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BOUNDED SOLUTIONS TO A SYSTEM OF SECOND ORDER ODES AND THE WHITNEY PENDULUM

Abstract. We give an existence theorem for bounded solutions to a system of second order ODEs. Dynamical applications are considered.

1. Introduction. Courant and Robbins [2] formulated a problem set up by H. Whitney. The problem is as follows.

> Suppose a train travels from station A to station B along a straight section of track. The journey need not be of uniform speed or acceleration. The train may act in any manner, speeding up, slowing down, coming to a halt, or even backing up for a while, before reaching B. But the exact motion of the train is supposed to be known in advance; that is, the function $s = w(t)$ is given, where s is the distance of the train from station A, and t is the time, measured from the instant of departure. On the floor of one of the cars a rod is pivoted so that it may move without friction either forward or backward until it touches the floor. If it does touch the floor, we assume that it remains on the floor henceforth; this will be the case if the rod does not bounce. Is it possible to place the rod in such a position that, if it is released at the instant when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey from A to B?

The authors gave a positive answer to this question. Their argument was informal. V. Arnold [1] considered this problem to be open.

A complete solution has been given by I. Polekhin in his Ph.D. thesis (unpublished) (see also [3]). He solved the problem by a direct application of results from [4].

In this article we prove a simple and general theorem which implies in particular that there are continuum many never-falling solutions to Whitney's problem; we also believe that this theorem describes many other similar

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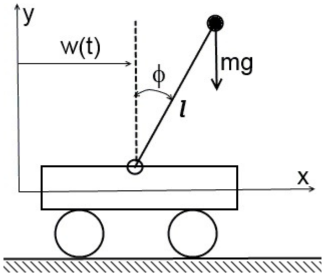

Fig. 1. The train with a pendulum; xy is the inertial frame.

effects. Our technique is based on Ważewski's ideas [5] which are applied to a second order ODE.

2. Main theorem. We introduce some notation. Let $M \subseteq \mathbb{R}^m$ be a domain and $\mathbb{R}_+$ stand for the non-negative reals. A function

$$f : H \to \mathbb{R}^m, \qquad H = \mathbb{R}_+ \times M \times \mathbb{R}^m,$$

is assumed to be C^2-smooth in H.

The main object of our study is the system of ordinary differential equations

$$\ddot{x} = f(t, x, \dot{x}). \tag{2.1}$$

Here $x = (x^1, \ldots, x^m)$ is the standard coordinate system in $\mathbb{R}^m$.

We will use scalar valued functions F on M and the sets

$$D_c = \{x \in M \mid F(x) < c\}, \quad \partial D_c = \{x \in M \mid F(x) = c\}.$$

THEOREM 2.1. *Suppose that there exists $F \in C^4(M)$ such that*

(1) *for some constant c the set D_c is homeomorphic to the open ball of $\mathbb{R}^m$;*
(2) *$\overline{D}_c$ is compact and $\overline{D}_c \subset M$;*
(3) *if $(t, x) \in \mathbb{R}_+ \times \partial D_c$ and $\xi \in \mathbb{R}^m$ then $dF(x)[\xi] = 0$ implies*

$$dF(x)[f(t, x, \xi)] + d^2F(x)[\xi, \xi] > 0; \tag{2.2}$$

(4) *if a solution $x(t)$ to problem (2.1) with $x(0) \in D_c$ is not defined for all $t \geq 0$ then it leaves the domain D_c, i.e. for some $t' > 0$ one has $x(t') \in \partial D_c$.*

Take any continuous vector field v on M such that

$$dF(x)[v(x)]\big|_{x \in \partial D_c} \geq 0.$$

Then there exists $y \in D_c$ such that system (2.1) has a solution

$$x(\cdot) \in C^4(\mathbb{R}_+), \quad x(0) = y, \quad \dot{x}(0) = v(y), \tag{2.3}$$

and $x(t) \in D_c$ for all $t \geq 0$.

REMARK 2.2. Actually there is no need to demand D_c to be homeomorphic to the ball. The theorem remains valid if we replace (1) with the following one: $\overline{D}_c$ is not continuously retractable to ∂D_c.

REMARK 2.3. From (3) it follows that $dF|_{\partial D_c} \neq 0$ and thus ∂D_c is a smooth manifold provided $m \geq 2$.

Suppose that (2.1) has the Lagrangian form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} - \frac{\partial T}{\partial x^i} = Q_i(t, x, \dot{x}), \quad T = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j.$$

Here g_{ij} is a Riemannian metric on M. Then inequality (2.2) takes the following invariant shape:

$$(\nabla_i\nabla_j F(x))\xi^i\xi^j + dF(x)[R(t, x, \xi)] > 0, \quad R^i = g^{ij}Q_j.$$

2.1. Continuum of never-falling solutions to the Whitney pendulum. By a suitable choice of units one can set $g = 1$, $l = 1$, $m = 1$. Then the motion of the pendulum is described by the equation

$$\ddot{\phi} = \sin\phi - \ddot{w}(t)\cos\phi, \quad w \in C^4(\mathbb{R}_+). \tag{2.4}$$

Set the initial conditions for this equation as follows:

$$\phi(0) = \psi, \quad \dot{\phi}(0) = \lambda((\pi/2)^2 - \psi^2). \tag{2.5}$$

PROPOSITION 2.4. *For any $\lambda \in \mathbb{R}$ there exists an angle $\psi \in (-\pi/2, \pi/2)$ such that problem* (2.4)–(2.5) *has a solution*

$$\phi(t) \in C^4(\mathbb{R}_+), \qquad |\phi(t)| < \pi/2 \quad \forall t \geq 0.$$

Indeed, this follows immediately from Theorem 2.1: just set

$$F(\psi) = \psi^2, \quad c = (\pi/2)^2, \quad D_c = (-\pi/2, \pi/2), \quad v(\psi) = \lambda((\pi/2)^2 - \psi^2);$$

then $\partial D_c = \{\pm\pi/2\}$.

If the function $\ddot{w}$ is bounded: $|\ddot{w}(t)| < C$ for all $t \in \mathbb{R}_+$, then the proposition can be made more precise. Denote by $\phi_0 \in (0, \pi/2)$ the root of the equation

$$\tan(\phi_0) = C.$$

Then the pendulum has a continuum of solutions $\phi(t)$ such that

$$|\phi(t)| < \phi_0 \quad \forall t \geq 0.$$

The argument is the same.

2.2. The ring on the rotating rod. A long enough rod rotates around the point O in the vertical plane. The point O is the middle of the rod. Let ϕ be the angle between the rod and the horizontal axis x. The law of rotation $\phi = \phi(t) \in C^3(\mathbb{R}_+)$ is known.

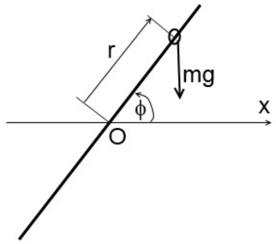

Fig. 2. The ring on the rotating rod.

The rod has a small ring put on it. The ring can slide over the rod without friction, also past the point O.

Setting $g = 1$ write the equation of motion as

$$\ddot{r} = (\dot{\phi}(t))^2 r - \sin(\phi(t)).$$

PROPOSITION 2.5. *Suppose that $\dot{\phi}^2(t) \geq C > 0$ for all $t \geq 0$. Then one can find an initial position of the ring such that the ring never slides off the rod, provided the rod is sufficiently long.*

Indeed, take a function $F(r) = r^2$. Then by Theorem 2.1 there exists a bounded solution $r(\cdot)$ with $|r(t)| < r_*$ and r_* can be chosen such that

$$r_* C > 1.$$

3. Proof of Theorem 2.1. Fix a vector field v on M. Denote by $x(\cdot, y)$ the solution with initial conditions (2.3).

Assume the converse: all the solutions to system (2.1) with initial conditions (2.3) leave the domain D_c.

Let $\tau(y)$, $y \in D_c$, be the first time when the solution $x(\cdot, y)$ meets ∂D_c. That is,

$$F(x(\tau(y), y)) = c. \tag{3.1}$$

For $y \in \partial D_c$ by definition set $\tau(y) = 0$.

LEMMA 3.1. *If $y \in D_c$ then $dF(x(\tau(y), y))[\dot{x}(\tau(y), y)] > 0$.*

Indeed, assume the converse: $dF(x(\tau(y), y))[\dot{x}(\tau(y), y)] \leq 0$. Then using the expansion

$$\begin{aligned} x(t, y) &= x(\tau(y), y) + \dot{x}(\tau(y), y)(t - \tau(y)) \\ &\quad + \frac{1}{2} f\big(\tau(y), x(\tau(y), y), \dot{x}(\tau(y), y)\big)(t - \tau(y))^2 + O(t - \tau(y))^3 \end{aligned}$$

we obtain

$$\begin{aligned}F(x(t,y)) = c &+ dF(x(\tau(y),y))[\dot{x}(\tau(y),y)]\cdot(t-\tau(y))\\ &+\frac{1}{2}\big(dF(x(\tau(y),y))[f(\tau(y),x(\tau(y),y),\dot{x}(\tau(y),y))]\\ &+d^2F(x(\tau(y),y))[\dot{x}(\tau(y),y),\dot{x}(\tau(y),y)]\big)\cdot(t-\tau(y))^2\\ &+O(t-\tau(y))^3.\end{aligned}$$

By condition (2.2) this implies that $F(x(t,y)) > c$ for small $\tau(y)-t>0$. But this is impossible since $x(t,y)$ is in D_c for $t<\tau(y)$.

Lemma 3.2. *The function $\tau(\cdot)$ is in $C(D_c)$.*

Indeed, by Lemma 3.1 this follows from the implicit function theorem being applied to equation (3.1).

Lemma 3.3. *The function $\tau(y)$ is a continuous function in $C(\overline{D}_c)$.*

Proof. Fix $\tilde{y}\in\partial D_c$, i.e. $F(\tilde{y})=c$. We have

$$x(t,y) = y+v(y)t+\frac{1}{2}f(0,y,v(y))t^2+\alpha(t,y). \tag{3.2}$$

Here $|\alpha(t,y)|\le Kt^3$ if $y\in D_c$ is sufficiently close to $\tilde{y}$ and t is small, with constant K independent of t,y. Substituting formula (3.2) to the equation $F(x(t,y))=c$ we obtain two substantially different situations.

The first one is when $dF(\tilde{y})[v(\tilde{y})]>0$. Then

$$\tau(y) = \frac{c-F(y)}{dF(y)[v(y)]}(1+\lambda(y)) \tag{3.3}$$

with $\lambda(y)\to 0$ as $y\to\tilde{y}$.

We do not give a detailed proof of (3.3) since the proof of an analogous fact but just more complicated is provided below.

From (3.3) it follows that $\tau(y)\to 0$ as $y\to\tilde{y}$.

The second case is $dF(\tilde{y})[v(\tilde{y})]=0$. By the assumptions of the theorem this implies

$$dF(\tilde{y})[f(0,\tilde{y},v(\tilde{y}))]+d^2F(\tilde{y})[v(\tilde{y}),v(\tilde{y})]>0.$$

Write

$$A(y)=\frac{1}{2}\big(dF(y)[f(0,y,v(y))]+d^2F(y)[v(y),v(y)]\big),\quad B(y)=dF(y)[v(y)],$$
$$C(y)=F(y)-c.$$

Recall that we assume that $y\in D_c$ is close to $\tilde{y}$, so that $C(y)<0$ and $A(y)\ge c'>0$ with some constant c'.

Equation (3.1) takes the form

$$A(y)\tau^2+B(y)\tau+C(y)+\gamma(\tau,y)=0. \tag{3.4}$$

Here $|\gamma(t,y)| \le Kt^3$ and $|\gamma_t(t,y)| \le Kt^2$, the constant K is positive, and $t \ge 0$ is small enough; and $B(y), C(y) \to 0$ as $y \to \tilde{y}$.

We seek for a solution to (3.4) in the form

$$\tau(y) = u(y)(1+\xi(y)), \quad u = \frac{-B+\sqrt{B^2-4AC}}{2A}.$$

Note that $u(y) > 0$ and $u(y) \to 0$ as $y \to \tilde{y}$.

The function ξ satisfies the equation

$$\xi + U(y)\xi^2 + p(y,\xi) = 0, \tag{3.5}$$

where

$$U(y) = \frac{1}{2}\Big(1 - \frac{B(y)}{\sqrt{B^2(y)-4A(y)C(y)}}\Big),$$

$$p(y,\xi) = \frac{\gamma(u(y)(1+\xi),y)}{u(y)\sqrt{B^2(y)-4A(y)C(y)}}.$$

It is easy to see that $p(y,\xi) \to 0$ as $y \to \tilde{y}$ and $0 \le U(y) \le 1$.

Consider the equation

$$\eta + s\eta^2 + p(y,\eta) = 0, \quad 0 \le s \le 1.$$

This equation implicitly defines a function $y \mapsto \eta(\cdot)$, $\eta(\cdot) \in C[0,1]$. Denote this function as $\eta(y,s)$. It exists by the implicit function theorem, and $\eta(y,s) \to 0$ uniformly in $s \in [0,1]$ as $y \to \tilde{y}$.

The solution to equation (3.5) takes the form $\xi(y) = \eta(y,U(y))$, and consequently $\xi(y) \to 0$ as $y \to \tilde{y}$.

The lemma is proved.

Now we can prove the theorem. By Lemma 3.3 the mapping $y \mapsto x(\tau(y),y)$ is a continuous retraction of the set $\overline{D}_c$ to its boundary. It is known that such a retraction does not exist.

This contradiction proves the theorem.

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References

[1] V. Arnold, *What is mathematics?*, MCCME, Moscow, 2002 (in Russian).

[2] R. Courant and H. Robbins, *What is Mathematics? An Elementary Approach to Ideas and Methods*, Oxford Univ. Press, 1996.

[3] I. Polekhin, *Inverted pendulum with moving pivot point: examples of topological approach*, arXiv:1407.4787 (2014).

[4] R. Srzednicki, K. Wójcik and P. Zgliczyński, *Fixed point results based on the Ważewski method*, in: Handbook of Topological Fixed Point Theory, R. Brown et al. (eds.), Springer, 2005, 903–941.

[5] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. 20 (1947), 279–313.

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