $v \in \mathcal{C}(\Omega)$ and $\operatorname{supp} v \subset \Omega \Subset \widetilde{\Omega}$. Hence, by Step 1 we have $\widetilde{u} = \widetilde{v} \in \mathcal{E}_m^0(\widetilde{\Omega}) \cap \mathcal{C}(\widetilde{\Omega})$ and $(dd^c\widetilde{u})^m \wedge \beta^{n-m} = 0$ on $\{\widetilde{v} < v\} = (\widetilde{\Omega} \setminus \Omega) \cup (\{\widetilde{u} < u\} \cap \Omega)$. Let K be a compact set in $\{\widetilde{u} = u\} \cap \Omega$. Then for $\varepsilon > 0$ we have $K \Subset \{\widetilde{u} + \varepsilon > u\} \cap \Omega$, and so

$$\begin{aligned} \int_K (dd^c\widetilde{u})^m \wedge \beta^{n-m} &= \int_K 1_{\{\widetilde{u}+\varepsilon>u\}} (dd^c\widetilde{u})^m \wedge \beta^{n-m} \\ &= \int_K 1_{\{\widetilde{u}+\varepsilon>u\}} (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m} \\ &\le \int_K (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m}, \end{aligned}$$

where the equality in the second line follows as in [BT2] (see also [Lu, proof of Theorem 3.23]). However, $\max(\widetilde{u}+\varepsilon, u) \searrow u$ on Ω as $\varepsilon \to 0$, therefore $(dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m}$ is weakly convergent to $(dd^c u)^m \wedge \beta^{n-m}$ as $\varepsilon \to 0$. On the other hand, 1_K is upper semicontinuous on Ω so we can approximate 1_K with a decreasing sequence of continuous functions φ_j. So,

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \int_\Omega 1_K (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m} \\ &\qquad = \limsup_{\varepsilon\to 0} \Big[\lim_j \int_\Omega \varphi_j (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m}\Big] \\ &\qquad \le \limsup_{\varepsilon\to 0} \Big(\int_\Omega \varphi_j (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m}\Big) \\ &\qquad \le \int_\Omega \varphi_j (dd^c u)^m \wedge \beta^{n-m} \searrow \int_K (dd^c u)^m \wedge \beta^{n-m} \end{aligned}$$

as $j \to \infty$. This yields $(dd^c\widetilde{u})^m \wedge \beta^{n-m} \le (dd^c u)^m \wedge \beta^{n-m}$ on $\{\widetilde{u} = u\} \cap \Omega$.

Step 3. Now, let $\mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega) \ni u_j \searrow u$ be such that

$$\sup_j \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

By Step 2, we have

$$\begin{aligned}\int_{\widetilde{\Omega}} (dd^c \widetilde{u}_j)^m \wedge \beta^{n-m} &= \int_{\{\widetilde{u}_j = u_j\} \cap \Omega} (dd^c \widetilde{u}_j)^m \wedge \beta^{n-m} \\ &\le \int_{\{\widetilde{u}_j = u_j\} \cap \Omega} (dd^c u_j)^m \wedge \beta^{n-m} \le \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m}.\end{aligned}$$

Hence,

$$\sup_j \int_{\widetilde{\Omega}} (dd^c \widetilde{u}_j)^m \wedge \beta^{n-m} \le \sup_j \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

Thus, $\widetilde{u} := \lim_{j\to\infty} \widetilde{u}_j \in \mathcal{F}_m(\widetilde{\Omega})$ and $\widetilde{u} \le u$ on Ω. ∎

The following result deals with the locality of membership in $\mathcal{E}_m(\Omega)$.

Theorem 3.3. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and m be an integer with $1 \le m \le n$. Assume that $u \in \mathrm{SH}_m^-(\Omega)$. Then the following statements are equivalent:*

(i) $u \in \mathcal{E}_m(\Omega)$.

(ii) *For all $K \Subset \Omega$, there exists a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega) \cap C(\Omega)$, $u_j \searrow u$ on K, such that for all $p = 0, 1, \ldots, m$ we have*

$$\sup_j \int_K (-u_j)^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty.$$

(iii) *For every $W \Subset \Omega$ such that W is a hyperconvex domain, we have $u|_W \in \mathcal{E}_m(W)$.*

(iv) *For every $z \in \Omega$ there exists a hyperconvex domain $V_z \Subset \Omega$ such that $z \in V_z$ and $u|_{V_z} \in \mathcal{E}_m(V_z)$.*

Proof. The proof here is due to Błocki [B2].

(i)⇒(ii). Let $K \Subset \Omega$ be given. Since $u \in \mathcal{E}_m(\Omega)$, there exists $v \in \mathcal{F}_m(\Omega)$ with $v = u$ on K. By the definition of the class $\mathcal{F}_m(\Omega)$ there exists a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega) \cap C(\Omega)$ with $u_j \searrow v$ on Ω such that

$$\sup_j \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m} < \infty. \tag{3.1}$$

Then $u_j \searrow u$ on K. We have to prove

$$\sup_j \int_K (-u_j)^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for $p = 0, 1, \ldots, m$. It is obvious that the conclusion holds for $p = 0$. Assume that $1 \le p \le m$. Choose $R > 0$ such that $\|z\|^2 - R^2 < 0$ on Ω and assume that $\varphi \in \mathcal{E}_m^0(\Omega)$ is given. Next, we choose $A > 0$ such that $\|z\|^2 - R^2 \ge A\varphi$ on K. Set $h = \max(\|z\|^2 - R^2, A\varphi)$. Then $h \in \mathcal{E}_m^0(\Omega)$ and $dd^c h = \beta$ on K. For each $p = 1, \ldots, m$ we define

$$I_p = \int_\Omega (-u_j)^p (dd^c u_j)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m}.$$

Then by integration by parts we get the chain of inequalities

$$\begin{aligned}
&\int_K (-u_j)^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} = \int_K (-u_j)^p (dd^c u_j)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m} \\
&\le I_p = \int_\Omega h (dd^c u_j)^{m-p} \wedge (dd^c h)^{p-1} \wedge (dd^c(-u_j)^p) \wedge \beta^{n-m} \\
&= \int_\Omega h (dd^c u_j)^{m-p} \wedge (dd^c h)^{p-1} [p(p-1) du_j \wedge d^c u_j - p(-u_j)^{p-1} dd^c u_j] \wedge \beta^{n-m} \\
&\le p \int_\Omega (-h)(-u_j)^{p-1} (dd^c u_j)^{m-p+1} \wedge (dd^c h)^{p-1} \wedge \beta^{n-m} \\
&\le p\|h\|_\Omega \int_\Omega (-u_j)^{p-1} (dd^c u_j)^{m-p+1} \wedge (dd^c h)^{p-1} \wedge \beta^{n-m} \\
&= p\|h\|_\Omega I_{p-1} \le \cdots \le p!\|h\|_\Omega^p I_0 = p!\|h\|_\Omega^p \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m}.
\end{aligned}$$

Hence, by (3.1) we get

$$\sup_j \int_K (-u_j)^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} \le p!\|h\|_\Omega^p \sup_j \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

(ii)$\Rightarrow$(iii). Let $W \Subset \Omega$ be a hyperconvex domain. Take $U \Subset W \Subset \Omega$ and a sequence $\mathcal{E}_m^0(\Omega) \ni u_j \searrow u$ on W such that

$$\sup_j \int_W (-u_j)^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for $p = 0, 1, \ldots, m$. Set $\widetilde{u}_j = \sup\{\varphi \in \mathrm{SH}_m^-(W) : \varphi \le u_j \text{ on } U\} \in \mathcal{E}_m^0(W)$. Next, choose $U \Subset \Omega_1 \Subset \cdots \Subset \Omega_m \Subset W$. Since $u_j \le \widetilde{u}_j$ on W and $(dd^c \widetilde{u}_j)^m \wedge \beta^{n-m} = 0$ on $W \backslash \overline{U}$, by applying Lemma 3.1 repeatedly we arrive at

$$\begin{aligned}
\int_W (dd^c \widetilde{u}_j)^m \wedge \beta^{n-m} &= \int_{\overline{U}} (dd^c \widetilde{u}_j)^m \wedge \beta^{n-m} \\
&\lesssim \int_{\Omega_1} (dd^c u_j + (-u_j)\beta) \wedge (dd^c \widetilde{u}_j)^{m-1} \wedge \beta^{n-m}
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\Omega_1} (dd^c\widetilde{u}_j)^{m-1} \wedge dd^c u_j \wedge \beta^{n-m} + \int\limits_{\Omega_1} (-u_j)(dd^c\widetilde{u}_j)^{m-1} \wedge \beta^{n-m+1} \\
&\lesssim \int\limits_{\Omega_2} (dd^c u_j + (-u_j)\beta) \wedge (dd^c\widetilde{u}_j)^{m-2} \wedge dd^c u_j \wedge \beta^{n-m} \\
&\quad + \int\limits_{\Omega_2} (-u_j)(dd^c u_j + (-u_j)\beta) \wedge (dd^c\widetilde{u}_j)^{m-2} \wedge \beta^{n-m+1} \\
&\lesssim \int\limits_{\Omega_2} [(-u_j)^2\beta^2 + (-u_j)\beta \wedge dd^c u_j + (dd^c u_j)^2] \wedge (dd^c\widetilde{u}_j)^{m-2} \wedge \beta^{n-m} \\
&\lesssim \cdots \\
&\lesssim \int\limits_{\Omega_m} [(-u_j)^m\beta^m + (-u_j)^{m-1} dd^c u_j \wedge \beta^{m-1} + \cdots + (dd^c u_j)^m] \wedge \beta^{n-m}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sup_j \int\limits_W (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} \\
&\lesssim \sup_j \int\limits_{\Omega_m} [(-u_j)^m\beta^m + (-u_j)^{m-1} dd^c u_j \wedge \beta^{m-1} + \cdots + (dd^c u_j)^m] \wedge \beta^{n-m} \\
&\lesssim \sup_j \int\limits_W [(-u_j)^m\beta^m + (-u_j)^{m-1} dd^c u_j \wedge \beta^{m-1} + \cdots + (dd^c u_j)^m] \wedge \beta^{n-m} < \infty.
\end{aligned}
$$

Thus, $u_{U,W} := \lim \widetilde{u}_j \in \mathcal{F}_m(W)$. Since $U \Subset W$ is arbitrary and $u_{U,W} = u$ on U, we conclude that $u \in \mathcal{E}_m(W)$.

(iii)⇒(iv). This is obvious.

(iv)⇒(i). Assume that $\Omega' \Subset \Omega$. Choose $z_j \in \Omega$, $j = 1, \ldots, s$, such that $\Omega' \Subset \bigcup_{j=1}^s V_{z_j}$, where V_{z_j} are hyperconvex domains. Let $W_{z_j} \Subset V_{z_j}$ be such that $\Omega' \Subset \bigcup_{j=1}^s W_{z_j}$. Since $u|_{V_{z_j}} \in \mathcal{E}_m(V_{z_j})$, there exists $v_j \in \mathcal{F}_m(V_{z_j})$ such that $v_j = u$ on W_{z_j}. By Lemma 3.2 there exists $\widetilde{v}_j \in \mathcal{F}_m(\Omega)$ such that $\widetilde{v}_j \le v_j$ on V_{z_j}. The convexity of the class $\mathcal{F}_m(\Omega)$ implies that $\widetilde{v} := \frac{1}{s}\widetilde{v}_1 + \cdots + \frac{1}{s}\widetilde{v}_s \in \mathcal{F}_m(\Omega)$, and hence $\max(\widetilde{v}, u) \in \mathcal{F}_m(\Omega)$. However, $\max(\widetilde{v}, u) = u$ on Ω' and therefore $u \in \mathcal{E}_m(\Omega)$. The proof of Theorem 3.3 is complete. ■

Remark 3.4. In [B2], Błocki proved that membership in the class $\mathcal{E}(\Omega)$ is a local property.

From Theorem 3.3 we get the following property of $\mathcal{E}_m(\Omega)$.

Corollary 3.5. *Assume that Ω is a bounded hyperconvex domain. Then $\mathcal{E}_m(\Omega) \subset \mathcal{E}_{m-1}(\Omega)$.*

Proof. Assume that $u \in \mathcal{E}_m(\Omega)$. Let $K \Subset \Omega$. Take a domain Ω' with $K \Subset \Omega' \Subset \Omega$. By Theorem 3.3 there exists a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega)$ such that $u_j \searrow u$ on Ω' and

$$\sup_j \int_{\Omega'} (-u_j)^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for $p = 0, 1, \ldots, m$. Choose $h \in \mathcal{E}^0_{m-1}(\Omega)$. For each $j > 0$ take $m_j > 0$ such that $u_j \geq m_j h$ on Ω'. Set $v_j = \max(u_j, m_j h) \in \mathcal{E}^0_{m-1}(\Omega)$ and $v_j = u_j$ on Ω'. Note that $v_j \searrow u$ on Ω' and $(dd^c v_j)^p \wedge \beta^q = (dd^c u_j)^p \wedge \beta^q$ on Ω' for $0 \leq p \leq m-1$ and $1 \leq q \leq n-m+1$. We may assume that $u|_{\Omega'} \leq -1$. By Hartogs' lemma (see [Ho, Theorem 3.2.13]) we conclude that $v_j|_{\Omega'} \leq -1$ for $j \geq j_0$ with some j_0. Without loss of generality, we may assume that $v_j|_{\Omega'} \leq -1$ for $j \geq 1$. Hence, $(-v_j)^k \geq (-v_j)^{k-1}$ on Ω' for all $j \geq 1$ and $k = 1, \ldots, m$. Now we have

$$\begin{aligned}
&\int_{\Omega'} [(-u_j)^m \beta^m + \cdots + (-u_j)(dd^c u_j)^{m-1} \wedge \beta + (dd^c u_j)^m] \wedge \beta^{n-m} \\
&\qquad \geq \int_{\Omega'} [(-u_j)^m \beta^m + \cdots + (-u_j)(dd^c u_j)^{m-1} \wedge \beta] \wedge \beta^{n-m} \\
&\qquad = \int_{\Omega'} [(-v_j)^m \beta^m + \cdots + (-v_j)(dd^c v_j)^{m-1} \wedge \beta] \wedge \beta^{n-m} \\
&\qquad = \int_{\Omega'} [(-v_j)^m \beta^{m-1} + \cdots + (-v_j)(dd^c v_j)^{m-1}] \wedge \beta^{n-m+1} \\
&\qquad \geq \int_{\Omega'} [(-v_j)^{m-1} \beta^{m-1} + \cdots + (dd^c v_j)^{m-1}] \wedge \beta^{n-m+1} \\
&\qquad \geq \int_{K} (-v_j)^p (dd^c v_j)^{m-p-1} \wedge \beta^{n-m+p+1}
\end{aligned}$$

for $p = 0, 1, \ldots, m-1$. Hence,

$$\sup_j \int_K (-v_j)^p (dd^c v_j)^{m-p-1} \wedge \beta^{n-m+p+1} < \infty$$

for $p = 0, 1, \ldots, m-1$. By Theorem 3.3, the desired conclusion follows. ∎

Now we prove an analytic characterization of $\mathcal{E}_m(\Omega)$ which we need in the next section. Note that part of the following arguments is due to Błocki [B2].

THEOREM 3.6. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathrm{SH}^-_m(\Omega), 1 \leq m \leq n$. Then the following are equivalent:*

(i) $u \in \mathcal{E}_m(\Omega)$.
(ii) *There exists a positive Radon measure μ on Ω such that if $\omega \Subset \Omega$ is a hyperconvex domain and a sequence $\{u_j\} \subset \mathrm{SH}^-_m(\omega) \cap L^\infty(\omega)$ satisfies $u_j \searrow u$ on ω, then the sequence of Hessian measures $H_m(u_j) = (dd^c u_j)^m \wedge \beta^{n-m}$ is weakly convergent to μ on ω.*

(iii) *For all hyperconvex domains* $\omega \Subset \Omega$ *and every sequence* $\{u_j\} \subset \mathrm{SH}_m^-(\omega) \cap L^\infty(\omega)$ *with* $u_j \searrow u$ *on* ω, *the sequence of Hessian measures* $H_m(u_j) = (dd^c u_j)^m \wedge \beta^{n-m}$ *is locally weakly bounded in* ω.

Proof. (i)⇒(ii). Assume $u \in \mathcal{E}_m(\Omega)$. Set $\mu = H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$. Then μ is a well defined positive Radon measure on Ω. Moreover, because $\omega \Subset \Omega$ is a hyperconvex domain, we have $u \in \mathcal{E}_m(\omega)$ by Theorem 3.3. Hence, if $\{u_j\} \subset \mathrm{SH}_m^-(\omega) \cap L^\infty(\omega)$ with $u_j \searrow u$ on ω then by [B1, p. 1736], $H_m(u_j) = (dd^c u_j)^m \wedge \beta^{n-m}$ is weakly convergent to $H_m(u) = \mu$ on ω, and we are done.

(ii)⇒(iii). Let $K \Subset \omega$. Then, by the hypothesis, if $\{u_j\} \subset \mathrm{SH}_m^-(\omega) \cap L^\infty(\omega)$ and $u_j \searrow u$ on ω then $H_m(u_j) = (dd^c u_j)^m \wedge \beta^{n-m}$ is weakly convergent to μ on ω. This yields

$$\sup_j \int_K H_m(u_j) \le \mu(K) < \infty,$$

and we get the desired conclusion.

(iii)⇒(i). Take a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega) \cap C(\Omega)$ with $u_j \searrow u$ on Ω. If for every $K \Subset \Omega$ we have

$$\sup_j \int_K (-u_j)^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for $p = 0, 1, \ldots, m$ then by (ii) of Theorem 3.3, $u \in \mathcal{E}_m(\Omega)$ and we are done.

Now we assume that the above claim is not true. Then there exists a ball $B \Subset \Omega$ such that

$$\sup_j \int_B (-u_j)^{p_0} (dd^c u_j)^{m-p_0} \wedge \beta^{n-m+p_0} = \infty$$

for some p_0, $0 \le p_0 \le m$. As in [B2], choose a sequence $\{\lambda_j\}$ of positive numbers increasing to 1 and set $v_j = \lambda_j u_j$. Then $v_j \in \mathcal{E}_m^0(\Omega) \cap C(\Omega)$, $v_j \searrow u$ on Ω and

$$\sup_j \int_B (-v_j)^{p_0} (dd^c v_j)^{m-p_0} \wedge \beta^{n-m+p_0} = \infty. \tag{3.2}$$

For $k \ge j+1$ we have $|v_j - v_k| \ge (1 - \lambda_j/\lambda_{j+1})|v_k|$. Hence,

$$\begin{aligned} &\int_B |v_j - v_k|^{p_0} (dd^c v_k)^{m-p_0} \wedge \beta^{n-m+p_0} \\ &\qquad \ge \left(1 - \frac{\lambda_j}{\lambda_{j+1}}\right) \int_B (-v_k)^{p_0} (dd^c v_k)^{m-p_0} \wedge \beta^{n-m+p_0}. \end{aligned} \tag{3.3}$$

By (3.3) and (3.2) we can find an increasing sequence $k = k(j) \ge j+1$ such that for every j we have

$$\int_B (v_j - v_k)^{p_0} (dd^c v_k)^{m-p_0} \wedge \beta^{n-m+p_0} \ge j.$$

Next, take balls $B \Subset B'' \Subset B' \Subset \Omega$. Set

$$\widetilde{u}_j = \sup\{w \in \mathrm{SH}_m^-(B') : w \le v_j \text{ on } B',\ w \le v_k \text{ on } B\}.$$

Then $\widetilde{u}_j \in \mathrm{SH}_m^-(B') \cap C(\overline{B'})$, $\widetilde{u}_j = v_j$ on $\partial B'$, $\widetilde{u}_j \le v_j$ on B', and $\widetilde{u}_j = v_k$ on B, $\widetilde{u}_j \searrow u$ on B'. By the hypothesis we have

$$\sup_j \int_{\overline{B''}} H_m(\widetilde{u}_j) = \sup_j \int_{\overline{B''}} (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} < \infty. \tag{3.4}$$

Next, choose $\phi \in \mathcal{E}_m^0(B')$ such that $dd^c\phi = \beta$ on B'. Now using integration by parts we get the following chain of inequalities:

$$\begin{aligned}
(3.5)\quad j &\le \int_B (v_j - v_k)^{p_0} (dd^c v_k)^{m-p_0} \wedge \beta^{n-m+p_0}\\
&= \int_B (v_j - \widetilde{u}_j)^{p_0} (dd^c\widetilde{u}_j)^{m-p_0} \wedge (dd^c\phi)^{p_0} \wedge \beta^{n-m}\\
&\le \int_{B'} (v_j - \widetilde{u}_j)^{p_0} (dd^c\widetilde{u}_j)^{m-p_0} \wedge (dd^c\phi) \wedge (dd^c\phi)^{p_0-1} \wedge \beta^{n-m}\\
&= \int_{B'} \phi (dd^c\widetilde{u}_j)^{m-p_0} \wedge [dd^c((v_j - \widetilde{u}_j)^{p_0})] \wedge (dd^c\phi)^{p_0-1} \wedge \beta^{n-m}\\
&\le p_0 \int_{B'} \phi (v_j - \widetilde{u}_j)^{p_0-1} (dd^c\widetilde{u}_j)^{m-p_0} \wedge [dd^c v_j - dd^c\widetilde{u}_j] \wedge (dd^c\phi)^{p_0-1} \wedge \beta^{n-m}\\
&\le p_0 \int_{B'} -\phi (v_j - \widetilde{u}_j)^{p_0-1} (dd^c\widetilde{u}_j)^{m-p_0+1} \wedge (dd^c\phi)^{p_0-1} \wedge \beta^{n-m}\\
&\le p_0 \|\phi\|_{B'} \int_{B'} (v_j - \widetilde{u}_j)^{p_0-1} (dd^c\widetilde{u}_j)^{m-p_0+1} \wedge (dd^c\phi)^{p_0-1} \wedge \beta^{n-m}\\
&\le \cdots \le p_0! \|\phi\|_{B'}^{p_0} \int_{B'} (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m}\\
&= \Big[\int_{\overline{B''}} (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} + \int_{B'\setminus\overline{B''}} (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m}\Big].
\end{aligned}$$

However, $(dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} \le (dd^c v_j)^m \wedge \beta^{n-m}$ on $B' \setminus \overline{B''}$ because on $\{\widetilde{u}_j < v_j\} \cap B' \setminus \overline{B''}$ we have $(dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} = 0$, and, on $\{\widetilde{u}_j = v_j\} \cap B' \setminus \overline{B''}$, by repeating the proof of Lemma 3.2, we get $(dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} \le (dd^c v_j)^m \wedge \beta^{n-m}$. Hence,

$$\begin{aligned}
&\int_{\overline{B''}} (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} + \int_{B'\setminus\overline{B''}} (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m}\\
&\qquad\le \int_{\overline{B''}} (dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} + \int_{B'\setminus\overline{B''}} (dd^c v_j)^m \wedge \beta^{n-m}.
\end{aligned}$$

Note that $v_j \searrow u$ on Ω and so by the hypothesis we have

$$\sup_j \int_{B'} (dd^c v_j)^m \wedge \beta^{n-m} < \infty. \tag{3.6}$$

Combining (3.4)–(3.6) we get a contradiction. The proof of Theorem 3.6 is complete. ■

4. A geometrical characterization of $\mathcal{E}_m(\Omega)$ and lower bounds for the log canonical threshold. In this section we give a geometrical description of the class $\mathcal{E}_m(\Omega)$ and then by using Theorem 3.3 we establish lower bounds for the log canonical threshold in $\mathrm{PSH}(\Omega) \cap \mathcal{E}_m(\Omega)$ and in $\mathrm{PSH}(\Omega) \cap L^\infty(\Omega \setminus E)$ when the Hausdorff measure $\mathcal{H}_{2(n-m)+2}(E)$ is 0.

The first result in this section is the following.

THEOREM 4.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{E}_m(\Omega) \cap \mathrm{PSH}^-(\Omega)$. Then for all $c > 0$, the upper level sets*

$$E(u,c) = \{z \in \Omega : \nu(u,z) \geq c\}$$

are analytic subsets in Ω of dimension $\leq n-m$.

Proof. Let $c > 0$ be given. By [Si] we know that $E(u,c)$ is an analytic subset in Ω. Let $T = (dd^c u)^m$. Proposition 4.3 below states that T is a closed positive current of bidimension $(n-m, n-m)$. Then, as in 2.4, the Lelong number $\nu(T,z)$ of T at $z \in \Omega$ is given by

$$\nu(T,z) = \int_{\{z\}} T \wedge (dd^c g_z)^{n-m} = \int_{\{z\}} (dd^c u)^m \wedge (dd^c g_z)^{n-m},$$

where g_z denotes the pluricomplex Green function. On the other hand, Corollary 6.5 in [D3] implies that for all $c > 0$, the upper level sets

$$E(T,c) = \{z \in \Omega : \nu(T,z) \geq c\}$$

are analytic subsets of dimension $\leq n-m$. Hence, the theorem is proved if we show that $E(u,c) \subset E(T,c^m)$ for $c > 0$. To do so, it suffices to prove

$$\nu(u,z) \leq \nu(T,z)^{1/m}$$

for all $z \in \Omega$.

The arguments here are due to Cegrell. First we use [C2, Lemma 5.3] to obtain an inequality for functions in $\mathcal{E}^0_m(\Omega)$. Next, using [C2, Proposition 5.1] we get the same inequality for functions in $\mathcal{F}_m(\Omega)$. Finally, combining all arguments we get the desired conclusion.

Now we give a detailed proof. Without loss of generality we may assume that $u \in \mathcal{F}_m(\Omega)$. For $\varepsilon > 0$, let $h = \max(\varepsilon g_z, -1) \in \mathcal{E}_0(\Omega)$. First, assume $\varphi \in \mathcal{E}^0_m(\Omega)$. By using a similar argument to that in [C2, proof of Lemma 5.3] we get

$$\int_{\Omega} -h(dd^c\varphi) \wedge (dd^c g_z)^{m-1} \wedge (dd^c g_z)^{n-m} \leq \Big(\int_{\Omega} -h(dd^c\varphi)^m \wedge (dd^c g_z)^{n-m}\Big)^{1/m} \Big(\int_{\Omega} -h(dd^c g_z)^n\Big)^{m-1/m}. \tag{4.1}$$

Now assume that $u \in \mathcal{F}_m(\Omega)$. Take a sequence $\mathcal{E}_m^0(\Omega) \ni \varphi_j \searrow u$ on Ω. As above, by similar arguments to those in [C2, proof of Proposition 5.1] we have

$$\lim_{j\to\infty} \int_{\Omega} -h(dd^c\varphi_j)^m \wedge (dd^c g_z)^{n-m} = \int_{\Omega} -h(dd^c u)^m \wedge (dd^c g_z)^{n-m} \tag{4.2}$$

for all $m \geq 1$. Hence, by replacing φ by φ_j in (4.1) and letting $m \to \infty$ we get

$$\int_{\Omega} -h(dd^c u) \wedge (dd^c g_z)^{m-1} \wedge (dd^c g_z)^{n-m} \leq \Big(\int_{\Omega} -h(dd^c u)^m \wedge (dd^c g_z)^{n-m}\Big)^{1/m} \Big(\int_{\Omega} -h(dd^c g_z)^n\Big)^{(m-1)/m}. \tag{4.3}$$

We rewrite (4.3) as follows:

$$\begin{aligned}
&\int_{\Omega\setminus\{z\}} -h(dd^c u) \wedge (dd^c g_z)^{m-1} \wedge (dd^c g_z)^{n-m} \\
&\qquad + \int_{\{z\}} -h(dd^c u) \wedge (dd^c g_z)^{m-1} \wedge (dd^c g_z)^{n-m} \\
&\quad \leq \Big(\int_{\Omega\setminus\{z\}} -h(dd^c u)^m \wedge (dd^c g_z)^{n-m} + \int_{\{z\}} -h(dd^c u)^m \wedge (dd^c g_z)^{n-m}\Big)^{1/m} \\
&\qquad \times \Big(\int_{\Omega} -h(dd^c g_z)^n\Big)^{(m-1)/m}.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\int_{\{z\}} (dd^c u) \wedge (dd^c g_z)^{n-1} \leq \Big(\int_{\{z\}} (dd^c u)^m \wedge (dd^c g_z)^{n-m}\Big)^{1/m},$$

or $\nu(u,z) \leq \nu(T,z)^{1/m}$, as we wanted. The proof of Theorem 4.1 is complete. ∎

Now from Theorem 3.3 and Corollary 3.5, and by relying on a recent result due to Demailly and Pham [DP] about a lower bound for the log canonical threshold of plurisubharmonic functions in $\widetilde{\mathcal{E}}(\Omega)$, we give a lower bound for the log canonical threshold of plurisubharmonic functions in $\mathcal{E}_m(\Omega) \cap \mathrm{PSH}(\Omega)$. We need the following lemma.

LEMMA 4.2. *Assume that $u \in \mathcal{E}_m(\Omega)$. Then for every $1 \le p \le m-1$ and for every $K \Subset \Omega$ we have*

$$\int_K u(dd^c u)^p \wedge \beta^{n-p} > -\infty.$$

Proof. By Corollary 3.5 we have $\mathcal{E}_{p+1}(\Omega) \subset \mathcal{E}_p(\Omega)$ for $1 \le p \le m-1$. Hence, it suffices to prove the lemma for $p = m-1$. Moreover, we can assume that $K = \mathbb{B}(0, r_0) \Subset \Omega$. Let $r_0 < r_1 < r_2$ be such that $\mathbb{B}(0, r_2) \Subset \Omega$. By Theorem 3.3 we have $u \in \mathcal{E}_m(\mathbb{B}(0, r_2))$. Choose $v \in \mathcal{F}_m(\mathbb{B}(0, r_2))$ such that $u = v$ in $\mathbb{B}(0, r_1)$. Then

$$\begin{aligned} \int_{\mathbb{B}(0,r_0)} u(dd^c u)^{m-1} \wedge \beta^{n-m+1} &\ge \int_{\mathbb{B}(0,r_2)} v(dd^c v)^{m-1} \wedge \beta^{n-m+1} \\ &= \int_{\mathbb{B}(0,r_2)} (|z|^2 - r_2^2)(dd^c v)^m \wedge \beta^{n-m} \\ &\ge -r_2^2 \int_{\mathbb{B}(0,r_2)} (dd^c v)^m \wedge \beta^{n-m} > -\infty. \ \blacksquare \end{aligned}$$

PROPOSITION 4.3. *Assume that $u \in \mathcal{E}_m(\Omega) \cap \mathrm{PSH}^-(\Omega)$. Then $(dd^c u)^p$ is a closed nonnegative current in Ω for $p = 1, \ldots, m$.*

Proof. For $p = 1$ the statement is clear. Assume that for $2 \le p \le m$, $(dd^c u)^{p-1}$ is well defined as a closed nonnegative current. Since $u(dd^c u)^{p-1}$ has locally bounded mass and the coefficients of $(dd^c u)^{p-1}$ are complex measures, u is locally integrable for these measures. Hence, as in [BT1] we can define $(dd^c u)^p := dd^c(u(dd^c u)^{p-1})$. Thus $(dd^c u)^p$ is a closed current. Moreover, since $u \in \mathrm{PSH}(\Omega)$, $(dd^c u)^p$ is nonnegative, and the desired conclusion follows. $\blacksquare$

Now we recall the following definitions introduced and investigated in [DK1] and [DP].

DEFINITION 4.4. Let $u \in \mathrm{PSH}(\Omega)$ and $0 \in \Omega$. As in [DK1], the *log canonical threshold* at $0 \in \Omega$ of u is defined by

$$c_u(0) = \sup\{c > 0 : e^{-2cu} \text{ is } L^1 \text{ on a neighbourhood of } 0\}.$$

Moreover, for $u \in \mathrm{PSH}(\Omega) \cap \mathcal{E}_m(\Omega)$ we define the *intersection numbers*

$$e_j(u) = \int_{\{0\}} (dd^c u)^j \wedge (dd^c \log|z|)^{n-j}, \quad j = 1, \ldots, m.$$

Note that by Proposition 4.3, Lemma 4.2 and [D2, Proposition 2.1 and Corollary 2.3] we have $e_j(u) < \infty$.

The main result of this section is the following estimate.

THEOREM 4.5. *Let $u \in \mathrm{PSH}(\Omega) \cap \mathcal{E}_m(\Omega), 1 \leq m \leq n-1$ and $0 \in \Omega$. Then*

$$c_u(0) \geq \sum_{j=1}^{m} \frac{e_{j-1}(u)}{e_j(u)},$$

where $e_0(u) = 1$.

Proof. For $k \geq 1$, set

$$u_k = \max\{u, k \log \|z\|\}.$$

Corollary 2.3 in [D2] implies that $(dd^c u_k)^n$ is well defined, and hence $u_k \in \mathcal{E}(\Omega) \subset \widetilde{\mathcal{E}}(\Omega)$. Theorem 1.2 in [DP] gives

$$c_{u_k}(0) \geq \sum_{j=1}^{n-1} \frac{e_{j-1}(u_k)}{e_j(u_k)} \geq \sum_{j=1}^{m} \frac{e_{j-1}(u_k)}{e_j(u_k)},$$

where $e_0(u_k) = 1$. On the other hand, since $u_k \geq u$ on Ω, by the comparison principle (see e.g. [D2]) we have $e_j(u) \geq e_j(u_k)$ for $j = 1, \ldots, m$. Let

$$D = \{(t_1, \ldots, t_m) \in [0, \infty)^m : t_1^2 \leq t_2,\ t_j^2 \leq t_{j-1}t_{j+1},\ \forall j = 1, \ldots, m-1\}.$$

Then D is a convex set in $\mathbb{R}^m$. Consider the function $f : \operatorname{int} D \to [0, \infty)$ given by

$$f(t_1, \ldots, t_m) = \frac{1}{t_1} + \frac{t_1}{t_2} + \cdots + \frac{t_{m-1}}{t_m}.$$

Then

$$\frac{\partial f}{\partial t_j}(t) = -\frac{t_{j-1}}{t_j^2} + \frac{1}{t_{j+1}} \leq 0, \qquad \forall t \in D.$$

For $a, b \in \operatorname{int} D$ with $a_j \geq b_j$, $j = 1, \ldots, m$, the function $[0,1] \ni t \mapsto f(b + t(a-b))$ is decreasing. Hence, for $e_j(u) \geq e_j(u_k), j = 1, \ldots, m$, we get

$$\sum_{j=1}^{m} \frac{e_{j-1}(u_k)}{e_j(u_k)} = f(e_1(u_k), \ldots, e_m(u_k)) \geq f(e_1(u), \ldots, e_m(u)) = \sum_{j=1}^{m} \frac{e_{j-1}(u)}{e_j(u)}.$$

Therefore,

$$c_{u_k}(0) \geq \sum_{j=1}^{m} \frac{e_{j-1}(u)}{e_j(u)}$$

for all $k \geq 1$. However, by [Ph, Lemma 2.1] we have $\lim_{k\to\infty} c_{u_k}(0) = c_u(0)$, and the desired conclusion follows. ∎

Finally, let $\mathcal{H}_\alpha$ denote the Hausdorff measure of dimension α in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. By using a result of [FS] we will prove the same lower bound as in Theorem 4.5 for the log canonical threshold of plurisubharmonic functions which are bounded outside a closed subset of small Hausdorff measure.

THEOREM 4.6. *Let Ω be an open subset in $\mathbb{C}^n$, $0 \in \Omega$ and $E \subset \Omega$ be a closed subset in Ω with $\mathcal{H}_{2(n-m)+2}(E) = 0$, where $1 \le m \le n-1$. Assume that $u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega \setminus E)$. Then*

$$c_u(0) \ge \sum_{j=1}^{m} \frac{e_{j-1}(u)}{e_j(u)},$$

where $e_0(u) = 1$.

Proof. Without loss of generality, we may assume that $u \in \mathrm{PSH}^-(\Omega)$. From the hypothesis $\mathcal{H}_{2(n-m)+2}(E) = 0$ and by [FS, Theorem 2.4], $(dd^c u)^j$ are closed positive currents with locally finite mass for $j = 1, \ldots, m$. Hence, as above, the intersection numbers $e_j(u)$ of u are well defined for $j = 1, \ldots, m$. Repeating the arguments from the proof of Theorem 4.5 we get the desired conclusion. ∎

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