ANNALES
POLONICI MATHEMATICI
115.3 (2015)

Attractor of a semi-discrete Benjamin–Bona–Mahony equation on $\mathbb{R}^1$

by Chaosheng Zhu (Chongqing)

Abstract. This paper is concerned with the study of the large time behavior and especially the regularity of the global attractor for the semi-discrete in time Crank–Nicolson scheme to discretize the Benjamin–Bona–Mahony equation on $\mathbb{R}^1$. Firstly, we prove that this semi-discrete equation provides a discrete infinite-dimensional dynamical system in $H^1(\mathbb{R}^1)$. Then we prove that this system possesses a global attractor $\mathcal{A}_\tau$ in $H^1(\mathbb{R}^1)$. In addition, we show that the global attractor $\mathcal{A}_\tau$ is regular, i.e., $\mathcal{A}_\tau$ is actually included, bounded and compact in $H^2(\mathbb{R}^1)$. Finally, we estimate the finite fractal dimensions of $\mathcal{A}_\tau$.

1. Introduction. This paper is concerned with the study of the large time behavior and especially the regularity of the global attractor for the following semi-discrete Benjamin–Bona–Mahony equation on $\mathbb{R}^1$:

$$u_t - u_{xxt} + \alpha u - \alpha u_{xx} + uu_x = f(x), \qquad t \in (0,\infty),\ x \in \mathbb{R}^1, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^1, \tag{1.2}$$

with the damping parameter $\alpha > 0$.

The Benjamin–Bona–Mahony equation was proposed in [10] as the model of propagation of long waves which incorporates nonlinear dispersive and dissipative effects. The existence and uniqueness, as well as the decay rates of solutions for this equation, were studied by many authors (see [5, 6, 11, 32] and the references therein). In addition, the long-time behavior of solutions for this equation was also considered (see [13, 24, 26, 28, 30, 31]).

When this equation is defined in a bounded domain, there exists a global attractor which has finite fractal dimensions (see [13, 26, 28]). In [13], the authors considered the periodic initial-boundary value problem for a multi-dimensional generalized Benjamin–Bona–Mahony equation, and proved the existence of a global attractor with finite fractal dimensions and the existence of the exponential attractor for the corresponding semigroup. The reg-

2010 *Mathematics Subject Classification*: Primary 35B41; Secondary 35B45, 35Q55.
Key words and phrases: Benjamin–Bona–Mahony equation, Crank–Nicolson scheme, global attractor, regularity, fractal dimension.

DOI: 10.4064/ap115-3-2

ularity of the global attractor was established in [27] when the forcing term f is in H^k with $k \geq 0$, and the Gevrey regularity was proved in [15] when f belongs to a Gevrey class. The authors of [15] also proved the existence of two determining nodes for the one-dimensional equation with periodic boundary conditions. In [30], we established the existence of the asymptotic attractor by the method of orthogonal decomposition, and obtained the dimensions estimate of this asymptotic attractor. When the equation is defined in an unbounded domain, there exists a global attractor which has finite fractal dimensions (see [24, 31]). In [24], the authors gave a sufficient condition for the asymptotic compactness of an evolution equation by using the Littlewood–Paley projection operators, and established the existence of an attractor for the Benjamin–Bona–Mahony equation in the phase space $H^1(\mathbb{R}^3)$ by showing that the solutions are point dissipative and asymptotically compact, and then proved that the attractor is regular and bounded in $H^2(\mathbb{R}^3)$.

There are fruitful results on the Crank–Nicolson scheme for infinite-dimensional dynamical systems. In the case where the space variable x belongs to a finite interval, the authors of [20] proved the existence of a global attractor under periodic boundary conditions. This result was also obtained when the space variable x belongs to $\mathbb{T}^2$ (see [17]). In addition, the discrete Crank–Nicolson scheme for both space and time variables for the classical nonlinear Schrödinger equation was studied in [29] and [3].

In this paper, we study the large time behavior and especially the regularity of the global attractor for the semi-discrete in time Crank–Nicolson scheme for the discretized Benjamin–Bona–Mahony equation on $\mathbb{R}^1$. First, we prove that this semi-discrete equation provides a discrete infinite-dimensional dynamical system in $H^1(\mathbb{R}^1)$. Then we prove that this system possesses a global attractor $\mathcal{A}_\tau$ in $H^1(\mathbb{R}^1)$. In addition, we show that the global attractor $\mathcal{A}_\tau$ is regular, i.e., $\mathcal{A}_\tau$ is actually included, bounded and compact in $H^2(\mathbb{R}^1)$. Finally, we estimate the finite fractal dimensions of $\mathcal{A}_\tau$.

Throughout this paper, we write L^r and H^s for $L^r(\mathbb{R}^1)$ and $H^s(\mathbb{R}^1)$ respectively, where $1 \leq r \leq \infty$ and $s \in \mathbb{R}^1$. The spaces L^2 and H^1 are Hilbert spaces with the inner products $(u,v) = \int_{\mathbb{R}^1} uv\,dx$ and $(u,v)_{H^1} = \int_{\mathbb{R}^1}(uv + u_x v_x)\,dx$. For general $s \in \mathbb{R}^1$, the Sobolev space H^s is endowed with the norm $\|u\|_{H^s} = \|(I-\Delta)^{s/2}u\|_{L^2}$, where we denoted by $\|\cdot\|_X$ the norm of a Banach space X.

Now we introduce a Crank–Nicolson scheme associated with (1.1): to begin, we recall that for a given $u_0 \in H^1(\mathbb{R}^1)$ there exists a unique solution $u \in C([0,\infty), H^1(\mathbb{R}^1)) \cap C^1([0,\infty), H^{-1})$ for (1.1) which is equivalent to

$$\frac{\partial}{\partial t}(e^{\alpha t}u(t)) - \Delta\frac{\partial}{\partial t}(e^{\alpha t}u(t)) + \frac{1}{2}(e^{\alpha t/2}u(t))_x^2 = e^{\alpha t}f, \tag{1.3}$$

where $\Delta u = \partial_{xx}u$. For a given time discretization $\tau > 0$, we consider the uniform time sequence $\{t^n\}_n$ defined by $t^n = n\tau$, and we integrate (1.3) over the time interval $[t^n, t^{n+1}]$ to get

$$(1.4)\qquad (e^{\alpha t^{n+1}}u(t^{n+1}) - e^{\alpha t^n}u(t^n)) - \Delta(e^{\alpha t^{n+1}}u(t^{n+1}) - e^{\alpha t^n}u(t^n)) + \frac{1}{2}\int_{t^n}^{t^{n+1}} (e^{\alpha t/2}u(t))_x^2\,dt = \int_{t^n}^{t^{n+1}} e^{\alpha t} f\,dt.$$

We recall the trapezoidal rule

$$(1.5)\qquad \int_{t^n}^{t^{n+1}} g(s)\,ds \sim \frac{\tau}{2}[g(t^{n+1}) + g(t^n)];$$

the error of the approximation (1.5) is

$$(1.6)\qquad \text{error} = \int_{t^n}^{t^{n+1}} g(s)\,ds - \frac{\tau}{2}[g(t^{n+1}) + g(t^n)] = -\frac{\tau^3}{12}g''(\xi);$$

Let $\{u^n\}_n$ be a real sequence such that $u^n \sim u(t^n)$, where u is the solution of the continuous form (1.1). Using (1.4) and the same trick as employed in [16], we derive the relevant Crank–Nicolson scheme as follows. For given $f \in L^2(\mathbb{R}^1)$, $u^0 \in H^1(\mathbb{R}^1)$ and $\beta = e^{-\alpha\tau}$ we solve recursively

$$(1.7)\qquad \frac{1}{\tau}(u^{n+1} - \beta u^n) - \frac{1}{\tau}\Delta(u^{n+1} - \beta u^n) + \frac{1}{8}\nabla(u^{n+1} + \beta u^n)^2 = \frac{1+\beta}{2}f.$$

It is well known that the Crank–Nicolson scheme is of second order (see [4]). In this paper, we assume that τ is small enough, more precisely, $\alpha\tau \le 2(1-\beta)$ and $\beta(1 - \alpha\tau/4)^{-1} \le 1$. In fact, the above two inequalities hold when $\alpha\tau < \ln 2$.

In Section 2, we prove that the Crank–Nicolson scheme is well posed. That is, we obtain the following theorem.

THEOREM 1.1. *Assume that $f \in L^2$. For any $u^n \in H^1$, there exists a solution u^{n+1} of (1.7). Let $S\colon H^1 \to H^1$, $u^n \mapsto u^{n+1}$, be the multivalued function defined by (1.7). There exists a bounded set $\mathcal{B} \subset H^1$ such that $S(\mathcal{B}) \subset \mathcal{B}$. Moreover, for $\tau > 0$ small enough, the map S is continuous and one-to-one on $\mathcal{B}$.*

REMARK. In order to ensure that $S\colon \mathcal{B} \to \mathcal{B}$ is one-to-one, the smallness assumption on τ in (2.10) below is

$$(1.8)\qquad C\tau^{3/4}K_1^2 < 1,$$

where $K_1 > 0$ depends on the size of $\mathcal{B}$ defined by Lemma 2.4 below. Thus we can define a discrete dynamical system $\{S^n\}_{n\in\mathbb{N}}$ on the set $\mathcal{B}$.

In Section 3, we prove the existence and regularity of a global attractor, that is, we prove the following theorem.

Theorem 1.2. *Assume that τ satisfies also* (1.8). *The discrete dynamical system $\{S^n\}_{n\in\mathbb{N}}$ defined on the set $\mathcal{B}$ by $Su^n = u^{n+1}$ possesses a global attractor $\mathcal{A}_\tau$ in H^1 that is a compact subset of $H^2(\mathbb{R}^1)$.*

It is well known that the existence of a global attractor for a dissipative evolution equation always relies on some kind of compactness of the semigroup generated by this equation. Usually, the compactness is obtained through some regularity properties of this equation together with the compact imbedding of the relevant Sobolev spaces (see [25] for instance). However, this approach is suitable only for bounded domains. As for unbounded domains, this approach does not work because of lack of compactness. To recover the compactness, one can consider weighted spaces (see for example F. Abergel [1, 2], A. V. Babin [7], A. V. Babin and M. I. Vishik [8] and E. Feireisl et al. [19]), but there is another drawback: the forcing term and in some cases even the initial condition must be restricted to the weighted spaces. Similarly, Huang [22] proves that there also exists a compact global attractor for the Schrödinger equation in a special weighted space, provided the effect of the zero-order dissipation is large enough.

In this paper, we use the methods in [18, 23] instead of weighted spaces. That is, we will establish the asymptotic compactness of the discrete semigroup S defined on $\mathcal{B}$. Now we recall the general existence results of [25] for global attractors in both continuous and discrete dynamical systems. Suppose $S = S_1 + S_2$, where $S_1, S_2\colon \mathcal{B} \to \mathcal{B}$, S_1 is relatively compact, and for every bounded set $B \subset \mathcal{B}$, $\sup_{u\in B} |S_2^n u|_{\mathcal{B}} \to 0$ as $n \to \infty$. By Theorem I.1.1 in [25], the ω-limit set $\omega(\mathcal{B})$ is a global attractor for S. Then we prove that the global attractor $\mathcal{A}_\tau$ is a compact set in $H^{3/2-\varepsilon}$ by the method of [9].

Moreover, we prove that the global attractor has finite fractal dimensions by the method of [14].

Theorem 1.3. *If $f \in L^2(\mathbb{R}^1, (1 + x^2)dx)$ then the global attractor $\mathcal{A}_\tau$ has finite fractal dimension in $H^1 \cap L^2(\mathbb{R}^1, (1 + x^2)dx)$.*

This paper is organized as follows. In Section 2, we prove that the Crank–Nicolson scheme is well posed. In Section 3, we prove the existence of a compact attractor and show that the global attractor has finite fractal dimensions.

2. Existence of solution. Let $\{\varphi_n\}_{n\in\mathbb{N}^+}$ be a regular Hilbertian basis of H^1, i.e., the φ_n are smooth functions. For any $N \in \mathbb{N}^+$, let $V_N = \operatorname{span}\{\varphi_1, \ldots, \varphi_N\}$, and consider the orthogonal projector $P_N\colon H^1 \to V_N$. We consider the sequence $\{w^N\}_N$ defined by the Faedo–Galerkin method, that is, a finite-dimensional reduction of (1.7). For $N \geq 1$, we seek a solution

w^N in V_N of

$$\frac{1}{\tau}\int_{\mathbb{R}^1}(w^N-\beta P_N u^n)\varphi\,dx+\frac{1}{\tau}\int_{\mathbb{R}^1}\nabla(w^N-\beta P_N u^n)\nabla\varphi\,dx \\ +\frac{1}{8}\int_{\mathbb{R}^1}\nabla(w^N+\beta P_N u^n)^2\varphi\,dx-\frac{1+\beta}{2}\int_{\mathbb{R}^1}P_N f\varphi\,dx=0,\ \forall\varphi\in V_N. \tag{2.1}$$

Lemma 2.1 ([12]). *Let X be a finite-dimensional space endowed with a scalar product $[\cdot,\cdot]$ and consider a continuous mapping $F\colon X\to X$. Suppose that there exists $R_0>0$ such that $[F(w),w]>0$ for all w with $[w,w]^{1/2}=R_0$. Then there exists w^* with $[w^*,w^*]^{1/2}\le R_0$ such that $F(w^*)=0$.*

Lemma 2.2. *The sequence $\{w^N\}_N$ is well defined by* (2.1).

Proof. Problem (2.1) defines a continuous map $F\colon V_N\to V_N$ such that

$$[F(w),\varphi]=\frac{1}{\tau}\int_{\mathbb{R}^1}(w^N-\beta P_N u^n)\varphi\,dx+\frac{1}{\tau}\int_{\mathbb{R}^1}\nabla(w^N-\beta P_N u^n)\nabla\varphi\,dx \\ +\frac{1}{8}\int_{\mathbb{R}^1}\nabla(w^N+\beta P_N u^n)^2\varphi\,dx-\frac{1+\beta}{2}\int_{\mathbb{R}^1}P_N f\varphi\,dx,$$

where the scalar product $[\cdot,\cdot]$ is defined by $[v,w]=\int_{\mathbb{R}^1}vw\,dx$ for $v,w\in V_N$. Now we verify that F satisfies the hypothesis of Lemma 2.1. Taking $\varphi=w=w^N+\beta P_N u^n$ in (2.1), we get

$$\begin{aligned}
[F(w),w]&=\frac{1}{\tau}(\|w^N\|_{L^2}^2-\beta^2\|P_N u^n\|_{L^2}^2)+\frac{1}{\tau}(\|\nabla w^N\|_{L^2}^2-\beta^2\|\nabla P_N u^n\|_{L^2}^2)\\
&\quad-\frac{1+\beta}{2}\int_{\mathbb{R}^1}P_N f(w^N+\beta P_N u^n)\,dx\\
&\ge\frac{1}{\tau}(\|w^N\|_{L^2}^2-\beta^2\|P_N u^n\|_{L^2}^2)+\frac{1}{\tau}(\|\nabla w^N\|_{L^2}^2-\beta^2\|\nabla P_N u^n\|_{L^2}^2)\\
&\quad-\frac{1}{2\tau}\|w^N\|_{L^2}^2-\frac{\beta^2}{2\tau}\|P_N u^n\|_{L^2}^2-\frac{(1+\beta)^2\tau}{4}\|f\|_{L^2}^2\\
&\ge\frac{1}{2\tau}\bigg[\|w^N\|_{L^2}^2+\|\nabla w^N\|_{L^2}^2-3\beta^2\|P_N u^n\|_{L^2}^2\\
&\qquad\qquad-2\beta^2\|\nabla P_N u^n\|_{L^2}^2-\frac{(1+\beta)^2\tau^2}{2}\|f\|_{L^2}^2\bigg].
\end{aligned}$$

Set

$$R_0^2=4\beta^2\|P_N u^n\|_{L^2}^2+4\beta^2\|\nabla P_N u^n\|_{L^2}^2+\frac{(1+\beta)^2\tau^2}{2}\|f\|_{L^2}^2.$$

For $w\in V_N$, $\sqrt{[w,w]}=R_0$, we have $[F(w),w]>0$. By Lemma 2.1, there exists $w^*\in V_N$ such that $\sqrt{[w^*,w^*]}\le R_0$ and $F(w^*)=0$. Hence $w^N=w+\beta P_N u^n\in V_N$ is a solution of (2.1) and satisfies (2.2). ∎

In order to let $N \to \infty$ in (2.1), we need some compactness argument. To this end, we prove:

LEMMA 2.3. *The sequence $\{w^N\}_N$ is bounded in H^1.*

Proof. Taking $\varphi = w^N + \beta P_N u^n$ in (2.1), we get

$$\begin{aligned}\|w^N\|_{L^2}^2 + \|\nabla w^N\|_{L^2}^2 &= \beta^2\|P_N u^n\|_{L^2}^2 + \beta^2\|\nabla P_N u^n\|_{L^2}^2 \\ &\quad + \frac{(1+\beta)\tau}{2}\int_{\mathbb{R}^1} P_N f(w^N + \beta P_N u^n)\,dx \\ &\le \beta^2\|P_N u^n\|_{L^2}^2 + \beta^2\|\nabla P_N u^n\|_{L^2}^2 + \frac{1}{2}\|w^N\|_{L^2}^2 \\ &\quad + \frac{(1+\beta)^2\tau^2}{8}\|f\|_{L^2}^2 + \frac{(1+\beta)\tau\beta}{2}\|f\|_{L^2}\|P_N u^n\|_{L^2}.\end{aligned}$$

Then

$$\begin{aligned}\|w^N\|_{L^2}^2 + \|\nabla w^N\|_{L^2}^2 &\le 2\beta^2\|P_N u^n\|_{L^2}^2 + 2\beta^2\|\nabla P_N u^n\|_{L^2}^2 \\ &\quad + \frac{(1+\beta)^2\tau^2}{4}\|f\|_{L^2}^2 + (1+\beta)\tau\beta\|f\|_{L^2}\|P_N u^n\|_{L^2} \le K,\end{aligned} \tag{2.2}$$

so $\{w^N\}_N$ is bounded in H^1. ∎

LEMMA 2.4. *For a given $u^n \in H^1$, there exists a solution $u^{n+1} \in H^1$ to* (2.1) *which is a weak limit in H^1 of a subsequence $\{w^{N'}\}_{N'}$.*

Proof. First, we pass to the limit in the linear parts of (2.1). Since $\{w^N\}_N$ is bounded in H^1, there exists a subsequence such that $w^N \rightharpoonup u^{n+1}$ weakly in H^1, that is,

$$\forall \varphi \in H^1, \qquad \int_{\mathbb{R}^1} w^N \varphi\,dx \to \int_{\mathbb{R}^1} u^{n+1}\varphi\,dx, \tag{2.3}$$

$$\forall \varphi \in H^1, \qquad \int_{\mathbb{R}^1} \nabla w^N \nabla\varphi\,dx \to \int_{\mathbb{R}^1} \nabla u^{n+1}\nabla\varphi\,dx. \tag{2.4}$$

Secondly, we pass to the limit in the nonlocal nonlinear part of (2.1). We consider now a smooth cut off function $\rho \in C_0^\infty(\mathbb{R}^1)$ such that $0 \le \rho \le 1$ and

$$\rho(s) = \begin{cases} 1 & \text{if } |s| \le 1, \\ 0 & \text{if } |s| \ge 2. \end{cases}$$

For each $r > 0$, let $\rho_r(s) = \rho(s/r)$. Then $w_r^N = \rho_r w^N \in H^1(-r, r)$. Since w^N is bounded in H^1, we deduce that w_r^N is bounded in $H^1(-r, r)$. We have $w_r^N \rightharpoonup u^{n+1}$ weakly in $H^1(-r, r)$, and $w_r^N \to u^{n+1}$ strongly in $L^2(-r, r)$.

Now we write

$$\int\limits_{\mathbb{R}^1} \nabla((w^N)^2-(u^{n+1})^2)\varphi\,dx = \int\limits_{-r}^{r} \nabla((w^N)^2-(u^{n+1})^2)\varphi\,dx + \int\limits_{|x|\ge r} \nabla((w^N)^2-(u^{n+1})^2)\varphi\,dx =: \Phi_1+\Phi_2.$$

We now majorize Φ_1 and Φ_2 separately. Let $\varepsilon>0$. Since $\varphi\in L^2$ and thanks to Lemma 2.3 we have, for $r>0$ large enough,

$$\Phi_2 \le K\Big(\int\limits_{|x|\ge r} |\varphi|^2\,dx\Big)^{1/2} \le \frac{\varepsilon}{2}.$$

On the other hand, since $w^N|_{(-r,r)} = w_r^N$, we have

$$\begin{aligned}\Phi_1 &= \int\limits_{-r}^{r} \nabla((w_r^N)^2-(u^{n+1})^2)\varphi\,dx = -\int\limits_{-r}^{r} (w_r^N+u^{n+1})(w_r^N-u^{n+1})\nabla\varphi\,dx\\ &\le \big(\|w_r^N\|_{L^\infty(-r,r)}+\|u^{n+1}\|_{L^\infty(-r,r)}\big)\|w_r^N-u^{n+1}\|_{L^2(-r,r)}\|\nabla\varphi\|_{L^2(-r,r)}\\ &\le K\|w_r^N-u^{n+1}\|_{L^2(-r,r)}\|\nabla\varphi\|_{L^2(-r,r)};\end{aligned}$$

that yields $\Phi_1\to0$ as $N\to\infty$, which is equivalent to saying that $|\Phi_1|\le\varepsilon/2$ if N large enough. Hence we deduce that

$$\forall\varphi\in P_NH^1,\qquad \Big|\int\limits_{\mathbb{R}^1} \nabla((w^N)^2-(u^{n+1})^2)\varphi\,dx\Big| \le \varepsilon \qquad \text{as } N\to\infty,$$

which allows us to conclude that

$$\forall\varphi\in P_NH^1,\quad \int\limits_{\mathbb{R}^1}\nabla(w^N)^2\varphi\,dx \to \int\limits_{\mathbb{R}^1}\nabla(u^{n+1})^2\varphi\,dx \quad \text{as } N\to\infty. \tag{2.5}$$

Thus u^{n+1} is a solution of (1.1) at least in the distribution sense. ■

We now prove that there exists a bounded absorbing set $\mathcal{B}\subset H^1$ that is positively invariant under S ($S(\mathcal{B})\subset\mathcal{B}$) and bounded in the H^1 topology. Set

$$M_0^2 = \frac{5(1-\beta^2)^2\|f\|_{L^2}^2}{\alpha^2(1-3\beta^2)},\qquad \mathcal{B}=\{u\in H^1\colon \|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\le M_0^2\}.$$

Lemma 2.5. *The set $\mathcal{B}$ is positively invariant under S, that is, $S(\mathcal{B})\subset\mathcal{B}$, and absorbing for S, that is, for all $u^0\in H^1$ there exists $n_0>0$ such that $S^nu^0\in\mathcal{B}$ for all $n\ge n_0$.*

Proof. Multiplying (1.7) by $u^{n+1}+\beta u^n$ and integrating on $\mathbb{R}^1$, we obtain

$$\begin{aligned}\|u^{n+1}\|_{L^2}^2 - \beta^2\|u^n\|_{L^2}^2 + \|\nabla u^{n+1}\|_{L^2}^2 - \beta^2\|\nabla u^n\|_{L^2}^2 \\ &= \frac{(1+\beta)\tau}{2}\int_{\mathbb{R}^1} f(u^{n+1}+\beta u^n)\,dx \\ &\le \frac{(1+\beta)\tau}{2}\|f\|_{L^2}(\|u^{n+1}\|_{L^2}+\beta\|u^n\|_{L^2}) \\ &\le \frac{1}{2}\|u^{n+1}\|_{L^2}^2 + \frac{\beta^2}{2}\|u^n\|_{L^2}^2 + \frac{(1+\beta)^2\tau^2}{4}\|f\|_{L^2}^2.\end{aligned}$$

Then

$$\begin{aligned}(2.6)\qquad \|u^{n+1}\|_{L^2}^2 + \|\nabla u^{n+1}\|_{L^2}^2 \\ &\le 3\beta^2\|u^n\|_{L^2}^2 + 2\beta^2\|\nabla u^n\|_{L^2}^2 + \frac{(1+\beta)^2\tau^2}{2}\|f\|_{L^2}^2 \\ &\le 3\beta^2(\|u^n\|_{L^2}^2 + \|\nabla u^n\|_{L^2}^2) + \frac{4(1-\beta^2)^2}{\alpha^2}\|f\|_{L^2}^2.\end{aligned}$$

We have $S(\mathcal{B})\subset\mathcal{B}$: indeed, from (2.6) we deduce that if $u^n\in\mathcal{B}$ then

$$\begin{aligned}\|u^{n+1}\|_{L^2} + \|\nabla u^{n+1}\|_{L^2}^2 &\le 3\beta^2M_0^2 + (1-3\beta^2)\frac{4(1-\beta^2)^2}{\alpha^2(1-3\beta^2)}\|f\|_{L^2}^2 \\ &\le \max\biggl\{M_0^2, \frac{4(1-\beta^2)^2}{\alpha^2(1-3\beta^2)}\|f\|_{L^2}^2\biggr\} = M_0^2.\end{aligned}$$

Secondly, $\mathcal{B}$ is an absorbing set in H^1 with respect to the H^1 topology. Indeed, by the discrete Gronwall lemma we see from (2.6) that for all $u^0\in H^1$,

$$\|u^n\|_{L^2}^2 + \|\nabla u^n\|_{L^2}^2 \le 3^n\beta^{2n}(\|u^0\|_{L^2}^2 + \|\nabla u^0\|_{L^2}^2) + \frac{4(1-\beta^2)^2}{\alpha^2(1-3\beta^2)}\|f\|_{L^2}^2.$$

Now, for $u^0\in H^1$ there exists n_0 such that for $n\ge n_0$,

$$3^n\beta^{2n}(\|u^0\|_{L^2}^2 + \|\nabla u^0\|_{L^2}^2) \le \frac{(1-\beta^2)^2}{\alpha^2(1-3\beta^2)}\|f\|_{L^2}^2,$$

which yields

$$\|u^n\|_{L^2}^2 + \|\nabla u^n\|_{L^2}^2 \le \frac{5(1-\beta^2)^2}{\alpha^2(1-3\beta^2)}\|f\|_{L^2}^2 = M_0^2.$$

The proof of Lemma 2.5 is complete. ■

LEMMA 2.6. *For τ small enough, the mapping $S\colon\mathcal{B}\to\mathcal{B}$ is one-to-one and continuous, that is, $\|Su - S\widetilde{u}\|_{H^1} \le C_0\|u-\widetilde{u}\|_{H^1}$, where $C_0>0$ is a universal constant.*

Proof. We establish the uniqueness of solution to (1.7) with the aid of the Banach fixed point theorem. We introduce the following functional and we will make some smallness restrictions on the size of the time step discretization τ. Let

$$\mathcal{F}_u(v) = \beta(I-\varDelta)^{-1}(I-\varDelta)u + \tau(I-\varDelta)^{-1}\left[\frac{1+\beta}{2}f - \frac{1}{8}\nabla(v+\beta u)^2\right].$$

Then any solution to (1.7) is a fixed point $u^{n+1} = \mathcal{F}_{u^n}(u^{n+1})$. Let $\varepsilon > 0$ be small enough. We recall that for τ small enough,

$$\forall s_1 < s_2, \quad \|(I-\varDelta)^{-1}\|_{\mathcal{L}(H^{s_1},H^{s_2})} \le \frac{C}{\tau^{(s_2-s_1)/2}}. \tag{2.7}$$

It is clear that for $u, v \in H^1$, (2.7) yields

$$\tau(I-\varDelta)^{-1}\left[\frac{1+\beta}{2}f - \frac{1}{8}\nabla(v+\beta u)^2\right] \in H^2, \tag{2.8}$$

and thus $\mathcal{F}_u(H^1) \subset H^1$. Let us prove that $\mathcal{F}_u$ is a contraction on $\mathcal{B}$. Let $v_1, v_2, u \in \mathcal{B}$. Then

$$\begin{aligned}\|\mathcal{F}_u(v_2) - \mathcal{F}_u(v_1)\|_{H^1} &\le C\tau\|(I-\varDelta)^{-1}\|_{\mathcal{L}(H^{-1/2},H^1)} \\ &\quad\times (\|v_2\|_{H^1}^2 + \|v_1\|_{H^1}^2 + \|u\|_{H^1}^2)\|v_2 - v_1\|_{H^1} \le C\tau^{3/4}K_1^2\|v_2 - v_1\|_{H^1}.\end{aligned} \tag{2.9}$$

Thus, we deduce from (2.9) that if $\tau > 0$ is small enough such that $C\tau^{3/4}K_1^2 < 1/2$, then $\mathcal{F}_u$ is a contraction on $\mathcal{B}$. The mapping $u \mapsto v$, where v is a fixed point of $\mathcal{F}_u$, is continuous on $\mathcal{B}$: if v and $\widetilde{v}$ are respectively the fixed points for $\mathcal{F}_u$ and $\mathcal{F}_{\widetilde{u}}$, then

$$\begin{aligned}\|v - \widetilde{v}\|_{H^1} &= \|\mathcal{F}_u(v) - \mathcal{F}_u(\widetilde{v})\|_{H^1} \\ &\le \beta\|u - \widetilde{u}\|_{H^1} + C\tau^{3/4}K_1^2\|u - \widetilde{u}\|_{H^1} + C\tau^{3/4}K_1^2\|v - \widetilde{v}\|_{H^1}.\end{aligned} \tag{2.10}$$

The result follows promptly if $\tau > 0$ is small enough as above then $\|v - \widetilde{v}\|_{H^1} \le (2\beta+1)\|u - \widetilde{u}\|_{H^1}$. The proof of Lemma 2.6 is complete. ∎

Proof of Theorem 1.1. From Lemmas 2.2–2.6, we easily obtain the desired conclusion.

3. Global attractor

3.1. Existence of a global attractor. In this subsection, we prove the existence of a compact global attractor in H^1. Now, we plan to describe the splitting $S = S_1 + S_2$ with the needed properties for S_1, S_2. For $L > 0$, we consider a C^∞-smooth cut-off function χ_L such that $0 \le \chi_L \le 1$ and

$$\chi_L(x) = \begin{cases} 1 & \text{if } |x| \le L, \\ 0 & \text{if } |x| \ge 1 + L. \end{cases}$$

For $\eta > 0$, that is chosen in $(0,1)$ without loss of generality, we consider a function $f\chi_L \in S$ such that $\|f(1-\chi_L)\|_{L^2} \le \eta$. We split the solution u^n of

(1.7) as $u^n = v^n + w^n$, where v^n is the solution to

$$(3.1)\qquad \frac{1}{\tau}(v^{n+1}-\beta v^n) - \frac{1}{\tau}\Delta(v^{n+1}-\beta v^n) = \frac{1+\beta}{2} f\chi_L - \frac{1}{8}[\nabla(u^{n+1}+\beta u^n)^2]\chi_L, \qquad v^0 = 0.$$

Then $w^n = u^n - v^n$ is the solution to

$$(3.2)\qquad \frac{1}{\tau}(w^{n+1}-\beta w^n) - \frac{1}{\tau}\Delta(w^{n+1}-\beta w^n) = \frac{1+\beta}{2} f(1-\chi_L) - \frac{1}{8}[\nabla(u^{n+1}+\beta u^n)^2](1-\chi_L), \qquad w^0 = u^0.$$

LEMMA 3.1. *The sequence $\{v^n\}_n$ is well defined and belongs to H^2 for $\varepsilon > 0$.*

Proof. Problem (3.1) is linear and elliptic on H^1. Using the Lax–Milgram theorem, we easily obtain the existence of a solution to (3.1). The estimate (2.7) shows that the solutions are in H^2. ∎

LEMMA 3.2. *There exists $K = K(\alpha)$ such that for all $\eta \in [0,1]$ there exists $n_0 = n_0(\|f\|_{L^2}, \eta)$ such that $\|w^n\|_{H^1} \le K\eta$ for all $u^0 \in \mathcal{B}$ and $n \ge n_0$.*

Proof. Multiplying (3.2) by $w^{n+1}+\beta w^n$ and integrating on $\mathbb{R}^1$, we obtain

$$\begin{aligned}
\|w^{n+1}\|_{L^2}^2 &- \beta^2\|w^n\|_{L^2}^2 + \|\nabla w^{n+1}\|_{L^2}^2 - \beta^2\|\nabla w^n\|_{L^2}^2 \\
&= \frac{(1+\beta)\tau}{2}\int_{\mathbb{R}^1} f(1-\chi_L)(w^{n+1}+\beta w^n)\,dx \\
&\quad - \frac{\tau}{8}\int_{\mathbb{R}^1}[\nabla(u^{n+1}+\beta u^n)^2](1-\chi_L)(w^{n+1}+\beta w^n)\,dx \\
&\le \frac{(1+\beta)\tau}{2}\|f(1-\chi_L)\|_{L^2}(\|w^{n+1}\|_{L^2}+\beta\|w^n\|_{L^2}) \\
&\quad + \frac{\tau}{8}\big\|[(u^{n+1}+\beta u^n)^2](1-\chi_L)\big\|_{L^2}(\|w^{n+1}\|_{L^2}+\beta\|w^n\|_{L^2}) \\
&\le \frac{(5+4\beta)\tau\eta}{8}(\|w^{n+1}\|_{L^2}+\beta\|w^n\|_{L^2}) \\
&\le \frac{1}{2}\|w^{n+1}\|_{L^2}^2 + \frac{\beta^2}{2}\|w^n\|_{L^2}^2 + \frac{(5+4\beta)^2\tau^2\eta^2}{64}.
\end{aligned}$$

We get

$$\|w^{n+1}\|_{L^2}^2 + \|\nabla w^{n+1}\|_{L^2}^2 \le 3\beta^2(\|w^n\|_{L^2}^2 + \|\nabla w^n\|_{L^2}^2) + \frac{(5+4\beta)^2\tau^2\eta^2}{32}.$$

The discrete Gronwall lemma gives

$$\begin{aligned}
&\|w^n\|_{L^2} + \|\nabla w^n\|_{L^2}^2 \\
&\qquad \le 3^n\beta^{2n}(\|u^0\|_{L^2}^2 + \|\nabla u^0\|_{L^2}^2) + \frac{(5+4\beta)^2\tau^2\eta^2}{32(1-3\beta^2)}, \\
&\qquad \le 3^n\beta^{2n}M_0^2 + \frac{(5+4\beta)^2(1-\beta)^2}{8(1-3\beta^2)}\cdot\frac{\eta^2}{\alpha^2} \quad (\text{since } \alpha\tau \le 2(1-\beta)).
\end{aligned}$$

Then for n large enough that $3^n\beta^{2n}M_0^2 \le \frac{7(5+4\beta)^2(1-\beta)^2}{8(1-3\beta^2)}\cdot\frac{\eta^2}{\alpha^2}$, we get

$$\|w^{n+1}\|_{L^2}^2 + \|\nabla w^{n+1}\|_{L^2}^2 \le \frac{(5+4\beta)^2(1-\beta)^2}{1-3\beta^2}\cdot\frac{\eta^2}{\alpha^2}. \tag{3.3}$$

The proof of Lemma 3.2 is complete. ■

Lemma 3.3. *The sequence $\{v^n\}_n$ is uniformly bounded in $H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx)$ and in H^2, that is, there exists a constant K such that for all $u^0 \in \mathcal{B}$ we have, for any $n \ge 0$,*

$$\|v^n\|_{H^1} \le K, \qquad \|xv^n\|_{L^2} \le K, \qquad \|v^n\|_{H^2} \le K/\tau. \tag{3.4}$$

Proof. Since $v^n = u^n - w^n$, Lemma 3.2 yields

$$\|v^n\|_{H^1} \le K, \tag{3.5}$$

that is, $\{v^n\}_n$ is bounded in H^1.

Now in order to prove the second inequality in (3.4), we have to verify that $xv^n \in L^2$. To this end we shall consider the sequence $\{x\varphi(x/N)v^n\}_N$, where $\varphi \in \mathcal{D}$ satisfies $\varphi(0) = 1$. Multiplying (3.1) by $x^2\varphi^2(x/N)(v^{n+1}+\beta v^n)$ and integrating on $\mathbb{R}^1$, we obtain

$$\begin{aligned}
&\|x\varphi(x/N)v^{n+1}\|_{L^2}^2 - \beta^2\|x\varphi(x/N)v^n\|_{L^2}^2 \\
&\quad + \|x\varphi(x/N)\nabla v^{n+1}\|_{L^2}^2 - \beta^2\|x\varphi(x/N)\nabla v^n\|_{L^2}^2 \\
&= -\int\limits_{\mathbb{R}^1}\Big[(2\varphi(x/N)+\frac{2}{N}x\varphi'(x/N))\nabla(v^{n+1}-\beta v^n)\Big][x\varphi(x/N)(v^{n+1}+\beta v^n)]\,dx \\
&\quad + \frac{(1+\beta)\tau}{2}\int\limits_{\mathbb{R}^1}[x\varphi(x/N)f\chi_L][x\varphi(x/N)(v^{n+1}+\beta v^n)]\,dx \\
&\quad - \frac{\tau}{8}\int\limits_{\mathbb{R}^1}\big[x\varphi(x/N)\chi_L[\nabla(v^{n+1}+\beta v^n)^2]\big][x\varphi(x/N)(v^{n+1}+\beta v^n)]\,dx \\
&\le \Big[2\|(\nabla v^{n+1}+\beta\nabla v^n)(\varphi(x/N)+(x/N)\varphi'(x/N))\|_{L^2} \\
&\qquad + \frac{(1+\beta)\tau}{2}\|x\varphi(x/N)f\chi_L\|_{L^2} + \frac{\tau}{8}\|x\varphi(x/N)\chi_L(\nabla(v^{n+1}+\beta v^n)^2)\|\Big] \\
&\quad \times[\|x\varphi(x/N)v^{n+1}\|_{L^2} + \beta\|x\varphi(x/N)v^n\|_{L^2}] \\
&\le \frac{1}{2}\|x\varphi(x/N)v^{n+1}\|_{L^2}^2 + \frac{\beta^2}{2}\|x\varphi(x/N)v^n\|_{L^2}^2 + K.
\end{aligned}$$

Then we easily get

$$\|x\varphi(x/N)v^{n+1}\|_{L^2}^2 + \|x\varphi(x/N)\nabla v^{n+1}\|_{L^2}^2 \\ \leq 3\beta^2(\|x\varphi(x/N)v^n\|_{L^2}^2 + \|x\varphi(x/N)\nabla v^n\|_{L^2}^2) + K.$$

Again thanks to the discrete Gronwall lemma, we deduce that

$$\|x\varphi(x/N)v^{n+1}\|_{L^2}^2 + \|x\varphi(x/N)\nabla v^{n+1}\|_{L^2}^2 \leq K/\tau.$$

Hence, the Fatou lemma yields

$$\|xv^{n+1}\|_{L^2}^2 + \|x\nabla v^{n+1}\|_{L^2}^2 \leq K/\tau. \tag{3.6}$$

In fact we can obtain a better bound than (3.6) for $\{xv^n\}_n$ in H^1. Multiplying (3.1) by $x^2(v^{n+1} + \beta v^n)$ and integrating on $\mathbb{R}^1$, we obtain

$$\begin{aligned}
&\|xv^{n+1}\|_{L^2}^2 - \beta^2\|xv^n\|_{L^2}^2 + \|x\nabla v^{n+1}\|_{L^2}^2 - \beta^2\|x\nabla v^n\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^1} [(2+2x)\nabla(v^{n+1} - \beta v^n)][x(v^{n+1} + \beta v^n)]\,dx \\
&\qquad + \frac{(1+\beta)\tau}{2}\int_{\mathbb{R}^1} [xf\chi_L][x(v^{n+1} + \beta v^n)]\,dx \\
&\qquad - \frac{\tau}{8}\int_{\mathbb{R}^1} [x\chi_L\nabla(u^{n+1} + \beta u^n)^2][x(v^{n+1} + \beta v^n)]\,dx \\
&\quad \leq \bigg(2\|(1+x)(\nabla v^{n+1} + \beta\nabla v^n)\|_{L^2} + \frac{(1+\beta)\tau}{2}\|xf\chi_L\|_{L^2} \\
&\qquad + \frac{\tau}{8}\|x\chi_L[\nabla(v^{n+1} + \beta v^n)^2]\|\bigg)(\|xv^{n+1}\|_{L^2} + \beta\|xv^n\|_{L^2}) \\
&\quad \leq \frac{1}{2}\|xv^{n+1}\|_{L^2}^2 + \frac{\beta^2}{2}\|xv^n\|_{L^2}^2 + K(\eta).
\end{aligned}$$

Then we easily get

$$\|xv^{n+1}\|_{L^2}^2 + \|x\nabla v^{n+1}\|_{L^2}^2 \leq 3\beta^2(\|xv^n\|_{L^2}^2 + \|x\nabla v^n\|_{L^2}^2) + K(\eta).$$

Since $v^0 = 0$, the discrete Gronwall lemma gives

$$\|xv^n\|_{L^2} + \|x\nabla v^n\|_{L^2}^2 \leq K(\eta). \tag{3.7}$$

Thus, we deduce that $\{v^n\}_n$ is bounded in $H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx)$.

Now we prove the third inequality in (3.4). From (3.1) we obtain

$$\Delta v^{n+1} = \beta\Delta v^n + v^{n+1} - \beta v^n - \frac{(1+\beta)\tau}{2}f\rho + \frac{\tau}{8}[\nabla(u^{n+1} + \beta u^n)^2]\rho.$$

Using the fact that u^n and v^n are bounded in H^1, we obtain

$$\begin{aligned}\|\Delta v^{n+1}\|_{L^2} &\le \beta\|\Delta v^n\|_{L^2} + \|v^{n+1}\|_{L^2} + \frac{(1+\beta)\tau}{2}\|f\chi_L\|_{L^2} \\ &\quad + \beta\|v^n\|_{L^2} + \frac{\tau}{8}\|\nabla(u^{n+1}+\beta u^n)^2\chi_L\|_{L^2} \\ &\le \beta\|\Delta v^n\|_{L^2} + K.\end{aligned}$$

Thus by the discrete Gronwall lemma we get

$$\|\Delta v^n\|_{L^2} \le K/\tau. \tag{3.8}$$

The proof of Lemma 3.3 is complete. ∎

We have constructed a splitting $u^n = v^n + w^n$ where w^n is small enough in H^1 and v^n belongs to a bounded set of $H^2 \cap L^2(\mathbb{R}^1, (1+x^2)dx) \subset H^1$. To establish the proof of compactness of the trajectories in H^1, we use the following compact embedding.

LEMMA 3.4 ([21]). *The embedding $H^2(\mathbb{R}^1)\cap L^2(\mathbb{R}^1,(1+x^2)dx) \hookrightarrow H^1(\mathbb{R}^1)$ is compact.*

Proof of Theorem 1.2. Now, the assumptions of [25, Theorem I.1.1] are satisfied. Indeed, we set $S_1 u^n = v^n$ and $S_2 u^n = w^n$, and use Lemmas 3.2–3.4. Thus, we have the existence of a global attractor A_τ in H^1 that is a bounded set in H^2. In order to prove that the global attractor A_τ is a compact set in H^2, we rely on the J. Ball argument [9]. Let $\{u_j^{n+1}\}_j$ be a sequence of points of $A_\tau \subset H^2$, and consider $\{u_j^n\}_j$ in A_τ such that $\{u_j^{n+1}\}_j = S\{u_j^n\}_j$. We are going to prove that there exists a subsequence of $\{u_j^{n+1}\}_j$ that converges strongly in H^2. Let $w_j^n = u_j^n - u^n$. Then w_j^n is a solution to

$$\begin{aligned}\frac{1}{\tau}(w_j^{n+1}-\beta w_j^n) - \frac{1}{\tau}\Delta(w_j^{n+1}-\beta w_j^n) \\ = \tfrac{1}{8}\nabla(u^{n+1}+\beta u^n)^2 - \tfrac{1}{8}\nabla(u_j^{n+1}+\beta u_j^n)^2.\end{aligned} \tag{3.9}$$

To go further, we reformulate (3.9) as follows

$$\begin{aligned}w_j^{n+1} &= \beta(I-\Delta)^{-1}(I-\Delta)w_j^n \\ &\quad + \frac{\tau}{8}(I-\Delta)^{-1}[\nabla(u^{n+1}+\beta u^n)^2 - \nabla(u_j^{n+1}+\beta u_j^n)^2].\end{aligned}$$

Now, it is easy to obtain

$$\|w_j^{n+1}\|_{H^2} \le \beta\|w_j^n\|_{H^2} + C\|\nabla(u^{n+1}+\beta u^n)^2 - \nabla(u_j^{n+1}+\beta u_j^n)^2\|_{H^{-1}}.$$

The continuity of the map S on H^1 given by Lemma 3.1 and the strong convergence of the sequences $\{u_j^{n+1}\}_j$ and $\{u_j^n\}_j$ to u^{n+1} and u^n respectively in H^1 implies that $u^{n+1} = Su^n$ and $\nabla(u_j^{n+1}+\beta u_j^n)^2 \to \nabla(u^{n+1}+\beta u^n)^2$ in H^{-1}. We set $\Pi_n = \limsup_{j\to\infty}\|w_j^n\|_{H^2}$. Then $\Pi_{n+1} \le \beta\Pi_n$, and hence by induction $\Pi_n \le \beta^{n-p}\Pi_p$. As $\{\Pi_p\}_p$ is uniformly bounded since $\{u_j^n\}^n$

and $\{u^n\}^n$ are trajectories on the global attractor, we let $p \to -\infty$ in $\Pi_n \le \beta^{n-p}\Pi_p$ to get $\Pi_n = 0$. The proof of Theorem 1.2 is complete.

3.2. Dimension of the global attractor. In this subsection we prove that the global attractor has finite fractal dimensions. For this purpose, we recall a result of [14].

LEMMA 3.5 ([14]). *Let X be a separable Hilbert space and M a bounded closed set in X. Assume that there exists a mapping $V\colon M \to X$ such that $M \subseteq VM$ and*

(i) *V is Lipschitz on M, i.e., there exists $L_0 > 0$ such that*

$$\|Vu_1 - Vu_2\|_X \le L_0\|u_1 - u_2\|_X, \qquad u_1, u_2 \in M.$$

(ii) *There exist compact seminorms $n_1(\cdot)$ and $n_2(\cdot)$ on X such that*

$$\|Vu_1 - Vu_2\|_X \le \mu\|u_1 - u_2\|_X + K(n_1(u_1 - u_2) + n_2(Vu_1 - Vu_2))$$

for $u_1, u_2 \in M$, where $0 < \mu < 1$ and $K > 0$ are constants.

(*A seminorm $n(x)$ in X is said to be* compact *if $n(x_m) \to 0$ for any sequence $\{x_m\}_m \subset X$ such that $x_m \rightharpoonup 0$ weakly in X.) Then M is a compact set in X with finite fractal dimensions.*

In order to apply Lemma 3.5 with $V = S$, the solution map defining the scheme (1.7), we have to impose some assumption on the force f. We prove the following:

LEMMA 3.6. *Let $f \in L^2(\mathbb{R}^1, (1+x^2)dx)$. Then the global attractor A_τ is in fact a bounded set in $H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx)$.*

Proof. Here we use again the splitting method. Let $\{u^n\}_n$ be a global trajectory that lies on the global attractor A_τ. Any u^n will split as follows:

$$\begin{aligned}(3.10)\qquad \frac{1}{\tau}(v^{n+1} - \beta v^n) - \frac{1}{\tau}\Delta(v^{n+1} - \beta v^n)\\ = \frac{1+\beta}{2}f - \frac{1}{8}\nabla(u^{n+1} + \beta u^n)^2, \qquad v^0 = 0.\end{aligned}$$

Then $w^n = u^n - v^n$ is the solution to

$$(3.11)\qquad \frac{1}{\tau}(w^{n+1} - \beta w^n) - \frac{1}{\tau}\Delta(w^{n+1} - \beta w^n) = 0, \qquad w^0 = u^0.$$

Problem (3.10) is linear and the existence of $\{v^n\}_n$ follows easily by the Lax–Milgram lemma. Moreover, since $xf \in L^2(\mathbb{R}^1)$, it is clear that $\{xv^n\}_n \subset L^2(\mathbb{R}^1)$. Then the sequence $\{v^n\}_n$ defined by (3.10) is contained in $H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx)$ and is bounded. On the other hand, it is easy to prove that $\{w^n\}_n$ is bounded in H^1 and satisfies $\|w^{n+1}\|_{L^2} = \beta\|w^n\|_{L^2}$. Thus A_τ is bounded in $H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx)$. ∎

Proof of Theorem 1.3. We apply Lemma 3.5 with $S = V$, $X = H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx)$ and $M = A_\tau$. The first assumption is valid with $L_0 = 2\beta + 1$ due to Lemma 3.1. On the other hand, considering the equation satisfied by the difference $w^n = u^n - v^n$ of two solutions $u^{n+1} = Su^n$ and $v^{n+1} = Sv^n$, we have

$$\|w^{n+1}\|_{H^1}^2 - \beta^2 \|w^n\|_{H^1}^2 \le M(\|w^{n+1}\|_{L^4}^2 + \|w^{n+1}\|_{L^\infty}^2 + \|w^n\|_{L^4}^2 + \|w^n\|_{L^\infty}^2).$$

Since the embeddings

$$H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx) \subset L^4 \quad \text{and} \quad H^1 \cap L^2(\mathbb{R}^1, (1+x^2)dx) \subset L^\infty$$

are compact, assumption (ii) in Lemma 3.5 is valid. The proof of Theorem 1.3 is complete. ■

Acknowledgements. The author gratefully acknowledges the helpful suggestions and comments of two anonymous referees and the editor, Professor Marta Kosek. This work was supported by the National Natural Sciences Foundation of China (No. 61273020 and No. 11571283) and the Postdoctoral Research Foundation of Chongqing (No. yuXM201102006).

References

[1] F. Abergel, *Attractors for a Navier–Stokes flow in an unbounded domain*, Math. Models Numer. Anal. 23 (1989), 359–370.

[2] F. Abergel, *Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains*, J. Differential Equations 83 (1990), 85–108.

[3] M. Abounouh, H. Al Moatassime, J. P. Chehab, S. Dumont and O. Goubet, *Discrete Schrödinger equations and dissipative dynamical systems*, Comm. Pure Appl. Anal. 7 (2008), 211–227.

[4] G. D. Akrivis, V. A. Dougalis and O. A. Karakashian, *On fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation*, Numer. Math. 59 (1991), 31–53.

[5] J. Albert, *Dispersion of low-energy waves for the generalized Benjamin–Bona–Mahony equation*, J. Differential Equations 63 (1986), 117–134.

[6] J. Avrin, *The generalized Benjamin–Bona–Mahony equation in $\mathbb{R}^n$ with singular initial data*, Nonlinear Anal. 11 (1987), 139–147.

[7] A. V. Babin, *The attractor of a Navier–Stokes system in an unbounded channel-like domain*, J. Dynam. Differential Equations 4 (1992), 555–584.

[8] A. V. Babin and M. I. Vishik, *Attractors of partial differential equations in an unbounded domain*, Proc. Roy. Soc. Edinburgh Sect. A 116 (1990), 221–243.

[9] J. Ball, *Global attractors for damped semilinear wave equations, partial differential equations and applications*, Discrete Contin. Dynam. Systems 10 (2004), 31–52.

[10] T. B. Benjamin, J. L. Bona and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London 272 (1972), 47–78.

[11] P. Biler, *Long time behavior of solutions of the generalized Benjamin–Bona–Mahony equation in two space dimensions*, Differential Integral Equations 5 (1992), 891–901.

[12] F. E. Browder, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, in: Applications of Partial Differential Equations, R. Finn (ed.), Amer. Math. Soc., Providence, RI, 1965, 24–29.

[13] A. O. Celebi, V. K. Kalantarov and M. Polat, *Attractors for the generalized Benjamin–Bona–Mahony equation*, J. Differential Equations 157 (1999), 439–451.

[14] I. Chueshov and I. Lasiecka, *Attractors for second-order evolution equations with a nonlinear damping*, J. Dynam. Differential Equations 16 (2004), 469–512.

[15] I. Chueshov, M. Polat and S. Siegmund, *Gevrey regularity of global attractor for generalized Benjamin–Bona–Mahony equation*, Mat. Fiz. Anal. Geom. 11 (2004), 226–242.

[16] M. Delfour, M. Fortin and G. Payre, *Finite-difference solutions of a nonlinear Schrödinger equation*, J. Comput. Phys. 44 (1981), 277–288.

[17] E. Ezzoug, O. Goubet and E. Zahrouni, *Semi-discrete weakly damped nonlinear 2-D Schrödinger equation*, Differential Integral Equations 23 (2010), 232–252.

[18] E. Feireisl, *Attractors for semilinear damped wave equations on* $\mathbf{R}^3$, Nonlinear Anal. 23 (1994), 187–195.

[19] E. Feireisl, P. Laurençot, F. Simondon, and H. Touré, *Compact attractors for reaction-diffusion equations in* $\mathbb{R}^N$, C. R. Acad. Sci. Paris Sér. I Math. 319 (1994), 147–151.

[20] O. Goubet and E. Zahrouni, *On a time discretization of a weakly damped forced nonlinear Schrödinger equation*, Comm. Pure Appl. Anal. 7 (2008), 1429–1442.

[21] B. L. Guo and Y. S. Li, *Attractor for dissipative Klein–Gordon–Schrödinger equations in* $\mathbf{R}^3$, J. Differential Equations 136 (1997), 356–377.

[22] H. Y. Huang, *The finite dimension behavior of the Schrödinger equation with nonlocal integral nonlinearity*, Acta Math. Appl. Sinica (English Ser.) 15 (1999), 34–44.

[23] P. Laurençot, *Long-time behaviour for weakly damped driven nonlinear Schrödinger equations in* $\mathbb{R}^N$, $N \leq 3$, Nonlinear Differential Equations Appl. 2 (1995), 357–369.

[24] M. Stanislavova, A. Stefanov and B. Wang, *Asymptotic smoothing and attractors for the generalized Benjamin–Bona–Mahony equation on* $\mathbb{R}^3$, J. Differential Equations 219 (2005), 451–483.

[25] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer, New York, 1998.

[26] B. Wang, *Strong attractors for the Benjamin–Bona–Mahony equation*, Appl. Math. Lett. 10 (1997), 23–28.

[27] B. Wang, *Regularity of attractors for the Benjamin–Bona–Mahony equation*, J. Phys. A 31 (1998), 7635–7645.

[28] B. Wang and W. Yang, *Finite dimensional behaviour for the Benjamin–Bona–Mahony equation*, J. Phys. A 30 (1997), 4877–4885.

[29] Y. Yan, *Attractors and dimensions for discretizations of a weakly damped Schrödinger equations and a sine-Gordon equation*, Nonlinear Anal. 20 (1993), 1417–1452.

[30] C. S. Zhu, *Asymptotic attractors of Benjamin–Bona–Mahony equations*, Eur. J. Pure Appl. Math. 1 (2008), 3–10.

[31] C. S. Zhu, *Global attractor for the damped Benjamin–Bona–Mahony equations on* $\mathbb{R}^1$, Appl. Anal. 86 (2007), 59–65.

[32] C. S. Zhu and C. L. Mu, *Exponential decay estimates for time-delayed Benjamin–Bona–Mahony equations*, Appl. Anal. 87 (2008), 401–407.

Chaosheng Zhu
School of Mathematics and Statistics
Southwest University
400715 Chongqing, P.R. China
E-mail: zcs@swu.edu.cn

Received 11.1.2015
and in final form 28.4.2015 (3600)