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Legendrian dual surfaces in hyperbolic 3-space

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Abstract. We consider surfaces in hyperbolic 3-space and their duals. We study flat dual surfaces in hyperbolic 3-space by using extended Legendrian dualities between pseudo-hyperspheres in Lorentz–Minkowski 4-space. We define the flatness of a surface in hyperbolic 3-space by the degeneracy of its dual, which is similar to the case of the Gauss map of a surface in Euclidean 3-space. Such surfaces are a kind of ruled surfaces. Moreover, we investigate the singularities of these surfaces and the dualities of the singularities.

1. Introduction. A theorem on Legendrian dualities for pseudo-hyperspheres (hyperbolic n-space, de Sitter n-space and n-dimensional lightcone) in Lorentz–Minkowski $(n+1)$-space, which gives a commutative diagram between contact manifolds defined by the dual relations, was shown in [11] (see also [5, 21]). This theorem was extended in [17] to one-parameter families of pseudo-hyperspheres in Lorentz–Minkowski $(n+1)$-space depending on a parameter $\phi \in [0, \pi/2]$. As an application of this extended duality theorem, one-parameter families of extrinsic differential geometries of spacelike hypersurfaces in hyperbolic n-space, de Sitter n-space and n-dimensional lightcone were constructed in [2, 16, 17]. Here, we point out that the principal curvatures and the Gaussian curvature of each of these spacelike hypersurfaces depend on ϕ. This geometry which depends on ϕ is called *slant geometry*. Here, we emphasize that this geometry connects continuously horizontal geometry which corresponds to $\phi = 0$ and vertical geometry which corresponds to $\phi = \pi/2$ (see [2] for the details).

In this paper, using the extended Legendrian dualities between hyperbolic 3-space and one-parameter families of pseudo-hyperspheres in Lorentz–Minkowski 4-space, we consider one-parameter families of flat surfaces in hyperbolic 3-space. It is well-known that a flat surface in Euclidean 3-space is characterized by the degeneracy of its Gauss map. Moreover, a surface

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is developable if the image of its Gauss map is a point or a curve (that is, all points of the surface are singularities of its Gauss map). Furthermore, the dual of a surface plays a similar role to the Gauss map of the surface [26]. Taking into account these cases in Euclidean 3-space, we can consider the Legendrian dual of a surface in hyperbolic 3-space as a kind of Gauss map of the surface. Therefore, a surface in hyperbolic 3-space is flat if its Legendrian dual is singular at any point of the surface. In particular, we consider the case when the Legendrian dual is a spacelike curve in the dual pseudo-hypersphere. We point out that a surface in hyperbolic 3-space whose dual is valued in the 3-dimensional lightcone was studied in [13]. In this case, the surface is called a *horospherical flat* (briefly, *horo-flat*) *surface*. Such surfaces are one-parameter families of horo-cycles.

In this study, we consider surfaces in hyperbolic 3-space which have similar properties to horo-flat surfaces. While these surfaces are one-parameter families of horo-cycles for $\phi = 0$, they are not one-parameter families of horo-cycles for $\phi \in (0, \pi/2]$. However, they are one-parameter families of equidistant curves for $\phi \in (0, \pi/2]$ since the radiuses of the dual pseudo-hyperspheres given in one-parameter families of Legendrian dualities are non-zero. Thus, these surfaces connect continuously two important classes of surfaces: the one-parameter family of horo-cycles and the one-parameter family of geodesics. Flat cases of these surfaces connect horo-flat surfaces and analogies of tangent developable surfaces in hyperbolic 3-space. Our main results in this paper classify the singularities of these surfaces and show dualities among them. These surfaces are *frontals*, that is, projections of isotropic maps into the base space of a Legendrian fibration. If the isotropic map is a Legendrian immersion, the frontal is called a *wave front* (or briefly a *front*).

Singularities of wave fronts were originally investigated by Zakalyukin [31, 32]. See also [1] for the details. It was shown that generic singularities of wave front surfaces are cuspidal edges and swallowtails. It is known that generic singularities of frontal surfaces can also be cuspidal cross caps, in addition to the above two [7, 10]. In this paper, we show that flat one-parameter families of equidistant curves are always frontals, and generic singularities of a certain class of them can also be a cuspidal beaks in addition to the above three (Theorem 5.6(2)). Moreover, we discuss generic singularities of other classes of one-parameter families of equidistant curves (Theorems 4.4, 5.3 and 5.6(1)). See [5, 19, 20, 22, 25] for some other investigations of surfaces in hyperbolic 3-space.

In this paper, we deal with singularities of maps up to $\mathcal{A}$-equivalence among map germs. Two map germs $g_1, g_2 : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are $\mathcal{A}$*-equivalent* if there exist diffeomorphism germs $S : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ and $T : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ satisfying $g_2 \circ S = T \circ g_1$. Here, a *cuspidal edge* is a map germ which is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at the origin, a *swallowtail* is a map germ which is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at the origin, and a *cuspidal cross cap* is a map germ which is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, uv^3)$ at the origin. Moreover, dual surfaces can have a more degenerate singularity called cuspidal beaks. A *cuspidal beaks* is a map germ which is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, -2v^3 - u^2v, 3v^4 - u^2v^2)$ at the origin. The images of these singularities are shown in Figure 1.

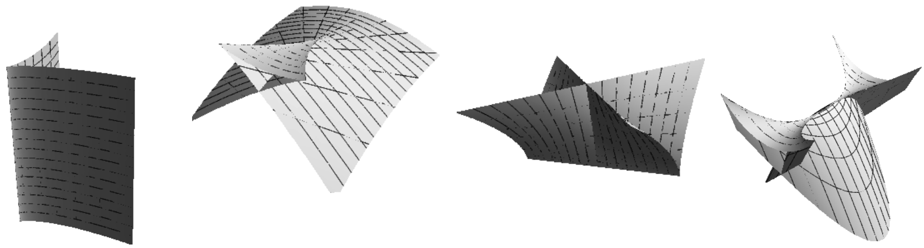

Fig. 1. Cuspidal edge, swallowtail, cuspidal cross cap and cuspidal beaks

On the other hand, there are many investigations of singularities of ruled surfaces and their analogies (see [12, 15, 23, 24] for example). Circular surfaces, that is, one-parameter families of circles are one of the analogies of ruled surfaces in Euclidean 3-space. They were investigated in [14]. Surfaces which we study here are one-parameter families of equidistant curves. These are also analogies of ruled surfaces.

Throughout this paper, we assume that all the maps and manifolds considered are of class C^∞.

2. One-parameter families of Legendrian dualities. Let $\mathbb{R}^{n+1}_1$ be the Lorentz–Minkowski $(n+1)$-space with the inner product $\langle\ ,\ \rangle = (-, +, \dots, +)$. Let $H^n(-c^2)$ (respectively, $S^n_1(c^2)$ and LC^*)$\subset \mathbb{R}^{n+1}_1$ denote *hyperbolic* n*-space with radius* c (respectively, *de Sitter* n*-space with radius* c and n*-dimensional* (*open*) *lightcone*) defined by

$$H^n(-c^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -c^2\},$$
$$S^n_1(c^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = c^2\},$$
$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0\}.$$

We denote $S^n_1(1)$ by S^n_1.

Recall that a point of a surface in a Riemannian space form is called *umbilical* if the shape operator at this point is a multiple of the identity. Moreover, a surface in a Riemannian space form is called *totally umbilic* if all of its points are umbilical. Furthermore, it is called *totally geodesic* if its second fundamental form is identically zero.

For a vector $\boldsymbol{p} \in S^n_1$ and $k \geq 0$, define

$$\Omega(\boldsymbol{p}, k) = \{\boldsymbol{x} \in H^n(-1) \mid \langle \boldsymbol{x}, \boldsymbol{p} \rangle = k\}.$$

It can be easily seen that $\Omega(\boldsymbol{p}, \sinh r)$ is a totally umbilic hypersurface of constant extrinsic curvature $\sinh^{n-1} r / \cosh^{n-1} r$ (cf. [4, p. 68]). We remark that $\Omega(\boldsymbol{p}, 0)$ is a totally geodesic hyperbolic $(n-1)$-space in $H^n(-1)$, and the distance from $\Omega(\boldsymbol{p}, 0)$ to $\Omega(\boldsymbol{p}, \sinh r)$ is r. The space $\Omega(\boldsymbol{p}, \sinh r)$ is said to be an *equidistant surface* (*from a geodesic plane*) when $n = 3$, and an *equidistant curve* (*from a geodesic line*) when $n = 2$.

On the other hand, for a vector $\boldsymbol{q} \in H^n(-1)$ and $k \leq -1$, we define

$$\Sigma(\boldsymbol{q}, k) = \{\boldsymbol{x} \in H^n(-1) \mid \langle \boldsymbol{x}, \boldsymbol{q} \rangle = k\}.$$

If $\cosh r \neq 1$, then $\Sigma(\boldsymbol{q}, -\cosh r)$ is a totally umbilic hypersurface of constant extrinsic curvature $\cosh^{n-1} r / \sinh^{n-1} r$ (see [4, p. 69]). We remark that $\Sigma(\boldsymbol{q}, -1)$ is a point and the distance from $\Sigma(\boldsymbol{q}, -1)$ to $\Sigma(\boldsymbol{q}, -\cosh r)$ is r. The space $\Sigma(\boldsymbol{q}, -\cosh r)$ is said to be a *sphere* (an *equidistant surface from a point*) when $n = 3$, and a *circle* (an *equidistant curve from a point*) when $n = 2$.

In order to give the Legendrian dualities which we will use throughout this paper, we now review some basic notions related to contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional manifold and K be a tangent hyperplane field on N. Locally such a field is defined as the field of zeros of a 1-form α. The field K is called *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case, K (respectively, α) is said to be a *contact structure* (respectively, a *contact form*).

A map $L : U \to N$ from an open set $U \subset \mathbb{R}^n$ to a contact manifold (N, K) is said to be *isotropic* if $dL_x(T_xU) \subset K_{L(x)}$ for any $x \in U$. An isotropic embedding $L : U \to N$ is said to be *Legendrian* if $\dim U = n$. Images of Legendrian embeddings are called *Legendrian submanifolds*.

A smooth fiber bundle $\pi : E \to N$ is said to be a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \to N$ be a Legendrian fibration. A map $f : U \to N$ is a *frontal* if there exists an isotropic map $L_f : U \to E$ such that $\pi \circ L_f = f$. If L_f is an immersion, then f is said to be a *wave front* (or briefly a *front*).

An example of a Legendrian fibration is the unit tangent bundle of a Riemannian manifold. Let N be an n-dimensional Riemannian manifold with a Riemannian metric g, and TN be its tangent bundle. Let $(x_1, \ldots, x_n)$ be the local coordinates on a neighborhood V of N and $(\xi_1, \ldots, \xi_n)$ be the coordinates on the fiber over V. Let g_{ij} be the components of the metric g with respect to the above coordinates. Let us define a one-form α locally by

$\alpha = \sum_{i,j} g_{ij}\xi_j dx_i$. Let T_1N be the unit tangent bundle with respect to the metric g. Then the restriction of α onto T_1N gives a contact structure, and $\tilde{\pi} : T_1N \to N$ is a Legendrian fibration (see [1, 3] for the details).

Now, set

$$\Delta_{21}^{-}(\phi) = \{(\boldsymbol{x}, \boldsymbol{y}) \in H^3(-1) \times S_1^3(\sin^2\phi) \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\cos\phi\}$$

for $0 < \phi < \pi/2$, and

$$\tilde{\Delta}_{21}^{-}(\phi) = \{(\boldsymbol{x}, \boldsymbol{y}) \in H^3(-1) \times H^3(-\sinh^2\phi) \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\cosh\phi\}$$

for $\phi > 0$. We follow [2, 16, 17] for some of these notations. Let Δ be $\Delta_{21}^{-}(\phi)$ or $\tilde{\Delta}_{21}^{-}(\phi)$. Then the restrictions of the standard one-forms

$$-x_0 dy_0 + \sum_{i=1}^{3} x_i dy_i \quad \text{and} \quad -y_0 dx_0 + \sum_{i=1}^{3} y_i dx_i$$

$$(\boldsymbol{x} = (x_0, \dots, x_3),\ \boldsymbol{y} = (y_0, \dots, y_3) \in \mathbb{R}_1^4)$$

to Δ define the same tangent hyperplane field over Δ. So, Δ is a contact manifold with contact form $(-x_0 dy_0 + \sum_{i=1}^{3} x_i dy_i)|_{\Delta}$. Let U be a domain in $\mathbb{R}^2$ and $L = (f, g) : U \to \Delta \subset \mathbb{R}_1^4 \times \mathbb{R}_1^4$ be a map. If L is isotropic, we say that f and g are Δ-*dual* to each other, g is a Δ-*dual* of f, and f is a Δ-*dual* of g. Let $L = (f, g) : U \to \Delta \subset \mathbb{R}_1^4 \times \mathbb{R}_1^4$ be a Legendrian immersion. Then each of f and g is a front. By comparing this duality with a surface in $\mathbb{R}^3$ and its unit normal vector, we define the extrinsic curvature of a surface in $H^3(-1)$ as follows:

Let $L = (f, g) : (U; (u, v)) \to \Delta \subset \mathbb{R}_1^4 \times \mathbb{R}_1^4$ be an isotropic map. We define

$$\begin{aligned} E &= \langle f_u, f_u \rangle, & F &= \langle f_u, f_v \rangle, & G &= \langle f_v, f_v \rangle, \\ L &= -\langle f_u, g_u \rangle, & M &= -\langle f_u, g_v \rangle, & N &= -\langle f_v, g_v \rangle, \end{aligned}$$

and call $E\,du^2 + 2F\,du\,dv + G\,dv^2$ the *first fundamental form*, $L du^2 + 2M du\,dv + N dv^2$ the *second fundamental form* (with respect to Δ), where $(\ )_u = \partial/\partial u$ and $(\ )_v = \partial/\partial v$. Moreover, we define

$$S = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

and call $\det S$ the *extrinsic curvature with respect to* Δ, or briefly the *extrinsic curvature* (cf. [11]).

Let us consider $\Delta_{21}^{-}(\phi)$. When $\phi = 0$, we have $\Delta_{21}^{-}(0) = \Delta_2^{-} = \{(\boldsymbol{x}, \boldsymbol{y}) \in H^3(-1) \times LC^* \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -1\}$, where LC^* denotes the 3-dimensional lightcone. Moreover, when $\phi = \pi/2$, we have $\Delta_{21}^{-}(\pi/2) = \Delta_1 = \{(\boldsymbol{x}, \boldsymbol{y}) \in H^3(-1) \times S_1^3 \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0\}$. Thus, $\Delta_{21}^{-}(\phi)$ connects the dualities Δ_2^{-} and Δ_1 (see [2, 11, 17]). So, we may regard that dual surfaces constructed by using $\Delta_{21}^{-}(\phi)$ connect dual surfaces constructed by using Δ_2^{-} (horo-flat surfaces [13]) and Δ_1 (usual extrinsic flat surfaces).

Let us now consider $\tilde{\Delta}_{21}^{-}(\phi)$. When $\phi = 0$, we have $\tilde{\Delta}_{21}^{-}(0) = \Delta_2^{-} = \{(\boldsymbol{x}, \boldsymbol{y}) \in H^3(-1) \times LC^* \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -1\}$. Furthermore, when $\phi = \sinh^{-1}(1) = \log(1+\sqrt{2})$, we get $\tilde{\Delta}_{21}^{-}(\sinh^{-1}(1)) = \{(\boldsymbol{x}, \boldsymbol{y}) \in H^3(-1) \times H^3(-1) \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\cosh(\sinh^{-1}(1)) = -\sqrt{2}\}$.

3. Dual surfaces. In this section, we construct dual surfaces in $H^3(-1)$ of spacelike curves in $S_1^3(\sin^2\phi)$ and $H^3(-\sinh^2\phi)$, respectively.

3.1. Frame. We now introduce a one-parameter frame field and its invariant. Let $\boldsymbol{a}_i : I \to \mathbb{R}_1^4$ $(i = 0, \dots, 3)$ be smooth maps such that

$$A(t) = {}^t(\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)) \in SO(3,1),$$

where $I \subset \mathbb{R}$ is an open interval. Here, a vector in $\mathbb{R}_1^4$ is a row vector and ${}^t(\cdot)$ denotes the transpose of a matrix. Let us define the following fundamental invariants:

$$\begin{aligned} c_1(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_1(t) \rangle, \quad c_2(t) = \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t) \rangle, \quad c_3(t) = \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_3(t) \rangle, \\ c_4(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_2(t) \rangle, \quad c_5(t) = \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_3(t) \rangle, \quad c_6(t) = \langle \boldsymbol{a}_2'(t), \boldsymbol{a}_3(t) \rangle, \end{aligned}$$

where $(\ )' = d/dt$. Then we can easily show that $C = A'A^{-1}$, where

$$C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ c_1(t) & 0 & c_4(t) & c_5(t) \\ c_2(t) & -c_4(t) & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \in \mathfrak{so}(3,1), \tag{3.1}$$

and $\mathfrak{so}(3,1)$ is the Lie algebra of $SO(3,1)$. In this sense, $C(t)$ is a Lorentzian invariant of $A(t)$. If we suppose that $\boldsymbol{a}_0$ is a unit speed curve in $H^3(-1) \subset \mathbb{R}_1^4$, $\boldsymbol{a}_1 = \boldsymbol{a}_0'$ and $\boldsymbol{a}_2$ (respectively, $\boldsymbol{a}_3$) is the principal-normal (respectively, binormal) vector of this curve, then we find that $c_1 = 1$ and $c_2 = c_3 = c_5 = 0$. Moreover, c_4 (respectively, c_6) coincides with the hyperbolic curvature (respectively, hyperbolic torsion) of this curve. Thus, $c_1, \dots, c_6$ represent the curvatures of the frame $\{\boldsymbol{a}_0, \dots, \boldsymbol{a}_3\}$.

For the converse, let $A : I \to SO(3,1)$ be a smooth curve. Then we can show that $A'A^{-1} \in \mathfrak{so}(3,1)$. Moreover, for any smooth curve $C : I \to \mathfrak{so}(3,1)$, if we apply the existence theorem for linear systems of ordinary differential equations, we conclude that there exists a unique curve $A : I \to SO(3,1)$ such that $C = A'A^{-1}$ with initial data $A(t_0)$.

3.2. $\Delta_{21}^{-}(\phi)$-dual surface of a spacelike curve in $S_1^3(\sin^2\phi)$. Let $\boldsymbol{\ell}_\phi : I \to S_1^3(\sin^2\phi)$ be a spacelike curve, and set $\boldsymbol{a}_3(t) = \boldsymbol{\ell}_\phi'(t)/|\boldsymbol{\ell}_\phi'(t)|$. Since $\boldsymbol{\ell}_\phi \in (\boldsymbol{a}_3)^\perp$, we have curves $\boldsymbol{a}_0(t)$ and $\boldsymbol{a}_2(t)$ satisfying $\boldsymbol{\ell}_\phi = \cos\phi\, \boldsymbol{a}_0 + \boldsymbol{a}_2$, $-\langle \boldsymbol{a}_0, \boldsymbol{a}_0 \rangle \equiv \langle \boldsymbol{a}_2, \boldsymbol{a}_2 \rangle \equiv 1$ and $\langle \boldsymbol{a}_0, \boldsymbol{a}_2 \rangle \equiv 0$. Let $\boldsymbol{a}_1(t) = \boldsymbol{a}_0(t) \wedge \boldsymbol{a}_2(t) \wedge \boldsymbol{a}_3(t)$

and $A(t) = {}^t\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$. Then $A \in SO(3,1)$ and $c_2 \equiv \cos\phi\, c_1 - c_4 \equiv 0$. Here, $\equiv$ means that the equality holds for any $t \in I$.

Consider the function

$$\Phi : H^3(-1) \times I \to \mathbb{R}, \qquad (\boldsymbol{x}, t) \mapsto \langle \boldsymbol{x}, \boldsymbol{\ell}_\phi(t)\rangle + \cos\phi.$$

Then for each $t \in I$, the first component of $\Phi^{-1}(0)$ coincides with the equidistant surface

$$\Omega\left(\frac{\boldsymbol{y}}{\sin\phi}, -\frac{\cos\phi}{\sin\phi}\right).$$

Set

$$\begin{aligned}(3.2)\quad f_A(s,t) &= f(s,t)\\ &= \frac{\cosh s - \cos^2\phi}{\sin^2\phi}\,\boldsymbol{a}_0(t) + \frac{\sinh s}{\sin\phi}\,\boldsymbol{a}_1(t) + \cos\phi\,\frac{\cosh s - 1}{\sin^2\phi}\,\boldsymbol{a}_2(t).\end{aligned}$$

Then the image of f is a part of the discriminant set of Φ. This means that f is the envelope of a one-parameter family of equidistant surfaces and it is a $\Delta_{21}^-(\phi)$-dual of $\boldsymbol{\ell}_\phi$. Moreover, for each $t \in I$, it is clear that $\{f(s,t) \mid s \in \mathbb{R}\} \subset \Omega(\boldsymbol{a}_3(t), 0) \cap \Omega(\boldsymbol{\ell}_\phi(t)/\sin\phi, -\cos\phi/\sin\phi)$. It can be easily seen that $s \mapsto f(s,t)$ is an equidistant curve in a hyperbolic plane in hyperbolic 3-space. Under the assumptions $c_2 \equiv \cos\phi\, c_1 - c_4 \equiv 0$, $\boldsymbol{\ell}_\phi$ is a $\Delta_{21}^-(\phi)$-dual of f. Thus, f is of constant extrinsic curvature zero with respect to $\Delta_{21}^-(\phi)$. We call f a *$\Delta_{21}^-(\phi)$-flat surface foliated by equidistant curves* or briefly a *flat surface foliated by equidistant curves*. Here, we remark that f has been constructed under the assumptions $c_2 \equiv \cos\phi\, c_1 - c_4 \equiv 0$ and $\boldsymbol{\ell}'_\phi \neq 0$. However, without these assumptions, we can define a surface f by a similar formula to that in (3.2). We then call f a *surface foliated by equidistant curves* when we do not have any assumptions on C.

Since $A(t) = {}^t(\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)) \in SO(3,1)$ is uniquely determined by $C(t)$ and initial data $A(t_0)$, we consider the space of each class of surfaces foliated by equidistant curves as follows: The space of surfaces foliated by equidistant curves is defined to be $C^\infty(I, \mathfrak{so}(3,1))$ with the Whitney C^∞-topology, and the space of $\Delta_{21}^-(\phi)$-flat surfaces foliated by equidistant curves is defined to be $C^\infty(I, \mathfrak{fes}(3,1))$ again with the Whitney C^∞-topology, where

$$\mathfrak{fes}(3,1) = \{C \in \mathfrak{so}(3,1) \mid c_2 = \cos\phi\, c_1 - c_4 = 0\}$$

and

$$(3.3)\qquad C = \begin{pmatrix} 0 & c_1 & c_2 & c_3\\ c_1 & 0 & c_4 & c_5\\ c_2 & -c_4 & 0 & c_6\\ c_3 & -c_5 & -c_6 & 0\end{pmatrix}.$$

Let us rewrite f as follows:

$$f(s,t) = -\cos\phi\,\frac{\cos\phi\,\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)}{\sin^2\phi} + \cosh s\,\frac{\boldsymbol{a}_0(t) + \cos\phi\,\boldsymbol{a}_2(t)}{\sin^2\phi} + \sinh s\,\frac{\boldsymbol{a}_1(t)}{\sin\phi}.$$

Furthermore, define

$$\boldsymbol{\gamma}(t) = -\cos\phi\,\frac{\cos\phi\,\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)}{\sin^2\phi},$$
$$\boldsymbol{d}_1(t) = \frac{\boldsymbol{a}_0(t) + \cos\phi\,\boldsymbol{a}_2(t)}{\sin^2\phi} \quad \text{and} \quad \boldsymbol{d}_2(t) = \frac{\boldsymbol{a}_1(t)}{\sin\phi}.$$

We call $\boldsymbol{\gamma}$ the *base curve*, $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ the *directrices*, and the equidistant curve $s \mapsto \boldsymbol{\gamma}(t) + \cosh s\,\boldsymbol{d}_1(t) + \sinh s\,\boldsymbol{d}_2(t)$ the *generating* (*equidistant*) *curve* of f. If f is flat, then $\mathrm{image}(f) = H^3(-1) \cap \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid d(\boldsymbol{\gamma}, \boldsymbol{x}) = -1/\sin^2\phi\}$. Thus, a flat surface foliated by equidistant curves is called a *canal surface*.

Example 3.1. Consider the matrix

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then we have the corresponding surface $f_C(s,t)$, foliated by equidistant curves, which is drawn in the Poincaré ball model of $H^3(-1)$ in Figure 2.

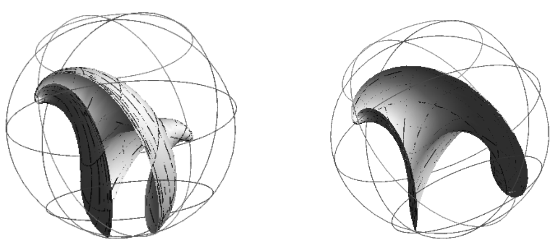

Fig. 2. The surface f_C foliated by equidistant curves; left: $\phi = \pi/12$, right: $\phi = \pi/4$

3.3. $\tilde{\Delta}^-_{21}(\phi)$-dual surface of a curve in $H^3(-\sinh^2\phi)$. Let $\boldsymbol{\ell}_\phi : I \to H^3(-\sinh^2\phi)$ be a curve, and take a frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ such that $\boldsymbol{a}_3(t) = \boldsymbol{\ell}'_\phi(t)/|\boldsymbol{\ell}'_\phi(t)|$, $\boldsymbol{\ell}_\phi = \cosh\phi\,\boldsymbol{a}_0 + \boldsymbol{a}_2$, and $A(t) = {}^t(\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)) \in SO(3,1)$. By the same arguments as in Subsection 3.2, we have the following $\tilde{\Delta}^-_{21}(\phi)$-dual surface $\tilde{f}$ of $\boldsymbol{\ell}_\phi$:

$$\tag{3.4} \begin{aligned}\tilde{f}_A(s,t) &= \tilde{f}(s,t)\\ &= \frac{\cos s+\cosh^2\phi}{\sinh^2\phi}\,\boldsymbol{a}_0(t) + \frac{\sin s}{\sinh\phi}\,\boldsymbol{a}_1(t) + \cosh\phi\,\frac{\cos s+1}{\sinh^2\phi}\,\boldsymbol{a}_2(t).\end{aligned}$$

It can be easily seen that $\tilde{f}$ is the envelope of a one-parameter family of spheres, and so $s \mapsto \tilde{f}(s,t)$ is a one-parameter family of circles. Moreover, $\tilde{f}$ is a $\tilde{\Delta}_{21}^{-}(\phi)$-dual of $\boldsymbol{\ell}_\phi$ under the assumptions $c_2 \equiv \cosh\phi\, c_1 - c_4 \equiv 0$. In this case, we call $\tilde{f}$ a *$\tilde{\Delta}_{21}^{-}(\phi)$-flat surface foliated by circles* (equidistant curves) or briefly a *flat surface foliated by circles*. We call $\tilde{f}$ a *surface foliated by circles* when we do not have any assumptions on C.

Now, define

$$\tilde{\gamma}(t) = \cosh\phi\,\frac{\cosh\phi\,\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)}{\sinh^2\phi},$$

$$\tilde{\boldsymbol{d}}_1(t) = \frac{\boldsymbol{a}_0(t) + \cosh\phi\,\boldsymbol{a}_2(t)}{\sinh^2\phi} \quad \text{and} \quad \tilde{\boldsymbol{d}}_2(t) = \frac{\boldsymbol{a}_1(t)}{\sinh\phi},$$

and call $\tilde{\gamma}$ the *base curve*, $\tilde{\boldsymbol{d}}_1$ and $\tilde{\boldsymbol{d}}_2$ the *directrices*, and the circle $s \mapsto \tilde{\gamma}(t) + \cos s\,\tilde{\boldsymbol{d}}_1(t) + \sin s\,\tilde{\boldsymbol{d}}_2(t)$ the *generating curve* (*circle*) of $\tilde{f}$. A flat surface foliated by circles is also called a *canal surface*. In what follows, the equidistant curves from a geodesic line or from a point are simply referred to as "equidistant curves".

We now consider the space of each class of surfaces foliated by circles. The space of surfaces foliated by circles is defined to be $C^\infty(I, \mathfrak{so}(3,1))$ with the Whitney C^∞-topology, and the space of $\tilde{\Delta}_{21}^{-}(\phi)$-flat surfaces foliated by circles is defined to be $C^\infty(I, \tilde{\mathfrak{fcs}}(3,1))$ again with the Whitney C^∞-topology, where

$$\tilde{\mathfrak{fcs}}(3,1) = \{C \in \mathfrak{so}(3,1) \mid c_2 = \cosh\phi\, c_1 - c_4 = 0\},$$

and C is a matrix as in (3.3).

4. Surfaces foliated by equidistant curves. In this section, we do not assume that $c_2 \equiv \cos\phi\, c_1 - c_4 \equiv 0$ (respectively, $c_2 \equiv \cosh\phi\, c_1 - c_4 \equiv 0$) for $\Delta_{21}^{-}(\phi)$ (respectively, $\tilde{\Delta}_{21}^{-}(\phi)$). The following definition corresponds to the non-cylindricity of a ruled surface in $\mathbb{R}^3$.

DEFINITION 4.1. A point (s,t) on a surface f_A (respectively, $\tilde{f}_A$), foliated by equidistant curves, is called a *non-$\Delta_{21}^{-}(\phi)$-flat point* (respectively, *non-$\tilde{\Delta}_{21}^{-}(\phi)$-flat point*), or briefly a *non-flat point*, if $(c_2, \cos\phi\, c_1 - c_4)(t) \neq (0,0)$ (respectively, $(c_2, \cosh\phi\, c_1 - c_4)(t) \neq (0,0)$). A surface f_A (respectively, $\tilde{f}_A$), foliated by equidistant curves, is *non-$\Delta_{21}^{-}(\phi)$-flat* (respectively, *non-$\tilde{\Delta}_{21}^{-}(\phi)$-flat*), or briefly *non-flat*, if any point on the surface f_A (respectively, $\tilde{f}_A$) is a non-flat point.

4.1. Striction curve. It is known that the singularities of a ruled surface are located on the striction curve, and there exists a unique striction curve on a non-cylindrical ruled surface (see [9, 15] for the details). In this subsection, we consider a curve on a surface foliated by equidistant curves with similar properties to those of the striction curve.

DEFINITION 4.2. Let f (respectively, $\tilde{f}$) be a surface foliated by equidistant curves as in (3.2) (respectively, (3.4)). A curve $\sigma : I \to U$ (or $f \circ \sigma$ (respectively, $\tilde{f} \circ \sigma$)) is a *striction curve* if $\langle (f \circ \sigma)', \boldsymbol{d}_1 \rangle \equiv \langle (f \circ \sigma)', \boldsymbol{d}_2 \rangle \equiv 0$ (respectively, $\langle (\tilde{f} \circ \sigma)', \tilde{\boldsymbol{d}}_1 \rangle \equiv \langle (\tilde{f} \circ \sigma)', \tilde{\boldsymbol{d}}_2 \rangle \equiv 0$).

We now suppose that f (respectively, $\tilde{f}$) is non-flat. For a function $s(t)$ (respectively, $\tilde{s}(t)$), the curve $\sigma(t) = (s(t), t)$ (respectively, $\tilde{\sigma}(t) = (\tilde{s}(t), t)$) is a striction curve if and only if

$$\sin\phi\, c_2(t)\cosh s(t) + (-\cos\phi\, c_1(t) + c_4(t))\sinh s(t) \equiv 0$$
$$\big(\text{respectively, } \sinh\phi\, c_2(t)\cos\tilde{s}(t) + (\cosh\phi\, c_1(t) - c_4(t))\sin\tilde{s}(t) \equiv 0\big).$$

Hence, in the case of f, if

$$|{-\cos\phi}\, c_1(t) + c_4(t)| > |\sin\phi\, c_2(t)|,$$

then there exists a striction curve of f, where

$$s(t) = \sinh^{-1}\left(\frac{\operatorname{sgn}\big(\cos\phi\, c_1(t) - c_4(t)\big)\sin\phi\, c_2(t)}{\sqrt{(\cos\phi\, c_1(t) - c_4(t))^2 - (\sin\phi\, c_2(t))^2}}\right).$$

On the other hand, since f is non-flat, there exist two striction curves of f. For a surface foliated by equidistant curves, the base curve is an image of the striction curve if and only if $c_2 \equiv 0$. Thus, if a striction curve exists, then we can assume that $c_2 \equiv 0$.

If f is flat, then any curve $t \mapsto (s(t), t)$ satisfies the condition for a striction curve.

4.2. Singularity. In this subsection, we study conditions for a point on a surface foliated by equidistant curves to be a Whitney umbrella, and show that a Whitney umbrella is a generic singularity of a surface foliated by equidistant curves. A map-germ $g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ at the origin is a *Whitney umbrella* if g is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, uv)$ at the origin.

By a direct calculation, we obtain

$$f_s(s,t) = \frac{1}{\sin^2\phi}\big(\sinh s\, \boldsymbol{a}_0(t) + \sin\phi\cosh s\, \boldsymbol{a}_1(t) + \cos\phi\sinh s\, \boldsymbol{a}_2(t)\big),$$

$$
\begin{aligned}
f'(s,t) = \frac{1}{\sin^2\phi}\Big[&\big(\sin\phi\, c_1(t)\sinh s + \cos\phi\, c_2(t)\,(\cosh s - 1)\big)\,\boldsymbol{a}_0(t)\\
&+ \big(c_1(t)(-\cos^2\phi + \cosh s) - \cos\phi\, c_4(t)\,(\cosh s - 1)\big)\,\boldsymbol{a}_1(t)\\
&+ \big(c_2(t)(-\cos^2\phi + \cosh s) + \sin\phi\, c_4(t)\sinh s\big)\,\boldsymbol{a}_2(t)\\
&+ \big(c_3(t)(-\cos^2\phi + \cosh s) + \sin\phi\, c_5(t)\sinh s\\
&\qquad + \cos\phi\, c_6(t)(\cosh s - 1)\big)\,\boldsymbol{a}_3(t)\Big],
\end{aligned}
$$

where $(\ )_s = \partial/\partial s$ and $(\ )' = \partial/\partial t$. Set $f'(s,t) = \sum_{i=0}^{3}\alpha_i(s,t)\boldsymbol{a}_i(t)$ and denote the set of singular points of f by $S(f)$. Then $(s,t)\in S(f)$ if and only if

$$
\sin\phi\, c_2(t)\cosh s + (-\cos\phi\, c_1(t) + c_4(t))\sinh s = 0, \quad \alpha_3(s,t) = 0. \tag{4.1}
$$

If $|{-\cos\phi\, c_1(t)} + c_4(t)| > |\sin\phi\, c_2(t)|$, then there exists a function $s(t)$ such that $(s(t),t)$ satisfies the first equation of (4.1). Take $\kappa_{cr}(t) = \alpha_3(s(t),t)$. Then by a direct calculation, we get

$$
\begin{aligned}
\kappa_{cr}(t) = {}&(c_3(t) + \cos\phi\, c_6(t))(-\cos\phi\, c_1(t) + c_4(t)) - \sin^2\phi\, c_2(t)c_5(t)\\
&- \operatorname{sgn}(c_4(t) - \cos\phi\, c_1(t))\cos\phi\,(c_6(t) + \cos\phi\, c_3(t))\sqrt{\rho(t)},
\end{aligned} \tag{4.2}
$$

where $\rho(t) = (c_4(t) - \cos\phi\, c_1(t))^2 - \sin^2\phi\, c_2^2(t)$.

Let $S(\tilde f)$ denote the set of singular points of $\tilde f$. In a similar vein, $(s,t)\in S(\tilde f)$ if and only if

$$
\sinh\phi\, c_2(t)\cos s + (\cosh\phi\, c_1(t) - c_4(t))\sin s = 0, \quad \tilde\alpha_3(s,t) = 0, \tag{4.3}
$$

where $\tilde f'(s,t) = \sum_{i=0}^{3}\tilde\alpha_i(s,t)\boldsymbol{a}_i(t)$. If $\tilde f$ is non-flat, then there exists a function $\tilde s(t)$ such that $(\tilde s(t),t)$ satisfies the first equation of (4.3). Put $\tilde\kappa_{cr}(t) = \tilde\alpha_3(\tilde s(t),t)$.

We now state conditions for a point on the surface f (respectively, $\tilde f$) to be a Whitney umbrella by using the function κ_{cr} (respectively, $\tilde\kappa_{cr}$):

Theorem 4.3.

(1) *The map-germ f at a non-flat point (s_0,t_0) is a Whitney umbrella if and only if*

$$
\sin\phi\, c_2(t_0)\cosh s_0 + (-\cos\phi\, c_1(t_0) + c_4(t_0))\sinh s_0 = 0,
$$
$$
\kappa_{cr}(t_0) = 0 \quad \textit{and} \quad \kappa'_{cr}(t_0) \neq 0.
$$

(2) *The map-germ $\tilde f$ at a non-flat point (s_0,t_0) is a Whitney umbrella if and only if*

$$
\sinh\phi\, c_2(t_0)\cos s_0 + (\cosh\phi\, c_1(t_0) - c_4(t_0))\sin s_0 = 0,
$$
$$
\tilde\kappa_{cr}(t_0) = 0 \quad \textit{and} \quad \tilde\kappa'_{cr}(t_0) \neq 0.
$$

Proof. The characterization of a Whitney umbrella is very well-known [30, p. 161] (see also [27, Remark 2.3]). Let $\varphi = \det(f_s, \eta f, \eta\eta f, f)$. For f, set $\eta = \alpha_1\,\partial_s - \sin\phi\cosh s\,\partial t$. Then f at (s_0,t_0) is a Whitney umbrella if and only if $\varphi(s_0,t_0)=0$ and $\varphi_s(s_0,t_0)\neq 0$. It can be easily seen that η spans the kernel of df at (s_0,t_0), and it is tangent to the striction curve $\sigma(t)$. Thus, on σ we have $\eta\det(\pi_0(f_s),\pi_0(\eta f),\pi_0(f))\equiv 0$, where $\pi_0(\sum_{i=0}^3 \delta_i\boldsymbol{a}_i)=\sum_{i=0}^2\delta_i\boldsymbol{a}_i$. Hence, $\det(\pi_0(f_s),\pi_0(\eta\eta f),\pi_0(f)) = 0$ at (s_0,t_0). Moreover, by a direct calculation, $\det(\pi_0(f_s),\,\pi_0(\eta f_s),\pi_0(f))\neq 0$ at (s_0,t_0). Thus, it is clear that $\varphi_s(s_0,t_0)\neq 0$ is equivalent to $\beta_3(s_0,t_0)\neq 0$, where $\eta\eta f=\sum_{i=0}^3\beta_i\boldsymbol{a}_i$. By a direct calculation, we find that $\beta_i(s_0,t_0)=(d/dt)\alpha_i(\sigma(t))|_{t=t_0}$. Thus, we have (1). We can obtain (2) in the same way. ∎

Taking into account the above characterization of a Whitney umbrella, we prove the following theorem:

THEOREM 4.4. *There exists a residual subset $\mathcal{O}_1\subset C^\infty(I,\mathfrak{so}(3,1))$ (respectively, $\tilde{\mathcal{O}}_1\subset C^\infty(I,\mathfrak{so}(3,1))$) such that the germ of the surface $f_A(s,t)$ (respectively, $\tilde f_{\tilde A}(s,t)$), foliated by equidistant curves, at any point (s_0,t_0) is an immersion or a Whitney umbrella for any $C\in\mathcal{O}_1$ (respectively, $\tilde C\in\tilde{\mathcal{O}}_1$). Here, $A: I\to SO(3,1)$ (respectively, $\tilde A: I\to SO(3,1)$) is the smooth curve corresponding to $C\in C^\infty(I,\mathfrak{so}(3,1))$ (respectively, $\tilde C\in C^\infty(I,\mathfrak{so}(3,1))$).*

Proof. Let us identify the 1-jet space as Euclidean space by

$$\begin{aligned}J^1(I,\mathfrak{so}(3,1)) &\simeq I\times\mathbb{R}^6\times\mathbb{R}^6\\ &= \{(t,c,d)\mid t\in I,\ c=(c_1,\dots,c_6)\in\mathbb{R}^6,\ d=(d_1,\dots,d_6)\in\mathbb{R}^6\},\end{aligned}$$

where $I\subset\mathbb{R}$, and define a polynomial $P(c)$ (respectively, $Q(c,d)$) of 6 variables $c_1,\dots,c_6$ (respectively, 12 variables $c_1,\dots,c_6,d_1,\dots,d_6$) as follows:

$$\begin{aligned}P(c) &= \big((c_3+\cos\phi\,c_6)(-\cos\phi\,c_1+c_4)-\sin^2\phi\,c_2c_5\big)^2\\ &\quad -\cos^2\phi\,(c_6+\cos\phi\,c_3)^2\big((c_4-\cos\phi\,c_1)^2-\sin^2\phi\,c_2^2\big),\\ Q(c,d) &= \big((c_3+\cos\phi\,c_6)(-\cos\phi\,c_1+c_4)-\sin^2\phi\,c_2c_5\big)\\ &\quad\times\big((d_3+\cos\phi\,d_6)(-\cos\phi\,c_1+c_4)\\ &\qquad\qquad + (c_3+\cos\phi\,c_6)(-\cos\phi\,d_1+d_4)-\sin^2\phi\,(d_2c_5+c_2d_5)\big)\\ &\quad -\cos^2\phi\,(c_6+\cos\phi\,c_3)(d_6+\cos\phi\,d_3)\big((c_4-\cos\phi\,c_1)^2-\sin^2\phi\,c_2^2\big)\\ &\quad -\cos^2\phi\,(c_6+\cos\phi\,c_3)^2\big((c_4-\cos\phi\,c_1)(d_4-\cos\phi\,d_1)-\sin^2\phi\,c_2d_2\big).\end{aligned}$$

Moreover, define the following subsets of $J^1(I,\mathfrak{so}(3,1))$:

$$\begin{aligned}S_1 &= \{(t,c,d)\in J^1(I,\mathfrak{so}(3,1))\mid c_2=\cos\phi\,c_1-c_4=0\},\\ S_2 &= \{(t,c,d)\in J^1(I,\mathfrak{so}(3,1))\mid P(c)=0\},\\ S_3 &= \{(t,c,d)\in J^1(I,\mathfrak{so}(3,1))\mid Q(c,d)=0\}.\end{aligned}$$

Then we can easily see that S_1 is a submanifold of codimension two, and S_2 and S_3 are algebraic subsets of codimension one. We remark that ϕ is a fixed number, and the coefficients of d_5 and d_6 in $Q(c,d)$ are

$$c_2\big(-c_3c_4+\cos\phi\, c_1c_3-\cos\phi\, c_4c_6+\cos^2\phi\, c_1c_6+(1-\cos^2\phi)\, c_2c_5\big)$$

and

$$(c_3c_4^2-2\cos\phi\, c_1c_3c_4+\cos^2\phi\, c_1^2c_3) \\ +(-c_4c_5+\cos\phi\, c_1c_5)c_2+(\cos\phi\, c_6+\cos^2\phi\, c_3)c_2^2,$$

respectively. We see that these coefficients do not have any common factors and they do not appear in $P(c)$. Thus, $S_2\cap S_3$ is an algebraic subset of codimension two. Therefore, we have stratifications of S_2, S_3 and $S_2\cap S_3$. We say that j^1C is transverse to S_2 (or S_3 or $S_2\cap S_3$) if j^1C is transverse to all of these stratifications. By Thom's jet transversality theorem (see [8, Theorem 4.9, p. 54], for example), $\mathcal{O}_1=\{C\in C^\infty(I,\mathfrak{so}(3,1))\mid j^1C$ is transverse to S_1,S_2,S_3 and $S_2\cap S_3\}$ is a residual subset in $C^\infty(I,\mathfrak{so}(3,1))$. On the other hand, $P(c)$ is constructed by taking the difference of the squares of two terms of κ_{cr} (see (4.2)), and $Q(c,d)$ is constructed from the differentiation of it. Furthermore, since the codimensions of S_1 and $S_2\cap S_3$ are both two, which is greater than $\dim I=1$, we conclude that j^1C is transverse to both S_1 and $S_2\cap S_3$. This implies that $j^1C(I)$ intersects neither S_1 nor $S_2\cap S_3$. Consequently, Theorem 4.3 implies that $\mathcal{O}_1$ satisfies the required conditions. We can obtain $\tilde{\mathcal{O}}_1$ in the same way. ■

5. Special classes of surfaces foliated by equidistant curves.

In this section, we consider a special class of surfaces foliated by equidistant curves other than the flat surfaces. Although these surfaces are not flat, they have similar properties to flat surfaces.

5.1. The cut-end surface of a circular saw.

In the case of ruled surfaces in Euclidean 3-space, the condition of flatness is equivalent to the condition that all generating lines are lines of curvature. Here we study the surface $f=f_A$, foliated by equidistant curves, such that all generating equidistant curves are lines of curvature. Since s is the parameter of generating equidistant curves, this condition is equivalent to

$$r(s,t):=\det(f_s,\nu,\nu_s,f)(s,t)\equiv 0\quad(\nu=f_s\wedge f'\wedge f).$$

Let us consider $r(s,t)+r_s(s,t)\equiv 0$ and $r(s,t)-r_s(s,t)\equiv 0$. Then

$$c_2(t)\equiv\cos\phi\, c_1(t)-c_4(t)\equiv 0\quad\text{or}\quad \cos\phi\, c_3(t)+c_6(t)\equiv 0.$$

The first condition gives $r(s,t)\equiv 0$. Under the second condition, $r(s,t)\equiv 0$ is equivalent to $c_5(t)\,c_2(t)+c_3(t)(\cos\phi\, c_1(t)-c_4(t))\equiv 0$. Thus, we have the following conditions:

$$\begin{aligned} &c_2(t)\equiv 0,\quad \cos\phi\, c_1(t)-c_4(t)\equiv 0\quad\text{or}\\ &\cos\phi\, c_3(t)+c_6(t)\equiv 0,\quad c_5(t)c_2(t)+c_3(t)(\cos\phi\, c_1(t)-c_4(t))\equiv 0.\end{aligned}\tag{5.1}$$

The first condition implies flatness. On the other hand, the second condition is interesting, because under this condition, $(s,t) \in S(f)$ if and only if

$$\sin\phi\, c_2(t)\cosh s + (-\cos\phi\, c_1(t) + c_4(t))\sinh s = 0.$$

This means that all points on the striction curve are singular points. Moreover, all of the generating equidistant curves are tangent to the striction curve.

Since $\cos\phi\, c_3(t) + c_6(t) = 0$, we have $\boldsymbol{\gamma}' \in \langle \boldsymbol{d}_1, \boldsymbol{d}_2\rangle_{\mathbb{R}}$. Let $\boldsymbol{\gamma}$ be a unit speed curve. If $\boldsymbol{\gamma}$ satisfies $|\boldsymbol{\gamma}'' - \boldsymbol{\gamma}| \neq 0$, then f can be written as

$$\boldsymbol{\gamma} + \frac{\sinh s\,\boldsymbol{\gamma}' + \cosh s\,(\cos\theta(t)\,\boldsymbol{n} + \sin\theta(t)\,\boldsymbol{e})}{\sin\phi},$$

where $\boldsymbol{n} = (\boldsymbol{t}' - \boldsymbol{\gamma})/\kappa_h$ and $\boldsymbol{e} = \boldsymbol{\gamma}\wedge\boldsymbol{t}\wedge\boldsymbol{n}$. Here, we denote $|\boldsymbol{t}' - \boldsymbol{\gamma}|$ by κ_h. The condition $c_5(t)c_2(t) + c_3(t)(\cos\phi\, c_1(t) - c_4(t)) = 0$ implies that $\theta' + \tau \equiv 0$, where $\tau = |\boldsymbol{\gamma}, \boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''|/\kappa_h^2$ is the torsion of $\boldsymbol{\gamma}$. So, this is an analogy of roller coaster surfaces in $\mathbb{R}^3$ ([14]).

This surface is a locus swept out by equidistant curves. Taking into account the Poincaré ball model of $H^3(-1)$, we can say that equidistant curves are parts of circles. As a result, we call this surface the *cut-end surface of a circular saw* (see Figure 3).

Since we are interested in the singularities of this surface, we consider it together with the striction curve. Taking the base curve as the striction curve, we can suppose that $c_2 \equiv 0$. Then (5.1) is equivalent to $c_2 \equiv c_3 \equiv c_6 \equiv 0$. Thus, the space of cut-end surfaces of circular saws is defined to be $C^\infty(I, \mathfrak{cs}(3,1))$ with the Whitney C^∞-topology, where

$$\mathfrak{cs}(3,1) = \{C \in \mathfrak{so}(3,1) \mid c_2 = c_3 = c_6 = 0\}$$

and C is a matrix as in (3.3).

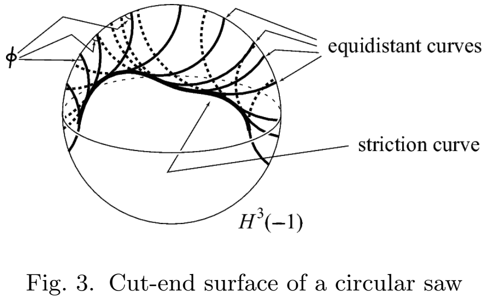

Fig. 3. Cut-end surface of a circular saw

We now consider the singularities of a non-flat cut-end surface of a circular saw. We fix a pseudo-orthonormal frame $A(t) = {}^t\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ corresponding to $C(t) \in \mathfrak{cs}(3,1)$ and suppose that $(\cos\phi\, c_1(t) - c_4(t), c_5(t))$

$\neq (0,0)$ for any $t \in I$. Moreover, set $f = f_A$ and take

$$\begin{aligned} \nu_c(s,t) = \delta(t)\biggl[&\frac{\cos\phi\, c_5(t)(1-\cosh s)}{\sin^2\phi}\,\boldsymbol{a}_0(t) - \frac{\cos\phi\, c_5(t)\sinh s}{\sin\phi}\,\boldsymbol{a}_1(t) \\ &+ \frac{c_5(t)(1-\cos^2\phi\cosh s)}{\sin^2\phi}\,\boldsymbol{a}_2(t) + (\cos\phi\, c_1(t) - c_4(t))\boldsymbol{a}_3(t)\biggr], \\ \delta(t) = \bigl(&(\cos\phi\, c_1(t) - c_4(t))^2 + c_5^2(t)\bigr)^{-1/2}. \end{aligned} \tag{5.2}$$

Then ν_c is an S^3_1-valued normal vector field of f. This means that the cut-end surface of a circular saw with $(\cos\phi\, c_1(t) - c_4(t), c_5(t)) \neq (0,0)$ is a frontal.

There are some criteria for the singularities of frontals. Let $(U; u, v)$ be an open domain in $\mathbb{R}^2$ and $h : U \to N$ be a front, where (N, g) is a 3-dimensional Riemannian manifold. The function defined by

$$\lambda(u,v) = \varOmega(h_u, h_v, \nu)$$

is said to be the *signed area density function*, where $\varOmega$ is a non-vanishing 3-form of N and $\nu = h_u \wedge h_v \wedge h$. Let $p \in U$ be a singular point of h. If $\operatorname{rank} dh_p = 1$, then p is said to be a *corank one singular point*. Assume that p is a corank one singular point. Then there exists a vector field η on U satisfying $dh_q(\eta_q) = 0$ for all singular points q. A vector field η with this property is said to be a *null vector field*. Let p be a singular point of h. If $d\lambda_p \neq 0$, then p is said to be a *non-degenerate* singular point. Let p be such a point. By the implicit function theorem, there exists a regular curve $\xi : (-\varepsilon, \varepsilon) \to U$ such that $\xi(0) = p$, and $\operatorname{image}(\xi) = S(h)$ in a neighborhood of p. Set

$$\psi(t) = \varOmega\left(\frac{dh(\xi(t))}{dt}, \nu(\xi(t)), (\eta\nu)(\xi(t))\right), \tag{5.3}$$

where $\eta\nu = D^h_\eta \nu$ is the canonical covariant derivative along the map h induced from the Levi-Civita connection on N. See [6] for the details. In the above notation, we state the following characterizations of the singularities of frontals [6, 13, 18, 28]:

Lemma 5.1. *Let h be a frontal and p be a corank one singular point of h. The map-germ h at p is*

1. *a cuspidal edge if and only if h is a front at p and $\eta\lambda(p) \neq 0$;*
2. *a swallowtail if and only if h is a front at p, $\eta\lambda(p) = 0$, $\eta\eta\lambda(p) \neq 0$ and $d\lambda_p \neq 0$;*
3. *a cuspidal beaks if and only if h is a front at p, $d\lambda(p) = 0$, $\det \operatorname{Hess} \lambda(p) < 0$ and $\eta\eta\lambda(p) \neq 0$;*
4. *a cuspidal cross cap if and only if $\eta\lambda(p) \neq 0$, $\psi(0) = 0$ and $\psi'(0) \neq 0$.*

See [18, Proposition 1.3] (see also [28, Corollary 2.5]), [13, Theorem A.1], [6, Corollary 1.5] for the details. We remark that if a function $\hat{\lambda}$ (respectively, $\hat{\psi}$) is proportional to λ (respectively, ψ), then we can use $\hat{\lambda}$ (respectively, $\hat{\psi}$) instead of λ (respectively, ψ) in the above lemma. Thus, we can characterize the singularities of non-flat cut-end surfaces of circular saws by the following theorem:

THEOREM 5.2. *Let* $c_2(t)=c_3(t)=c_6(t)=0$ *and* $(\cos\phi\, c_1(t) - c_4(t), c_5(t)) \neq (0,0)$ *for all* $t \in I$, *and set* $f = f_A$ *and* $\bar{\psi}(t) = c_5(t)(\cos\phi\, c_1'(t) - c_4'(t)) - c_5'(t)(\cos\phi\, c_1(t) - c_4(t))$. *Then* f *at* $(0, t_0) \in S(f)$ *is*

(1) *a cuspidal edge if and only if* $c_1(t_0) \neq 0$ *and* $\bar{\psi}(t_0) \neq 0$;
(2) *a swallowtail if and only if* $c_1(t_0) = 0$, $c_1'(t_0) \neq 0$ *and* $\bar{\psi}(t_0) \neq 0$;
(3) *a cuspidal cross cap if and only if* $c_1(t_0) \neq 0$, $\bar{\psi}(t_0) = 0$ *and* $\bar{\psi}'(t_0) \neq 0$.

Proof. From (5.2) and $S(f) = \{s = 0\}$, the signed area density function is proportional to $\lambda(s,t) = s$. Moreover, $\eta(t) = -\sin\phi\, c_1(t)\,\partial_s + \partial_t$ and $\psi(t) = c_1(t)\bar{\psi}(t)$ for the function ψ in (5.3). In view of Lemma 5.1, we get the desired result. ■

By means of this theorem, we have the following theorem:

THEOREM 5.3. *There exists a residual subset* $\mathcal{O}_{cs} \subset C^\infty(I, \mathfrak{cs}(3,1))$ *such that the germ of the cut-end surface of a circular saw* $f_A(s,t)$ *at any point* (s_0, t_0) *is an immersion, a cuspidal edge, a swallowtail or a cuspidal cross cap for any* $C \in \mathcal{O}_{cs}$. *Here,* $A : I \to SO(3,1)$ *is the smooth curve corresponding to* $C \in C^\infty(I, \mathfrak{cs}(3,1))$.

The proof of this theorem is the same as that of Theorem 4.4. So, we omit it.

5.2. Singularities of flat surfaces foliated by equidistant curves. In this section, we study the singularities of a $\Delta_{21}^-(\phi)$-flat surface foliated by equidistant curves. We recall that f_A is flat if and only if $c_2 \equiv \cos\phi\, c_1 - c_4 \equiv 0$. The space of flat surfaces foliated by equidistant curves is defined to be $C^\infty(I, \mathfrak{fe}(3,1))$ with the Whitney C^∞-topology, where

$$\mathfrak{fe}(3,1) = \{C \in \mathfrak{so}(3,1) \mid c_2 = \cos\phi\, c_1 - c_4 = 0\}$$

and C is a matrix as in (3.3).

Let $C \in \mathfrak{fe}(3,1)$ and take $f = f_A$, where A is the pseudo-orthonormal frame corresponding to C.

As mentioned in Sections 1 and 2, f can be lifted as an isotropic map to a contact manifold: $(f, \boldsymbol{\ell}_\phi) : U \to \Delta_{21}^-(\phi)$. Hence, f is always a frontal. By Lemma 5.1, we have the following theorem:

THEOREM 5.4. *Under the conditions* $c_2 \equiv \cos\phi\, c_1 - c_4 \equiv 0$, f *at* $(s_0, t_0) \in S(f)$ *is*

(1) *a cuspidal edge if and only if*

$$\cos\phi\, c_3(t_0)+c_6(t_0)\neq 0 \quad \textit{and} \quad -\sin\phi\, c_1(t_0)\,\lambda_s(s_0,t_0)+\lambda'(s_0,t_0)\neq 0;$$

(2) *a swallowtail if and only if*

$$\cos\phi\, c_3(t_0)+c_6(t_0)\neq 0,\quad -\sin\phi\, c_1(t_0)\,\lambda_s(s_0,t_0)+\lambda'(s_0,t_0)=0,$$

$\lambda_s(s_0,t_0)\neq 0$ *and* $\sin^2\phi\, c_1^2(t_0)\,\lambda_{ss}(s_0,t_0)-2\sin\phi\, c_1(t_0)\,\lambda_s'(s_0,t_0)-\sin\phi\, c_1'(t_0)\,\lambda_s(s_0,t_0)+\lambda''(s_0,t_0)\neq 0$*;*

(3) *a cuspidal cross cap if and only if* $\cos\phi\, c_3(t_0)+c_6(t_0)=0$*,* $\cos\phi\, c_3'(t_0)+c_6'(t_0)\neq 0$ *and* $-\sin\phi\, c_1(t_0)\,\lambda_s(s_0,t_0)+\lambda'(s_0,t_0)\neq 0$*;*

(4) *a cuspidal beaks if and only if*

$$\lambda_s(s_0,t_0)=\lambda'(s_0,t_0)=0,\quad \det\operatorname{Hess}\lambda(s_0,t_0)<0$$

and

$$\sin^2\phi\, c_1^2(t_0)\,\lambda_{ss}(s_0,t_0)-2\sin\phi\, c_1(t_0)\,\lambda_s'(s_0,t_0) \\ -\sin\phi\, c_1'(t_0)\,\lambda_s(s_0,t_0)+\lambda''(s_0,t_0)\neq 0.$$

Here,

$$\lambda(s,t)=\cos\phi\, c_6(t)(-1+\cosh s)+c_3(t)(-\cos^2\phi+\cosh s)+\sin\phi\, c_5(t)\sinh s.$$

Proof. Since the function λ defined as above is proportional to the signed area density function and the null vector field is $\eta(s,t)=-\sin\phi\, c_1(t)\,\partial_s+\partial_t$, Lemma 5.1 yields the desired result. ∎

If we assume that $c_2\equiv c_3\equiv \cos\phi\, c_1-c_4\equiv 0$, then f is a $\Delta_{21}^-(\phi)$-flat surface foliated by equidistant curves all of whose generating equidistant curves are tangent to the striction curve. Thus, we call this surface a *flat tangent surface foliated by equidistant curves*. This means that $S(f)\supset\{s=0\}\cup\{\cos\phi\, c_6\sinh s+\sin\phi\, c_5\,(1+\cosh s)\}$. The space of flat tangent surfaces foliated by equidistant curves is defined to be $C^\infty(I,\mathfrak{fte}(3,1))$ with the Whitney C^∞-topology, where

$$\mathfrak{fte}(3,1)=\{C\in\mathfrak{so}(3,1)\mid c_2=c_3=\cos\phi\, c_1-c_4=0\}$$

and C is a matrix as in (3.3).

In this case, we have the following corollary:

COROLLARY 5.5. *Suppose that* $c_2\equiv c_3\equiv\cos\phi\, c_1-c_4\equiv 0$*. Then* f *at* $(0,t_0)\in S(f)$ *is*

(1) *a cuspidal edge if and only if* $c_1(t_0)c_5(t_0)c_6(t_0)\neq 0$*;*

(2) *a swallowtail if and only if* $c_1(t_0)=0$ *and* $c_1'(t_0)c_5(t_0)c_6(t_0)\neq 0$*;*

(3) *a cuspidal cross cap if and only if* $c_6(t_0)=0$ *and* $c_1(t_0)c_5(t_0)c_6'(t_0)\neq 0$*;*

(4) *a cuspidal beaks if and only if* $c_5(t_0)=0$ *and* $c_1(t_0)(\cos\phi\, c_1(t_0)c_6(t_0)-2c_5'(t_0))\,c_5'(t_0)\,c_6(t_0)\neq 0$*.*

On the other hand, if $s_0 \neq 0$ *and*

$$(s_0, t_0) \in \{(s,t) \mid \cos\phi\, c_6(t) \sinh s + \sin\phi\, c_5(t)\,(1+\cosh s) = 0\},$$

then f *at* (s_0,t_0) *is a cuspidal edge if and only if*

$$\begin{aligned}-\sin\phi\, c_1(t_0)\big(\cos\phi\, c_6(t_0)\cosh s_0 + \sin\phi\, c_5(t_0)\sinh s_0\big)&\\ + \cos\phi\, c_6'(t_0)\sinh s_0 + \sin\phi\, c_5'(t_0)\,(1+\cosh s_0) &\neq 0.\end{aligned}$$

Using Theorem 5.4 and Corollary 5.5, we have the following theorem:

THEOREM 5.6. (1) *There exists a residual subset* $\mathcal{O}_3 \subset C^\infty(I, \mathfrak{fe}(3,1))$ *such that the germ of the flat surface* $f_A(s,t)$*, foliated by equidistant curves, at any point* (s_0,t_0) *is an immersion, a cuspidal edge, a swallowtail or a cuspidal cross cap for any* $C \in \mathcal{O}_3$*. Here,* $A : I \to SO(3,1)$ *is the smooth curve corresponding to* $C \in C^\infty(I, \mathfrak{fe}(3,1))$*.*

(2) *There exists a residual subset* $\mathcal{O}_4 \subset C^\infty(I, \mathfrak{fte}(3,1))$ *such that the germ of the flat tangent surface* $f_A(s,t)$*, foliated by equidistant curves, at any point* (s_0,t_0) *is an immersion, a cuspidal edge, a swallowtail, a cuspidal beaks or a cuspidal cross cap for any* $C \in \mathcal{O}_4$*. Here,* $A : I \to SO(3,1)$ *is the smooth curve corresponding to* $C \in C^\infty(I, \mathfrak{fte}(3,1))$*.*

The proof of this theorem is the same as that of Theorem 4.4. Thus, we omit it. We remark that a cuspidal beaks does not appear as a generic singularity of fronts. Therefore, we can say that flat tangent surfaces foliated by equidistant curves have a different geometric property from those of usual fronts.

6. Duality of singularities. The image of singular points of a $\Delta_{21}^-(\phi)$-flat tangent surface foliated by equidistant curves coincides with the image of $\boldsymbol{a}_0(t) \in H^3(-1)$. As in Subsection 3.2, using $\Delta_{21}^-(\phi)$, we can construct the dual surface in $S_1^3(\sin^2\phi)$ of $\boldsymbol{a}_0(t)$ as follows:

Let $\Psi : I \times S_1^3(\sin^2\phi) \to \mathbb{R}$ be defined by

$$\Psi(t, \boldsymbol{x}) = \langle \boldsymbol{a}_0(t), \boldsymbol{x}\rangle + \cos\phi.$$

Taking into account the discriminant set of Ψ, under the assumptions $c_2 \equiv c_3 \equiv 0$ and $\boldsymbol{a}_0'(t) \neq 0$, we have a parametrization of the dual surface g as follows:

$$g(s,t) = \cos\phi\, \boldsymbol{a}_0(t) + \cos s\, \boldsymbol{a}_2(t) + \sin s\, \boldsymbol{a}_3(t).$$

This is flat with respect to $\Delta_{21}^-(\phi)$ provided that $c_2 \equiv c_3 \equiv 0$. If we assume that $\cos\phi\, c_1 - c_4 \equiv 0$, then all of the generating circles are tangent to the striction curve. As in Subsection 5.1, we have the following theorem:

THEOREM 6.1. *Under the conditions* $c_2 \equiv c_3 \equiv \cos\phi\, c_1 - c_4 \equiv 0$*,* g *at* $(0,t_0) \in S(f)$ *is*

(1) *a cuspidal edge if and only if* $c_1(t_0)c_5(t_0)c_6(t_0) \neq 0$*;*

(2) *a swallowtail if and only if* $c_6(t_0) = 0$ *and* $c_1(t_0)c_5(t_0)c_6'(t_0) \neq 0$*;*
(3) *a cuspidal cross cap if and only if* $c_1(t_0) = 0$ *and* $c_1'(t_0)c_5(t_0)c_6(t_0) \neq 0$*;*
(4) *a cuspidal beaks if and only if* $c_5(t_0) = 0$ *and* $c_1(t_0)(\cos\phi\, c_1(t_0)c_6(t_0) + 2c_5'(t_0))c_5'(t_0)c_6(t_0) \neq 0$*.*

Proof. Since the method is the same as in the proofs of Theorems 5.2 and 5.4, we only show the fundamental data. Since

$$\begin{aligned} g_s(s,t) &= -\sin s\, \boldsymbol{a}_2(t) + \cos s\, \boldsymbol{a}_3(t), \\ g'(s,t) &= \big(-\cos\phi\, c_1(t)(-1+\cos s) - c_5(t)\sin s\big)\boldsymbol{a}_1(t) \\ &\qquad - c_6(t)\sin s\, \boldsymbol{a}_2(t) + c_6(t)\cos s\, \boldsymbol{a}_3(t), \end{aligned}$$

the signed area density function λ, the null vector field η and the function ψ of g are as follows: $\lambda(s,t) = \sin s(\cos\phi\, c_1(t)\sin s - c_5(t)(1+\cos s))$, $\eta(s,t) = c_6(t)\,\partial_s + \partial_t$ and $\psi(t) = c_1(t)c_6(t)$. ∎

Under the conditions $c_2 \equiv c_3 \equiv \cos\phi\, c_1 - c_4 \equiv 0$, the singular value of f is $\boldsymbol{a}_0(t)$, and the singular value of g is $\cos\phi\,\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)$. Moreover, $\boldsymbol{a}_0(t)$ is the $\Delta_{21}^-(\phi)$-dual of g, and $\cos\phi\,\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)$ is the $\Delta_{21}^-(\phi)$-dual of f. Consequently, we have the following diagram:

$$\begin{tikzcd} f \arrow[r, "\text{taking singular value}"] & \boldsymbol{a}_0(t) \arrow[d, "\Delta_{21}^-(\phi)\text{-dual}"'] \\ \cos\phi\,\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t) \arrow[u, "\Delta_{21}^-(\phi)\text{-dual}"] & g \arrow[l, "\text{taking singular value}"'] \end{tikzcd}$$

Thus, it is natural to expect a geometric relation between f and g. Table 1 gives the conditions defining the singularities of f and g.

Table 1. The duality of singularities, where C is $c_1c_5c_6 \neq 0$

	dual	$s=0$	ce	sw	ccr	cbk
f	$c_2 = 0$ $\cos\phi\, c_1 - c_4 = 0$	$c_3 = 0$	C	$c_1 = 0$ $c_1'c_5c_6 \neq 0$	$c_6 = 0$ $c_1c_5c_6' \neq 0$	$c_5 = 0$ $c_1(\cos\phi\, c_1c_6$ $-2c_5')c_5'c_6 \neq 0$
g	$c_2 = c_3 = 0$	$\cos\phi\, c_1 - c_4 = 0$	C	$c_6 = 0$ $c_1c_5c_6' \neq 0$	$c_1 = 0$ $c_1'c_5c_6 \neq 0$	$c_5 = 0$ $c_1(\cos\phi\, c_1c_6$ $+2c_5')c_5'c_6 \neq 0$

Here, we remark that there is a duality between the conditions for swallowtails and cuspidal cross caps. Such dualities have been studied by many researchers (for example, see [6, 12, 13, 29]). The duality which we show here is of the same type as in [12].

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References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps, Vol.* 1, Monogr. Math. 82, Birkhäuser Boston, Boston, MA, 1985.

[2] M. Asayama, S. Izumiya, A. Tamaoki and H. Yıldırım, *Slant geometry of spacelike hypersurfaces in hyperbolic space and de Sitter space*, Rev. Mat. Iberoamer. 28 (2012), 371–400.

[3] D. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progr. Math. 203, Birkhäuser, 2010.

[4] T. E. Cecil and P. J. Ryan, *Distance functions and umbilic submanifolds of hyperbolic space*, Nagoya Math. J. 74 (1979), 67–75.

[5] J. M. Espinar, J. A. Gálvez and P. Mira, *Hypersurfaces in* $\mathbb{H}^{n+1}$ *and conformally invariant equations: the generalized Christoffel and Nirenberg problems*, J. Eur. Math. Soc. 11 (2009), 903–939.

[6] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, Math. Z. 259 (2008), 827–848.

[7] A. B. Givental', *Singular Lagrangian manifolds and their Lagrangian mappings*, Itogi Nauki Tekh. Ser. Sovrem. Probl. Mat. Nov. Dostizh. 33 (1988), 55–112 (in Russian).

[8] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Grad. Texts in Math. 14, Springer, New York, 1973.

[9] A. Gray, *Modern Differential Geometry of Curves and Surfaces*, CRC Press, Boca Raton, FL, 1993.

[10] G. Ishikawa, *Infinitesimal deformations and stability of singular Legendre submanifolds*, Asian J. Math. 9 (2005), 133–166.

[11] S. Izumiya, *Legendrian dualities and spacelike hypersurfaces in the lightcone*, Moscow Math. J. 9 (2009), 325–357.

[12] S. Izumiya and K. Saji, *The mandala of Legendrian dualities for pseudo-spheres in Lorentz–Minkowski space and "flat" spacelike surfaces*, J. Singularities 2 (2010), 92–127.

[13] S. Izumiya, K. Saji and M. Takahashi, *Horospherical flat surfaces in hyperbolic 3-space*, J. Math. Soc. Japan 62 (2010), 789–849.

[14] S. Izumiya, K. Saji and N. Takeuchi, *Circular surfaces*, Adv. Geom. 7 (2007), 295–313.

[15] S. Izumiya and N. Takeuchi, *Singularities of ruled surfaces in* $\mathbb{R}^3$, Math. Proc. Cambridge Philos. Soc. 130 (2001), 1–11.

[16] S. Izumiya and H. Yıldırım, *Slant geometry of spacelike hypersurfaces in the lightcone*, J. Math. Soc. Japan 63 (2011), 715–752.

[17] S. Izumiya and H. Yıldırım, *Extensions of the mandala of Legendrian dualities for pseudo-spheres in Lorentz–Minkowski space*, Topology Appl. 159 (2012), 509–518.

[18] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic 3-space*, Pacific J. Math. 221 (2005), 303–351.

[19] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space and their caustics*, J. Math. Soc. Japan 59 (2007), 265–299.

[20] M. Kokubu, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space*, Pacific J. Math. 216 (2004), 149–175.

[21] H. Liu and S. D. Jung, *Hypersurfaces in lightlike cone*, J. Geom. Phys. 58 (2008), 913–922.

[22] H. Liu, M. Umehara and K. Yamada, *The duality of conformally flat manifolds*, Bull. Braz. Math. Soc. 42 (2011), 131–152.

[23] R. Martins and J. J. Nuño-Ballesteros, *Finitely determined singularities of ruled surfaces in* $\mathbb{R}^3$, Math. Proc. Cambridge Philos. Soc. 147 (2009), 701–733.

[24] S. Murata and M. Umehara, *Flat surfaces with singularities in Euclidean 3-space*, J. Differential Geom. 82 (2009), 279–316.

[25] P. Roitman, *Flat surfaces in hyperbolic 3-space as normal surfaces to a congruence of geodesics*, Tohoku Math. J. 59 (2007), 21–37.

[26] M. C. Romero Fuster, *Sphere stratifications and the Gauss map*, Proc. Roy. Soc. Edinburgh Sect. A 95 (1983), 115–136.

[27] K. Saji, *Criteria for cuspidal* S_k *singularities and their applications*, J. Gökova Geom. Topol. 4 (2010), 67–81.

[28] K. Saji, M. Umehara and K. Yamada, A_k *singularities of wave fronts*, Math. Proc. Cambridge Philos. Soc. 146 (2009), 731–746.

[29] O. P. Shcherbak, *Projectively dual space curves and Legendre singularities*, Sel. Math. Sov. 5 (1986), 391–421.

[30] H. Whitney, *The general type of singularitiy of a set of* $2n-1$ *smooth functions of* n *variables*, Duke Math. J. 10 (1943), 161–172.

[31] V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Funct. Anal. Appl. 10 (1976), 26–36.

[32] V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, J. Soviet Math. 27 (1984), 2713–2735.

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