APPLICATIONES MATHEMATICAE
42,2-3 (2015), pp. 183–191

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SYMMETRY BREAKING IN THE MINIMIZATION OF THE FIRST EIGENVALUE FOR THE COMPOSITE CLAMPED PUNCTURED DISK

Abstract. Let $D_0 = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$ be the unit punctured disk. We consider the first eigenvalue $\lambda_1(\rho)$ of the problem $\Delta^2 u = \lambda\rho u$ in D_0 with Dirichlet boundary condition, where ρ is an arbitrary function that takes only two given values $0 < \alpha < \beta$ and is subject to the constraint $\int_{D_0} \rho\, dx = \alpha\gamma + \beta(|D_0| - \gamma)$ for a fixed $0 < \gamma < |D_0|$. We will be concerned with the minimization problem $\rho \mapsto \lambda_1(\rho)$. We show that, under suitable conditions on α, β and γ, the minimizer does not inherit the radial symmetry of the domain.

1. Introduction. We consider a vibrating plate Ω with clamped boundary $\partial\Omega$ and density ρ. The displacement u satisfies the following eigenvalue problem:

$$(1.1)\qquad \begin{cases} \Delta^2 u = \lambda\rho u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the bi-Laplacian (or biharmonic) operator, $\Omega \subset \mathbb{R}^2$ is an open bounded connected set with C^4 boundary $\partial\Omega$, n denotes the outer normal on $\partial\Omega$, $0 < \rho \in L^\infty(\Omega)$ and $\lambda \in \mathbb{R}$; recall that $u \in H_0^2(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega \Delta u \Delta v\, dx = \lambda \int_\Omega \rho u v\, dx \qquad \forall v \in H_0^2(\Omega).$$

The reader can find the justification of this mathematical setting in [18, Appendix A].

2010 *Mathematics Subject Classification*: 35J40, 47A75.
Key words and phrases: eigenvalue problem, minimization, symmetry breaking, punctured disk.

DOI: 10.4064/am42-2-5

Note that densities ρ that differ from each other on a subset of null measure give the same eigenvalue problem. The constant λ associated to a nontrivial solution u is called an *eigenvalue*, and u is a corresponding *eigenfunction*. The set of all eigenfunctions corresponding to an eigenvalue λ is a linear space, called the *eigenspace* associated to λ; its dimension is the *multiplicity* of λ. If the eigenspace is one-dimensional, λ is called *simple*.

In this paper we study the first (or fundamental) eigenvalue $\lambda_1(\rho)$ that, physically, represents the fundamental vibration of the clamped plate Ω.

We restrict our attention to plates made with only two homogeneous materials of densities α and β, with $0 < \alpha < \beta$. Moreover, we require that the portion of the plate with density α has a fixed Lebesgue measure $0 < \gamma < |\Omega|$, where $|\Omega|$ denotes the measure of Ω.

Mathematically, this means to consider $\rho = \alpha\chi_E + \beta(1 - \chi_E)$, where χ_E is the characteristic function of an arbitrary set $E \subset \Omega$ such that $|E| = \gamma$. We denote this class of densities by ad_γ.

An interesting question is to understand how these two materials must be placed in Ω to minimize or maximize $\lambda_1(\rho)$. In the following, $\check{\rho}_1$ (resp. $\hat{\rho}_1$) stands for a minimizer (resp. maximizer) of $\rho \mapsto \lambda_1(\rho)$.

The analogous problem in the case of a rod (one-dimensional case) was solved by Banks [4, 5] and Schwarz [19], who explicitly found the (unique) minimizer $\check{\rho}_1$ and the (unique) maximizer $\hat{\rho}_1$ (a survey on these problems and their history can be found in [16]). We remark that these extremizers are symmetric with respect to the midpoint of the rod.

In higher dimensions, the existence of a minimizer and its characterization in terms of the level sets of the associated eigenfunction have been proved in [10, 3]. From this characterization it follows that in order to minimize the fundamental frequency, we should place the more dense material where the plate displacement is the greatest.

Moreover, if the plate is a ball, in [3, Theorem 3.3] it is shown that $\check{\rho}_1$ is radially symmetric and decreasing. This result relies on the fact that the first eigenfunction is of one sign, which is a consequence of the positivity preserving property (see [15]) of problem (1.1) when Ω is a ball and $\rho = 1$. It is well known that, for general domains Ω, the first eigenfunction of problem (1.1) when $\rho = 1$ is sign-changing; classical examples of this phenomenon are the annulus, the square and the punctured disk (see [7, 8, 9, 11]). A survey on the positivity preserving property and on the positivity of the first eigenfunction for the biharmonic operator can be found in [13].

In this paper we consider the minimization of $\lambda_1(\rho)$ when Ω is the unit punctured disk $D_0 = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$. We show that, in this case, under some constraints on α, β and γ, every minimizer $\check{\rho}_1$ is not radially symmetric. We note that a similar phenomenon happens for the composite circular membrane. Mathematically the eigenvalue problem is

$$\begin{cases} -\Delta u = \lambda \rho u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a ball and the density ρ is defined as above.

In this context the first natural frequency of vibration is minimized when ρ is radially symmetric; this can be shown by using standard rearrangement methods. Symmetry breaking happens in the minimization of the second eigenvalue (see [2]). Other eigenvalue problems related to a composite membrane that exhibit symmetry breaking can be found in [6]: here the first eigenvalue in thin annuli and dumbbells with narrow handle is considered.

This article is organized in this way: in Section 2 we give some preliminary tools; Section 3 contains the main result.

2. Preliminaries. We list some further regularity properties of the first eigenfunctions.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded connected set with C^4 boundary $\partial\Omega$; if u_1 is a first eigenfunction of (1.1), then

$$u_1 \in H_0^2(\Omega) \cap H^4(\Omega) \cap C(\overline{\Omega}) \cap C_B^2(\Omega),$$

where $C_B^2(\Omega)$ denotes the set of functions with bounded and continuous first and second derivatives, and the equation in (1.1) holds almost everywhere (see [1, 12, 14]).

We denote by $\mathcal{S}_1(\rho)$ the eigenspace associated to $\lambda_1(\rho)$.

Now we recall a well known variational characterization of the fundamental eigenvalue.

Rayleigh's Principle. *Let $\lambda_1(\rho)$ the first eigenvalue of* (1.1). *Then*

$$\lambda_1(\rho) = \min_{\substack{u \in H_0^2(\Omega) \\ u \neq 0}} \frac{\int_\Omega (\Delta u)^2\,dx}{\int_\Omega \rho u^2\,dx}.$$

The quantity

$$\frac{\int_\Omega (\Delta u)^2\,dx}{\int_\Omega \rho u^2\,dx}$$

is called *Rayleigh's quotient.*

We define the set

$$ad_\gamma = \{\alpha\chi_E + \beta(1-\chi_E) : E \subset \Omega,\ |E| = \gamma\},$$

with $0<\alpha<\beta$ and $0<\gamma<|\Omega|$. We denote by $\check{\rho}_1 \in ad_\gamma$ a minimizer of the map $\rho \mapsto \lambda_1(\rho)$, that is,

$$\lambda_1(\check{\rho}_1) = \inf_{\rho \in ad_\gamma} \lambda_1(\rho). \tag{2.1}$$

From the results in [10], such a minimizer exists.

Let u_1 be an eigenfunction of $\lambda_1(\check{\rho}_1)$; in [10] it has been proved that there exists an increasing function Φ such that $\check{\rho}_1 = \Phi(u_1^2)$. The previous characterization of a minimizer can be stated in a more useful way for our purpose.

Proposition 2.1. *Let $\check{\rho}_1$ be a minimizer of problem* (2.1) *and u_1 a first eigenfunction of* (1.1). *Then there exists a constant $l>0$ such that*

$$\check{\rho}_1(x) = \begin{cases} \beta & \text{if } |u_1(x)| > l, \\ \alpha & \text{if } |u_1(x)| < l. \end{cases}$$

Moreover, the set $\{x \in \Omega : |u_1(x)| = l\}$ is contained either in $\{x \in \Omega : \check{\rho}_1(x) = \beta\}$ or in $\{x \in \Omega : \check{\rho}_1(x) = \alpha\}$.

Remark 2.1. By using the equation in (1.1), it follows that a level set corresponding to a positive value of $|u_1|$ has null measure. It follows that the set $\{x \in \Omega : \check{\rho}_1(x) = \beta\}$ can be assumed to be open.

Consider now the biharmonic eigenvalue problem (1.1) with $\rho = 1$. Coffman, Duffin and Shaffer studied this problem in ring-shaped domains and in the punctured disk $D_0 = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$. In [11] it was shown that if Ω is an annulus with inner radius small enough, then the first eigenvalue is not simple and the corresponding eigenfunctions are not of one sign.

We are mainly interested in D_0. We observe that, as already noted in [8], the correct formulation of the boundary conditions of problem (1.1) with $\Omega = D_0$ is

$$\begin{cases} u(0) = 0, \\ u = \dfrac{\partial u}{\partial n} = 0 \quad \text{if } |x| = 1. \end{cases}$$

In the same paper the authors, by using the properties of the Bessel functions, proved that, also in D_0, the first eigenfunction changes sign. More precisely, given the problem

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } D_0, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{if } |x| = 1, \\ u(0) = 0, \end{cases} \tag{2.2}$$

they prove that if ν^4 denotes the first eigenvalue of (2.2) and μ^4 denotes the eigenvalue of (2.2) with positive associated eigenfunction, then the associated eigenfunction to ν^4 is sign-changing and

$$\nu^4 < \mu^4.$$

The approximate values μ and ν found in [9] are

$$\mu = 4.768309396, \qquad \nu = 4.61089980. \tag{2.3}$$

3. Symmetry breaking. We obtain the main result in two steps. First, we show that the fundamental eigenvalue associated to a minimizing density must be simple. Second, relying on the result of [8], we get a contradiction by proving that, under suitable conditions on α and β, every radially symmetric density gives a multiple first eigenvalue.

THEOREM 3.1. *Let $\check\rho_1$ be a minimizer of problem* (2.1). *If* $0 < \gamma < |\Omega|$ *then $\lambda_1(\check\rho_1)$ is simple.*

Proof. Let $D = \{x \in \Omega : \check\rho_1(x) = \beta\}$. Given $u_1, \tilde u_1 \in \mathcal{S}_1(\check\rho_1)$, we show that u_1 and $\tilde u_1$ are linearly dependent. By Proposition 2.1 there exists a constant $l > 0$ such that $\{|u_1| > l\} \subseteq D \subseteq \{|u_1| \geq l\}$. From the continuity of u_1 it follows that $|u_1| \geq l$ on ∂D. It turns out that $|u_1| = l$ on ∂D; indeed, if there were $x \in \partial D$ such that $u_1(x) > l$, then there would exist a ball B centered at x such that $B \subseteq \{|u_1| > l\} \subseteq D$. This contradicts $x \in \partial D$.

Similarly, there exists $\tilde l > 0$ with $|\tilde u_1| = \tilde l$ on ∂D.

Now let $t, \tilde t \in \mathbb{R}$ be such that $tu_1 = \tilde t \tilde u_1 = 1$ on ∂D. Setting $v = tu_1 - \tilde t \tilde u_1$, we obtain $v = 0$ on ∂D. We have either $v \equiv 0$ or v is a first eigenfunction. In the latter case there would exist $\bar l > 0$ such that $v = \bar l$ on ∂D, which leads to a contradiction. Therefore, since $tu_1 - \tilde t \tilde u_1 = 0$ in Ω, u_1 and $\tilde u_1$ are linearly dependent. ∎

THEOREM 3.2. *Let $\Omega = D_0$, let $\rho \in ad_\gamma$ be radially symmetric and μ, ν defined as in the previous section. If*

$$\frac{\beta}{\alpha} < \frac{\mu^4}{\nu^4} \tag{3.1}$$

and $\lambda_1(\rho)$ is the first eigenvalue of (1.1), *then $\lambda_1(\rho)$ has multiplicity greater than one.*

Proof. By contradiction, suppose that $\lambda_1(\rho)$ is simple.

By Rayleigh's Principle we have

$$\lambda_1(\rho) \leq \lambda_{1,\alpha} = \frac{\nu^4}{\alpha}, \tag{3.2}$$

where $\lambda_{1,\alpha}$ is the first eigenvalue of the Dirichlet problem (1.1) with $\Omega = D_0$ and density $\rho = \alpha$.

Let $u_1 \in \mathcal{S}_1(\rho)$ be a first eigenfunction of (1.1). Let T_θ be the counterclockwise rotation of angle θ, $0 \le \theta < 2\pi$, around the origin. Note that, by assumption, $\rho \circ T_\theta = \rho$ for all θ. We define $u_{1,\theta} = u_1 \circ T_\theta$. By a change of variable we find

$$\int_\Omega (\Delta u_{1,\theta})^2\,dx = \int_\Omega (\Delta u_1)^2\,dx \quad \text{and} \quad \int_\Omega \rho u_{1,\theta}^2\,dx = \int_\Omega \rho u_1^2\,dx.$$

Then the Rayleigh quotients of $u_{1,\theta}$ and u_1 are the same, that is, $u_{1,\theta}$ is a first eigenfunction relative to $\lambda_1(\rho)$ for each θ. By the simplicity of $\lambda_1(\rho)$, there exists a constant c_θ such that $u_{1,\theta} = c_\theta u_1$; therefore $c_\theta u_1 = u_1 \circ T_\theta$. For fixed $x_0 \in \Omega$ with $u_1(x_0) \neq 0$, we find that

$$c_\theta = \frac{u_1 \circ T_\theta(x_0)}{u_1(x_0)}$$

is a continuous function of θ. On the other hand, from

$$\int_\Omega \rho u_{1,\theta}^2\,dx = \int_\Omega \rho u_1^2\,dx,$$

it follows that $c_\theta^2 = 1$. Note that $c_0 = 1$; then $c_\theta = 1$ and $u_1 = u_1 \circ T_\theta$ for all θ. It follows that u_1 is radially symmetric ($u_1 = u_1(r)$, $r = |x|$). Then the pair $(\lambda_1(\rho),\ u_1(r))$ is a solution of the problem

$$\begin{cases} \frac{1}{r}[(ru'')'' - (r^{-1}u')'] = \lambda\rho u & \text{in } (0,1), \\ u(0) = u(1) = u'(1) = 0. \end{cases} \tag{3.3}$$

Note that $\lambda_1(\rho) \ge \eta_1(\rho)$, where $\eta_1(\rho)$ is the first eigenvalue of (3.3).

Now we compare the fundamental eigenvalue of problem (3.3) with that of

$$\begin{cases} \frac{1}{r}[(ru'')'' - (r^{-1}u')'] = \lambda\beta u & \text{in } (0,1), \\ u(0) = u(1) = u'(1) = 0. \end{cases} \tag{3.4}$$

First, we compute the fundamental eigenvalue of (3.4). After setting $z(s) = u(s/\sqrt[4]{\beta\lambda})$, the equation in (3.4) becomes

$$\frac{1}{s}[(sz'')'' - (s^{-1}z')'] = z, \tag{3.5}$$

which can be factorized as follows (see [17]):

$$\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + 1\right)\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} - 1\right) z = 0.$$

A solution of this equation can be written in the form $z = z_1 + z_2$, where z_1

and z_2 satisfy

$$\frac{\partial^2 z_1}{\partial s^2}+\frac{1}{s}\frac{\partial z_1}{\partial s}+z_1=0 \quad \text{and} \quad \frac{\partial^2 z_2}{\partial s^2}+\frac{1}{s}\frac{\partial z_2}{\partial s}-z_2=0,$$

i.e. the Bessel equation and the modified Bessel equation of order zero. Therefore the complete solution of (3.5) is

$$z(s)=AJ_0(s)+BY_0(s)+CI_0(s)+DK_0(s),$$

where J_0, Y_0 and I_0, K_0 are, respectively, the Bessel functions and the modified Bessel functions of first and second kind (see, for instance, [20]). Then

$$u(r)=AJ_0(\sqrt[4]{\beta\lambda}\,r)+BY_0(\sqrt[4]{\beta\lambda}\,r)+CI_0(\sqrt[4]{\beta\lambda}\,r)+DK_0(\sqrt[4]{\beta\lambda}\,r).$$

Now we use the boundary conditions in (3.4). From $u(0)=0$ and the behavior of the Bessel functions near the origin it follows that $D=2C/\pi$ (see [17, p. 135] for more details) and $A+B=0$. The conditions $u(1)=u'(1)=0$ are satisfied if and only if λ is a root of the equation

$$\begin{aligned}(3.6)\qquad &\bigg[Y_0(\sqrt[4]{\beta\lambda})+\frac{2K_0(\sqrt[4]{\beta\lambda})}{\pi}\bigg]\big[I_0'(\sqrt[4]{\beta\lambda})-J_0'(\sqrt[4]{\beta\lambda})\big]\\ &\qquad=\bigg[Y_0'(\sqrt[4]{\beta\lambda})+\frac{2K_0'(\sqrt[4]{\beta\lambda})}{\pi}\bigg]\big[I_0(\sqrt[4]{\beta\lambda})-J_0(\sqrt[4]{\beta\lambda})\big].\end{aligned}$$

Then the first eigenvalue of (3.4) is the least positive root of (3.6), which, by [8], coincides with μ^4/β. By using the Rayleigh Principle and (3.2) we obtain

$$\frac{\mu^4}{\beta}\le\eta_1(\rho)\le\lambda_1(\rho)\le\lambda_{1,\alpha}=\frac{\nu^4}{\alpha},$$

which contradicts (3.1) and concludes the proof. ■

REMARK 3.1. We observe that the above proof also shows that

$$\min_{\substack{u\in H_0^2(\Omega)\\ u\,\text{radial},\,u\neq0}}\frac{\int_\Omega(\Delta u)^2\,dx}{\int_\Omega u^2\,dx}=\mu^4.$$

REMARK 3.2. Note that $\mu^4/\nu^4>1$ and so condition (3.1) is not meaningless. Moreover by (2.3) this ratio is close to 1 and in our model (which involves only the density) does not depend on the material properties.

THEOREM 3.3. *Let $\Omega=D_0$. If $0<\gamma<|\Omega|$ and*

$$\frac{\beta}{\alpha}<\frac{\mu^4}{\nu^4},$$

and $\check\rho_1$ is a minimizer of problem (2.1), *then $\check\rho_1$ cannot be radially symmetric.*

Proof. Compare Theorems 3.1 and 3.2. ∎

REMARK 3.3. We point out that in the case of the ball there are symmetry preservation results that do not depend on the density quotient (see [3] and [10]). Therefore an open problem is to understand if hypothesis (3.1) can be removed from Theorem 3.3.

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Received on 22.9.2014;
revised version on 4.9.2015 (2243)