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Jump conditions for a metaharmonic double layer potential on rectifiable closed Jordan curves in $\mathbb{R}^2$

by Ricardo Abreu Blaya (Holguín) and
Juan Bory Reyes (México, DF)

Abstract. This paper is concerned with jump conditions for the double layer potential associated with the two-dimensional Helmholtz equation for Hölder continuous boundary data on arbitrary rectifiable Jordan closed curves in $\mathbb{R}^2$.

1. Introduction. Layer potentials play a central role in the study of numerous boundary value problems for the Helmholtz equation arising in mathematical physics. Important examples of such problems have been treated in connection with such areas as the Maxwell equations and the scattering of electromagnetic waves.

For a deeper description of the use of layer potentials for boundary value problems for the Helmholtz equation on smooth domains we refer the reader to [CK1, CK2].

It has long been understood that there are intimate connections between the holomorphic Cauchy type integral, supported on a smooth enough curve, and the logarithmic layer potentials (see for instance [K, Mu]), and the advantage of using holomorphic functions theory to derive jump conditions for layer potentials is well established (see e.g. [Ga, Epigraph 10, p. 73]).

In the last years, the theory of hyperholomorphic quaternion-valued functions of two real variables [GBS, GS1, GS2, KS, ST1, ST2] has been developed along many interesting directions, including some applications in potential theory [GSO, GKS] and in physical problems with elliptic geometries [LPRS, LRS] as well.

In [GSO] the authors proposed a modification of the metaharmonic double layer potential in the case of rectifiable Jordan closed curves in $\mathbb{R}^2$, and examined its boundary values by taking advantage of the connection be-

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tween the hyperholomorphic Cauchy type integral and the double layer potential. Their aim was to present an appropriate modification of the double layer potential that efficiently works on rectifiable curves. Their results are based on their previous work [GS2] regarding a sufficient condition for the existence of the limit boundary value of the hyperholomorphic Cauchy type integral that implicitly restricts the class of admissible rectifiable curves.

A jump condition for a hyperholomorphic Cauchy type integral for Hölder continuous boundary data on arbitrary rectifiable Jordan closed curves can then be invoked in order to propose a general description of jump conditions for the double layer potential associated with the two-dimensional Helmholtz equation on arbitrary rectifiable Jordan closed curves; this constitutes the main purpose of this work.

2. Preliminaries. We let $\mathbb{H}(\mathbb{C})$ denote the set of complex quaternions. Each quaternion $a \in \mathbb{H}(\mathbb{C})$ can be written as $a = \sum_{k=0}^{3} a_k i_k$ where $\{a_k\} \subset \mathbb{C}$, i_0 is the multiplicative unit and $\{i_k \mid k = 1,2,3\}$ are the quaternionic imaginary units, which satisfy the rules $i_k^2 = -1$, $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$, $i_3 i_1 = -i_1 i_3 = i_2$.

The imaginary unit in $\mathbb{C}$ is denoted as usual by i, and by definition,

$$i \cdot i_k = i_k \cdot i, \qquad k = 0,1,2,3.$$

The set $\mathbb{H}(\mathbb{C})$ is a complex non-commutative, associative algebra with zero divisors.

The quaternionic conjugation, denoted by $\overline{a}$, acts only on the quaternionic units, not on i. The module of a quaternion a coincides with its Euclidean norm: $|a| = \|a\|_{\mathbb{R}^8}$. For $a, b \in \mathbb{H}(\mathbb{C})$ the inequality $|ab| \le \sqrt{2} \cdot |a| \cdot |b|$ holds.

The scalar and vector parts of $a \in \mathbb{H}(\mathbb{C})$, $\mathrm{Sc}(a)$ and $\vec{a}$, are defined to be a_0 and $\sum_{k=0}^{3} a_k i_k$, respectively. This enables us to write $a = \mathrm{Sc}(a) + \vec{a}$ and $\overline{a} = \mathrm{Sc}(a) - \vec{a}$.

For any complex quaternions a and b,

$$ab = a_0 b_0 - \langle \vec{a}, \vec{b} \rangle + a_0 \vec{b} + b_0 \vec{a} + [\vec{a}, \vec{b}], \tag{2.1}$$

where

$$\langle \vec{a}, \vec{b} \rangle = \sum_{k=1}^{3} a_k b_k, \qquad [\vec{a}, \vec{b}] = \begin{vmatrix} i_1 & i_2 & i_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If $a_0 = b_0 = 0$ then (2.1) takes the most impressive form:

$$ab = -\langle \vec{a}, \vec{b} \rangle + [\vec{a}, \vec{b}]. \tag{2.2}$$

Throughout the paper, $\Omega \subset \mathbb{R}^2$ denotes a bounded Jordan domain with boundary a rectifiable curve Γ; let us also introduce the temporary notation $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \setminus \{\Omega_+ \cup \Gamma\}$.

As follows directly from Rademacher's theorem (see [EG, p. 81]), for rectifiable curves there exists the conventional unit normal vector $\vec{n}_{st}$ almost everywhere on Γ.

Typical points of the Euclidean space $\mathbb{R}^2$ will be denoted by $z := xi_1+yi_2$, $\zeta := \xi i_1 + \eta i_2$, etc.

We shall consider $\mathbb{H}(\mathbb{C})$-valued functions:

$$f : E \subset \mathbb{R}^2 \to \mathbb{H}(\mathbb{C}),$$

where E could be $\Omega_\pm$, Γ or $\mathbb{R}^2$.

Identifying $\mathbb{H}(\mathbb{C})$ with $\mathbb{C}^4$ in the usual way, we denote by $C^s(E, \mathbb{H}(\mathbb{C}))$, $s \in \mathbb{N} \cup \{0\}$, $C^{0,\nu}(E, \mathbb{H}(\mathbb{C}))$, $0 < \nu \leq 1$, $L_p(E, \mathbb{H}(\mathbb{C}))$, $p > 1$, respectively the complex linear spaces of s times continuously differentiable, Hölder continuous and p-integrable functions. All have the usual componentwise meaning.

For a function f defined on $\Omega_\pm$, its limit boundary values will be denoted by

$$[f]^\pm(\varsigma) := \lim_{\Omega_\pm \ni z \to \varsigma \in \Gamma} [f](z).$$

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and let α denote an arbitrary fixed solution in $\mathbb{H}(\mathbb{C})$ of the equation $\alpha^2 = \lambda$. This λ generates the two-dimensional Helmholtz operator with a quaternionic wave, which acts on $C^2(\Omega, \mathbb{H}(\mathbb{C}))$:

$${}_\lambda\Delta := \Delta_{\mathbb{R}^2} + {}^\lambda M, \qquad \Delta_\lambda := \Delta_{\mathbb{R}^2} + M^\lambda,$$

where $\Delta_{\mathbb{R}^2} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian and $M^\lambda[f] := f\lambda$, ${}^\lambda M[f] := \lambda f$, for any $\lambda \in \mathbb{H}(\mathbb{C})$.

We shall be considering the following partial differential operators with quaternionic coefficients:

$$
\begin{aligned}
{}_{st}\partial &:= i_1 \cdot \frac{\partial}{\partial x} + i_2 \cdot \frac{\partial}{\partial y}, & {}_{st}\overline{\partial} &:= \overline{i_1} \cdot \frac{\partial}{\partial x} + \overline{i_2} \cdot \frac{\partial}{\partial y},\\
\partial_{st} &:= \frac{\partial}{\partial x} \circ M^{i_1} + \frac{\partial}{\partial y} \circ M^{i_2}, & \overline{\partial}_{st} &:= \frac{\partial}{\partial x} \circ M^{\bar{i}_1} + \frac{\partial}{\partial y} \circ M^{\bar{i}_2}.
\end{aligned}
$$

Thus,

$${}_{st}\partial^2 = \partial_{st}^2 = -\Delta_{\mathbb{R}^2}.$$

Set

$${}_\alpha\partial := \partial_{st} + {}^\alpha M, \qquad \partial_\alpha := {}_{st}\partial + M^\alpha.$$

Then we have the following factorizations of the Helmholtz operator:

(2.3) $$\Delta_\lambda = -\partial_\alpha \circ \partial_{-\alpha} = -\partial_{-\alpha} \circ \partial_\alpha.$$

A function $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ is said to be *hyperholomorphic* if $\partial_\alpha f \equiv 0$ in Ω.

For further use, we recall that, for any $f \in C^{0,\nu}(\Gamma, \mathbb{R})$, there exists $\tilde{f}$ in $C^\infty(\mathbb{R}^2 \setminus \Gamma, \mathbb{R}) \cap C^{0,\nu}(\mathbb{R}^2, \mathbb{R})$, in general not unique, for which $\tilde{f}|_\Gamma = f$ and

$$\left|\frac{\partial \tilde{f}}{\partial x}(z)\right| \leq c\text{dist}(z, \Gamma)^{\nu-1}, \qquad \left|\frac{\partial \tilde{f}}{\partial y}(z)\right| \leq c\text{dist}(z, \Gamma)^{\nu-1}, \qquad \text{for } z \in \mathbb{R}^2 \setminus \Gamma.$$

Here and below, c stands for a generic positive constant, not necessarily the same in different occurrences. In fact, this extension result is based upon the so-called Whitney decomposition of the open set $\mathbb{R}^2 \setminus \Gamma$ (see e.g. [St]).

For our purposes, it suffices to consider a direct quaternionic reformulation of this Whitney extension result as stated in the following theorem.

THEOREM 2.1 (Whitney extension). *Let $f \in C^{0,\nu}(\Gamma, \mathbb{H}(\mathbb{C}))$. Then there exists a function $\tilde{f}$ satisfying*

(i) $\tilde{f}|_\Gamma = f$;
(ii) $\tilde{f} \in C^\infty(\mathbb{R}^2 \setminus \Gamma, \mathbb{H}(\mathbb{C}))$;
(iii) $|{}_{st}\partial \tilde{f}(z)| \leq c\,\text{dist}(z, \Gamma)^{\nu-1}$ *for* $z \in \mathbb{R}^2 \setminus \Gamma$.

REMARK 2.2. We remark that property (iii) permits estimating the integrability exponent of ${}_{st}\partial \tilde{f}$. Indeed, ${}_{st}\partial \tilde{f} \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$ for $p < 1/(1-\nu)$, as is easy to check. Note that if $\nu = 1$, then ${}_{st}\partial \tilde{f}$ is bounded in Ω.

Throughout the paper, we fix $\alpha = \alpha_0 \in \mathbb{C}$ with a strictly positive imaginary part. It is well known (see e.g. [V]) that a fundamental solution θ_{α_0} of Δ_λ is given by

$$\theta_{\alpha_0}(z) := -\frac{i}{4} H_0^{(1)}(\alpha_0|z|),$$

where $H_s^{(1)}$ is the Hankel function of first kind of order s.

The hyperholomorphic Cauchy kernel $\mathcal{K}_{st,\alpha_0}$, i.e. the fundamental solution of the operator ∂_{α_0}, can be calculated from

$$\mathcal{K}_{st,\alpha_0}(z) = -\partial_{-\alpha_0}\, \theta_{\alpha_0}(z), \qquad z \in \mathbb{R}^2 \setminus \{0\}. \tag{2.4}$$

Hence, explicitly,

$$\mathcal{K}_{st,\alpha_0}(z) = -\frac{i\alpha_0}{4}\left(H_1^{(1)}(\alpha_0|z|)\frac{z}{|z|} + H_0^{(1)}(\alpha_0|z|)\right). \tag{2.5}$$

The following vector property of the Hankel functions will be used:

$$\nabla H_0^{(1)}(\alpha_0|z|) = -\alpha_0 H_1^{(1)}(|z|)\frac{z}{|z|}, \qquad z \in \mathbb{R}^2 \setminus \{0\}. \tag{2.6}$$

Here, we suppress the explicit dependence on z in ∇, but in what follows, when ambiguity can arise, we shall indicate by subscript the variable with respect to which differentiation is considered.

3. Hyperholomorphic Cauchy type integral. One of the most crucial facts of the theory of hyperholomorphic quaternion-valued functions of two real variables is the existence of a Stokes formula, which can be found in [ST2]. Here we present this formula by looking more closely at the assumptions required.

THEOREM 3.1. *Let* $f, g \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C^0(\Omega \cup \Gamma, \mathbb{H}(\mathbb{C}))$. *Then*

$$\int_\Gamma g(\zeta)\sigma_{st}(\zeta)f(\zeta) = \int_\Omega \left[{}_{\alpha_0}\partial[g]\cdot f + g\cdot\partial_{-\alpha_0}[f] - (\alpha_0 g\cdot f - g\cdot f\alpha_0)\right] d\xi\wedge d\eta, \tag{3.1}$$

provided the double integral exists.

Clearly the double integral exists if $f, g \in C^1(\Omega \cup \Gamma, \mathbb{H}(\mathbb{C}))$. Also, if $f, g \in C^{0,\nu}(\Gamma, \mathbb{H}(\mathbb{C}))$ then

$$\tilde{f}, \tilde{g} \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C^{0,\nu}(\Omega \cup \Gamma, \mathbb{H}(\mathbb{C})),$$

and according to Remark 2.2 the double integral

$$\int_\Omega \left[{}_{\alpha_0}\partial[\tilde{g}]\cdot \tilde{f} + \tilde{g}\cdot\partial_{-\alpha_0}[\tilde{f}] - (\alpha_0 \tilde{g}\cdot \tilde{f} - \tilde{g}\cdot \tilde{f}\alpha_0)\right] d\xi\wedge d\eta$$

exists and the following Stokes formula holds:

$$\int_\Gamma g(\zeta)\sigma_{st}(\zeta)f(\zeta) = \int_\Omega \left[{}_{\alpha_0}\partial[\tilde{g}]\cdot \tilde{f} + \tilde{g}\cdot\partial_{-\alpha_0}[\tilde{f}] - (\alpha_0 \tilde{g}\cdot \tilde{f} - \tilde{g}\cdot \tilde{f}\alpha_0)\right] d\xi\wedge d\eta. \tag{3.2}$$

The hyperholomorphic Cauchy kernel $\mathcal{K}_{st,\alpha_0}$ generates, as usual, two important integrals related to the two-dimensional Helmholtz operator in Jordan domains with rectifiable boundary: the hyperholomorphic Cauchy type integral

$$K_{\alpha_0}[f](z) := -\int_\Gamma \mathcal{K}_{st,\alpha_0}(z-\zeta)\sigma_{st}(\zeta)f(\zeta), \qquad z \in \mathbb{R}^2 \setminus \Gamma, \tag{3.3}$$

where $\sigma_{st} := d\eta\, i_1 - d\xi\, i_2$, and the Teodorescu transform

$$T_{\alpha_0}[f](z) := \int_\Omega \mathcal{K}_{st,\alpha_0}(z-\zeta)f(\zeta)\, d\xi\wedge d\eta, \qquad z \in \mathbb{R}^2. \tag{3.4}$$

REMARK 3.2. It is well known that $\sigma_{st}(\zeta) = n_{st}(\zeta)\, d\Gamma_\zeta$ with $d\Gamma_\zeta$ the arc-length measure and $n_{st}(\zeta) = \vec{n}_{st}(\zeta) := n_1(\zeta)i_1 + n_2(\zeta)i_2$ the unit outward normal vector to Γ at the point ζ.

The operators (3.3) and (3.4) are connected by the Borel–Pompeiu type formula, which may be stated as follows:

$$K_{\alpha_0}[f](z) + T_{\alpha_0}[\partial_{\alpha_0} f](z) = \begin{cases} f(z) = \partial_{\alpha_0}\cdot T_{\alpha_0}[f](z) & \text{if } z \in \Omega_+, \\ 0 & \text{if } z \in \Omega_-. \end{cases} \tag{3.5}$$

This formula goes back to [ST2, Theorems 4.1 and 4.4], and its validity relies critically on the Stokes Theorem 3.1. However, in [ST2, Theorem 4.1] the necessity to guarantee the existence of the double integral in (3.1) is overlooked. For this reason, to justify the application of (3.1), we will assume that (3.5) is obtained under the assumption that the double integral in (3.1) exists.

THEOREM 3.3. *Let $f \in C^{0,\nu}(\Gamma, \mathbb{H}(\mathbb{C}))$. Then*

$$K_{\alpha_0}[f](z) + T_{\alpha_0}[\partial_{\alpha_0}\tilde{f}](z) = \begin{cases} \tilde{f}(z) = \partial_{\alpha_0} \cdot T_{\alpha_0}[\tilde{f}](z) & \text{if } z \in \Omega_+, \\ 0 & \text{if } z \in \Omega_-. \end{cases} \tag{3.6}$$

This Borel–Pompeiu type formula provides a tool for proving jump conditions for the hyperholomorphic Cauchy type integral (see Theorem 3.5 below), which yields a direct consequence for the metaharmonic double layer potentials to be considered in the next section.

REMARK 3.4. (i) By (3.5) it is clear that

$$K_{\alpha_0}[T_{\alpha_0}[f]](z) = 0, \quad z \in \mathbb{R}^2 \setminus \Gamma.$$

(ii) Using standard techniques, and basically following the ideas of [GS, Subsection 3.1], [GHS, Subsection 8.1] and [GSS], one can prove that for $p > 2$,

$$T_{\alpha_0}[f] \in C^{0,(p-2)/p}(\mathbb{R}^2, \mathbb{H}(\mathbb{C})) \quad \text{when } f \in L_p(\Omega, \mathbb{H}(\mathbb{C})).$$

THEOREM 3.5. *Let $f \in C^{0,\nu}(\Gamma, \mathbb{H}(\mathbb{C}))$ and let*

$$\nu > 1/2. \tag{3.7}$$

Then the limit boundary values of $K_{\alpha_0}[f](z)$ are given by

$$[K_{\alpha_0}[f]]^+(\varsigma) = f(\varsigma) + \int_\Omega \mathcal{K}_{st,\alpha_0}(\zeta - \varsigma) \cdot \partial_{\alpha_0}\tilde{f}(\zeta)\, d\xi \wedge d\eta,$$

$$[K_{\alpha_0}[f]]^-(\varsigma) = \int_\Omega \mathcal{K}_{st,\alpha_0}(\zeta - \varsigma) \cdot \partial_{\alpha_0}\tilde{f}(\zeta)\, d\xi \wedge d\eta.$$

Furthermore, $[K_{\alpha_0}[f]]^\pm \in C^{0,\mu}(\Gamma, \mathbb{H}(\mathbb{C}))$ whenever $\mu < 2\nu - 1$.

Proof. Our proof starts with the observation that by the Borel–Pompeiu formula (3.6) we have

$$K_{\alpha_0}[f](z) = \begin{cases} \tilde{f}(z) - T_{\alpha_0} \cdot \partial_{\alpha_0}[\tilde{f}](z) & \text{if } z \in \Omega_+, \\ -T_{\alpha_0} \cdot \partial_{\alpha_0}[\tilde{f}](z) & \text{if } z \in \Omega_-. \end{cases} \tag{3.8}$$

Hence, it suffices to prove that $T_{\alpha_0} \cdot \partial_{\alpha_0}[\tilde{f}]$ is continuous through Γ.

In fact, we might choose p such that $2 < p < 1/(1-\nu)$ since condition (3.7) implies that $2 < 1/(1-\nu)$; then ${}_{st}\partial\tilde{f} \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$.

On account of Remark 3.4(ii), together with the fact that $(p-2)/p < 2\nu - 1$, we obtain the desired result.

REMARK 3.6. The proof strongly depended on condition (3.7). If $\nu = 1$, which we may assume, the theorem gains in interest if we realize that in this case the restriction on the index μ can be relaxed.

If we allow $f \equiv 1$ then equation (3.8) gives a direct proof for Lemma 3.9 of [GS2].

4. Metaharmonic potentials. For $f \in C^{0,\nu}(\Gamma, \mathbb{C})$ and $z \in \mathbb{R}^2 \setminus \Gamma$, define the *metaharmonic potentials*:

- The double layer potential
$$V_{\alpha_0}[f](z) = \int_{\Gamma} \langle \vec{n}_{st}, \nabla_{\zeta} \rangle \theta_{\alpha_0}(z-\zeta) f(\zeta)\, d\Gamma_{\zeta}.$$
- The single layer potential
$$U_{\alpha_0}[f](z) = \int_{\Gamma} \theta_{\alpha_0}(z-\zeta) f(\zeta)\, d\Gamma_{\zeta}.$$

One of the main motivations for the study of the single and double metaharmonic layer potential operators is that they solve the two-dimensional Helmholtz equation for $z \in \mathbb{R}^2 \setminus \Gamma$. Moreover, they play a basic role in many real-world problems.

From the definitions, it is routine to verify the identities
$$-\langle \nabla_z, U_{\alpha_0}[\vec{n}_{st} f] \rangle = V_{\alpha_0}[f], \qquad K_{\alpha_0}[\vec{n}_{st} f] = \nabla_z U_{\alpha_0}[f] - \alpha_0 U_{\alpha_0}[f].$$
It is also of interest to look at the following single layer type potential:
$$U^*_{\alpha_0}[f](z) = \int_{\Gamma} \theta_{\alpha_0}(z-\zeta) \vec{n}_{st}(\zeta) f(\zeta)\, d\Gamma_{\zeta}.$$
Notice that $U^*_{\alpha_0}$ differs from U_{α_0} by the operator of quaternionic multiplication by the unit normal.

Finally, we define a Newton type potential for $f \in L_2(\Omega, \mathbb{C})$,
$$W_{\alpha_0}[f](z) = \int_{\Omega} \theta_{\alpha_0}(z-\zeta) f(\zeta)\, d\xi \wedge d\eta,$$
which represents a solution of the equation

$$\Delta_{\lambda}[W_{\alpha_0}[f]] = \begin{cases} f(z) & \text{if } z \in \Omega_+, \\ 0 & \text{if } z \in \Omega_-. \end{cases} \tag{4.1}$$

Let us present a connection between the above potentials:

Theorem 4.1. *Let $f \in C^{0,\nu}(\Gamma, \mathbb{C})$. Then*
$$U^*_{\alpha_0}[f](z) = T_{\alpha_0}[\tilde{f}](z) + W_{\alpha_0}[\partial_{-\alpha_0} \tilde{f}](z), \qquad z \in \mathbb{R}^2 \setminus \Gamma.$$

Proof. Applying the Stokes formula (3.2), we get
$$\int_{\Gamma} \theta_{\alpha_0}(z-\zeta) \vec{n}_{st}(\zeta) f(\zeta) d\Gamma_{\zeta} = \int_{\Omega} \mathcal{K}_{st,\alpha_0}(z-\zeta) \tilde{f}(\zeta)\, d\xi \wedge d\eta + \int_{\Omega} \theta_{\alpha_0}(z-\zeta) \partial_{-\alpha_0} \tilde{f}(\zeta)\, d\xi \wedge d\eta,$$
which establishes the formula.

Corollary 4.2. *Under the assumption of Theorem* 4.1 *we have*

$$K_{\alpha_0}[f] = {}_{-\alpha_0}\partial U^*_{\alpha_0}[f], \qquad T_{\alpha_0}[\tilde{f}] = -{}_{-\alpha_0}\partial W_{\alpha_0}[\tilde{f}],$$
$$K_{\alpha_0}[U^*_{\alpha_0}[\tilde{f}]] = 0 \quad \text{for } \tilde{f} \in \operatorname{Ker} \partial_{-\alpha_0}.$$

Before stating our main result, observe that we can write (2.4) in the form

$$\mathcal{K}_{st,\alpha_0}(z-\zeta) = -\nabla_\zeta \theta_{\alpha_0}(z-\zeta) + \alpha_0\theta_{\alpha_0}(z-\zeta),$$

and direct calculation yields

$$\mathcal{K}_{st,\alpha_0}(z-\zeta)n_{st}(\zeta) = -\langle \vec{n}_{st}, \nabla_\zeta\rangle\theta_{\alpha_0}(z-\zeta) + [\nabla_\zeta, \vec{n}_{st}]\theta_{\alpha_0}(z-\zeta) - \alpha_0\theta_{\alpha_0}(z-\zeta)\vec{n}_{st}(\zeta).$$

Thus

$$\operatorname{Sc}[K_{\alpha_0}[f]](z) = V_{\alpha_0}[f](z), \tag{4.2}$$

which provides a way for getting jump conditions for $V_{\alpha_0}[f]$.

Remark 4.3. Note that for any $f \in C^{0,\nu}(\Gamma, \mathbb{C})$, the Cauchy type integral $K_{\alpha_0}[f]$ decomposes as follows:

$$\begin{aligned} K_{\alpha_0}[f](z) = {} & -\frac{i\alpha_0}{4}\int_\Gamma \frac{H_1^{(1)}(\alpha_0|\zeta - z|)}{|\zeta - z|}((\eta - y)\,d\xi - (\xi - x)\,d\eta) \\ & - \left(\frac{i\alpha_0}{4}\int_\Gamma H_0^{(1)}(\alpha_0|\zeta - z|)\,d\eta\right)i_1 + \left(\frac{i\alpha_0}{4}\int_\Gamma H_0^{(1)}(\alpha_0|\zeta - z|)\,d\xi\right)i_2 \\ & + \left(\frac{i\alpha_0}{4}\int_\Gamma \frac{H_1^{(1)}(\alpha_0|\zeta - z|)}{|\zeta - z|}((\xi - x)\,d\xi + (\eta - y)\,d\eta)\right)i_3. \end{aligned}$$

Thus

$$\operatorname{Sc}(K_{\alpha_0}[f](z)) = -\frac{i\alpha_0}{4}\int_\Gamma \frac{H_1^{(1)}(\alpha_0|\zeta - z|)}{|\zeta - z|}((\eta - y)\,d\xi - (\xi - x)\,d\eta),$$

which is exactly the modification of the double layer potential introduced in [GSO].

As an evidence of the intimate relation between the hyperholomorphic Cauchy type integral and the double layer potential for the two-dimensional Helmholtz equation, we will establish now the following jump conditions.

Theorem 4.4. *Let $f \in C^{0,\nu}(\Gamma, \mathbb{C})$ and ν satisfy* (3.7). *Then the limit boundary values of $V_{\alpha_0}[f](z)$ are given by*

$$[V_{\alpha_0}[f]]^+(\varsigma) = f(\varsigma) + \alpha_0 \cdot W_{\alpha_0}[\partial_{\alpha_0}\tilde{f}](\varsigma),$$
$$[V_{\alpha_0}[f]]^-(\varsigma) = \alpha_0 \cdot W_{\alpha_0}[\partial_{\alpha_0}\tilde{f}](\varsigma).$$

Furthermore, $[V_{\alpha_0}[f]]^\pm \in C^{0,\mu}(\Gamma, \mathbb{C})$ whenever $\mu < 2\nu - 1$.

Proof. The structure of the limit boundary values of $K_{\alpha_0}[f]$ in Theorem 3.5, and (4.2), show that

$$[V_{\alpha_0}[f]]^+(\varsigma) = f(\varsigma) + \mathrm{Sc}\Big(\int_{\Omega} \mathcal{K}_{st,\alpha_0}(\zeta - \varsigma) \cdot \partial_{\alpha_0}\tilde{f}(\zeta)\, d\xi \wedge d\eta\Big),$$

$$[V_{\alpha_0}[f]]^-(\varsigma) = \mathrm{Sc}\Big(\int_{\Omega} \mathcal{K}_{st,\alpha_0}(\zeta - \varsigma) \cdot \partial_{\alpha_0}\tilde{f}(\zeta)\, d\xi \wedge d\eta\Big).$$

The proof is completed by observing that $\mathrm{Sc}(\mathcal{K}_{st,\alpha_0}) = \alpha_0 \cdot \theta_{\alpha_0}$.

REMARK 4.5. For $\nu = 1$, Theorem 4.4 holds, and the limit values $[V_{\alpha_0}[f]]^\pm$ belong now to every Hölder space $C^{0,\mu}(\Gamma, \mathbb{C})$, $0 < \mu < 1$.

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Ricardo Abreu Blaya
Facultad de Informática y Matemática
Universidad de Holguín
Holguín 80100, Cuba
E-mail: rabreu@facinf.uho.edu.cu

Juan Bory Reyes
SEPI-ESIME-Zacatenco
Instituto Politécnico Nacional
México, DF 07738, Mexico
E-mail: juanboryreyes@yahoo.com

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