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On the Existence of Free Ultrafilters on ω and on Russell-sets in ZF

by

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Dedicated to the memory of Horst Herrlich

Summary. In **ZF** (i.e. Zermelo–Fraenkel set theory without the Axiom of Choice **AC**), we investigate the relationship between $\mathbf{UF}(\omega)$ (*there exists a free ultrafilter on* ω) and the statements "there exists a free ultrafilter on every Russell-set" and "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A". We establish the following results:

1. $\mathbf{UF}(\omega)$ implies that there exists a free ultrafilter on every Russell-set. The implication is not reversible in **ZF**.
2. The statement "there exists a free ultrafilter on every Russell-set" is not provable in **ZF**.
3. If there exists a Russell-set A and a free ultrafilter on A, then $\mathbf{UF}(\omega)$ holds. The implication is not reversible in **ZF**.
4. If there exists a Russell-set A and a free ultrafilter on A, then there exists a free ultrafilter on every Russell-set.

We also observe the following:

(a) The statements $\mathbf{BPI}(\omega)$ (*every proper filter on* ω *can be extended to an ultrafilter on* ω) and "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A" are independent of each other in **ZF**.
(b) The statement "there exists a Russell-set and there exists a free ultrafilter on every Russell-set" is, in **ZF**, equivalent to "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A". Thus, "there exists a Russell-set and there exists a free ultrafilter on every Russell-set" is also relatively consistent with **ZF**.

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1. Terminology, known results and aim

Definition 1. (a) A *Russell-set* is a countable disjoint union $A = \bigcup\{A_n : n \in \omega\}$ of 2-element sets such that the family $\{A_n : n \in \omega\}$ has no partial choice function (i.e. $\{A_n : n \in \omega\}$ has no infinite subfamily with a choice function). The notion of a Russell-set was introduced in [9]. The existence of a Russell-set is relatively consistent with **ZF**—see the second Cohen model $\mathcal{M}7$ in [11] (or in [12, Section 5.4]), in which there is a countable family of pairs of sets of reals with no partial choice function. For an extensive study on Russell-sets and generalizations of this notion, the reader is referred to [5]–[10].

(b) Let X be a non-empty set. A non-empty subcollection $\mathcal{F}$ of $\mathcal{P}(X)\setminus\{\emptyset\}$ is called a *filter* on X if

(i) if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$,
(ii) if $F \in \mathcal{F}$, $F' \in \mathcal{P}(X) \setminus \{\emptyset\}$ and $F \subseteq F'$ then $F' \in \mathcal{F}$.

A non-empty collection $\mathcal{H} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ is called a *filter base* if for any $H_1, H_2 \in \mathcal{H}$ there is an $H_3 \in \mathcal{H}$ such that $H_3 \subseteq H_1 \cap H_2$. A filter base $\mathcal{H} \subseteq \mathcal{P}(X)\setminus\{\emptyset\}$ is called *free* (or non-principal) if $\bigcap \mathcal{H} = \emptyset$. Otherwise, $\mathcal{H}$ is called *non-free* (or principal).

A maximal, with respect to inclusion, filter on X is called an *ultrafilter* on X.

Notation. As usual, ω denotes the set of all natural numbers.

$\mathbf{UF}(\omega)$ is the statement: *There exists a free ultrafilter on* ω.

$\mathbf{BPI}(\omega)$ is the statement: *Every proper filter on* ω *can be extended to an ultrafilter on* ω.

For a set X, $[X]^{<\omega}$ denotes the set of all finite subsets of X, and $\mathrm{Fn}(X, 2)$ denotes the set of all *finite partial functions* from X into $2 = \{0, 1\}$, i.e. $p \in \mathrm{Fn}(X, 2)$ if and only if p is a function, $\mathrm{dom}(p) \in [X]^{<\omega}$ and $\mathrm{ran}(p) \subseteq 2$.

It is clear that, in **ZF**, $\mathbf{BPI}(\omega) \to \mathbf{UF}(\omega)$. Recently, in [3], it has been proved that the above implication is not reversible in **ZF**. To show this, a symmetric model N of **ZF** was constructed [3, Theorem 5.2], in which $\mathbf{UF}(\omega)$ is true, whereas $\mathbf{BPI}(\omega)$ is false. We label this result as Theorem 1 below.

Theorem 1 ([3]). $\mathbf{UF}(\omega)$ *does not imply* $\mathbf{BPI}(\omega)$ *in* **ZF**.

For the prospective reader, who may need more information on the principles $\mathbf{UF}(\omega)$ and $\mathbf{BPI}(\omega)$, we also recall a couple of known results which link these principles with compactness and forms of compactness of the Tychonoff product $2^{\mathbb{R}}$, where $\mathbb{R}$ is the set of the real numbers and 2 is the discrete 2-element space $\{0,1\}$. Further information can also be found in [4], [11], and [12].

THEOREM 2.

(i) ([14]) $\mathbf{BPI}(\omega)$ *if and only if the Tychonoff product* $2^{\mathbb{R}}$ *is compact.*
(ii) ([3], [15]) $2^{\mathbb{R}}$ *fails to be countably compact in the second Cohen model* $\mathcal{M}7$ *and in the symmetric model constructed in* [3], *hence* $\mathbf{BPI}(\omega)$ *fails in these models.*
(iii) ([17]) $\mathbf{UF}(\omega)$ *if and only if every countably infinite subset of* $2^{\mathbb{R}}$ *has an accumulation point.*

The aim of this paper is to investigate the relationship—in **ZF**—between the existence of free ultrafilters on ω and the existence of free ultrafilters on Russell-sets. It turns out that there is a *strong* connection between the two situations. In particular, we will show that, in **ZF**,

$$\begin{aligned}&(\text{there exists a Russell-set } A \text{ and a free ultrafilter } \mathcal{F} \text{ on } A)\\ &\qquad \to \mathbf{UF}(\omega) \to (\text{there exists a free ultrafilter on every Russell-set}),\end{aligned}$$

and that none of these implications is reversible in **ZF** (Theorems 5 and 3, respectively). We also show (in Theorem 4) that "there exists a free ultrafilter on every Russell-set" is not provable in **ZF**. In particular, we prove that the latter statement is false in Blass' model constructed in [1], where all ultrafilters are principal.

Finally, in Theorem 7 we observe that $\mathbf{BPI}(\omega)$ and "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A" are independent of each other in **ZF**, and in Theorem 8 we show that the statement "there exists a Russell-set and there exists a free ultrafilter on every Russell-set" is, in **ZF**, equivalent to "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A". Hence, "there exists a Russell-set and there exists a free ultrafilter on every Russell-set" is also relatively consistent with **ZF**.

2. Main results

THEOREM 3. $\mathbf{UF}(\omega)$ *implies that there exists a free ultrafilter on every Russell-set. Thus, the statement "there exists a free ultrafilter on every Russell-set" is relatively consistent with* $\mathbf{ZF} + \neg\mathbf{AC}$. *Furthermore, the above implication is not reversible in* **ZF**.

Proof. Assume $\mathbf{UF}(\omega)$. Let $A = \bigcup\{A_n : n \in \omega\}$ be a Russell-set. Let $f : A \to \omega$ be defined by $f(a) = n$ if and only if $a \in A_n$, and let $\mathcal{F}$ be a free ultrafilter on ω. Let $\mathcal{G}$ be the filter on the set A, which is generated by the free filter base $f^{-1}(\mathcal{F}) = \{\bigcup\{A_n : n \in F\} : F \in \mathcal{F}\}$. ($f^{-1}(\mathcal{F})$ is indeed a filter base since $\mathcal{F}$ is a filter, and it is free since $\mathcal{F}$ is free: Assuming the contrary, let $x \in \bigcap f^{-1}(\mathcal{F})$. Let n_0 be the unique $n \in \omega$ such that $x \in A_n$.

By the definition of $f^{-1}(\mathcal{F})$, it follows that $A_{n_0} \subseteq H$ for all $H \in f^{-1}(\mathcal{F})$, hence $A_{n_0} \in \{A_n : n \in F\}$ for every $F \in \mathcal{F}$. Consequently, $n_0 \in \bigcap \mathcal{F}$, contradicting the fact that $\mathcal{F}$ is free.) We assert that $\mathcal{G}$ is a (free) ultrafilter on A. If not, suppose $B \subseteq A$ meets every element of $\mathcal{G}$ in an infinite set, but does not include any of them. Let $M = \{n \in \omega : B \cap A_n \neq \emptyset\}$.

If M is finite, then $B \subseteq \bigcup\{A_n : n \in M\}$. Since $\mathcal{F}$ is a free ultrafilter on ω, it must contain every cofinite subset of ω, hence $\omega \setminus M \in \mathcal{F}$. Thus, $U = \bigcup\{A_n : n \in \omega \setminus M\} \in f^{-1}(\mathcal{F}) \subseteq \mathcal{G}$. However, $U \cap B = \emptyset$, a contradiction.

Thus, we may assume that M is infinite. Since $\mathcal{A}$ does not have a partial choice function, we may also conclude that there is a subset $N \subseteq M$ which is cofinite in M and such that $A_n \subseteq B$ for all $n \in N$. We claim that $N \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Assume on the contrary that $N \cap F = \emptyset$ for some $F \in \mathcal{F}$. Then $(\bigcup\{A_n : n \in N\}) \cap (\bigcup\{A_n : n \in F\}) = \emptyset$. As $M \setminus N$ is a finite subset of ω, and $\mathcal{F}$ is a free ultrafilter on ω, it follows that $K = \omega \setminus (M \setminus N) \in \mathcal{F}$. Since $\mathcal{F}$ is a filter, $K \cap F \in \mathcal{F}$ and consequently $C = \bigcup\{A_n : n \in K \cap F\} \in f^{-1}(\mathcal{F}) \subseteq \mathcal{G}$. It can be easily verified that $C \cap B = \emptyset$. (If $x \in C \cap B$, then $x \in A_n$ for some $n \in K \cap F$. Since x is also in B and $\mathcal{A}$ is disjoint, we must have $n \in N$. Therefore, $N \cap F \neq \emptyset$, a contradiction.) But this contradicts the fact that B meets every element of $\mathcal{G}$. Hence, $N \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ and consequently $N \in \mathcal{F}$, since $\mathcal{F}$ is an ultrafilter on ω. It follows that $G = \bigcup\{A_n : n \in N\} \in f^{-1}(\mathcal{F}) \subseteq \mathcal{G}$. Since $G \subseteq B$, this contradicts our assumption that B does not include any element of $\mathcal{G}$. Thus, $\mathcal{G}$ is an ultrafilter on A, proving our assertion.

The second assertion follows from the above result and the fact that $\mathbf{UF}(\omega)$ is true in the basic Cohen model $\mathcal{M}1$ (see [11]).

For the third assertion, consider Solovay's forcing model $\mathcal{M}5(\aleph)$, in which the axiom of countable choice (i.e. **AC** restricted to countable families of non-empty sets) is true (see [11]), hence in $\mathcal{M}5(\aleph)$ there are no Russell-sets. It follows that the statement "there exists a free ultrafilter on every Russell-set" is trivially true in this model. On the other hand, it is known that $\mathbf{UF}(\omega)$ fails in $\mathcal{M}5(\aleph)$ (see [11]), hence the implication in the statement of the theorem is not reversible in **ZF**. This completes the proof of the theorem. ∎

Remark 1. (a) The proof of Theorem 3 can be used without alterations in order to prove that $\mathbf{UF}(\omega)$ implies "for a pairwise disjoint family $\mathcal{A} = \{A_n : n \in \omega\}$ with $|A_n| \geq 2$ for each $n \in \omega$, if there is no partial Kinna–Wagner selection function (i.e. $\mathcal{A}$ has no infinite subfamily $\mathcal{B}$ and a function f on $\mathcal{B}$ such that for every $B \in \mathcal{B}$, $f(B)$ is a non-empty proper subset of B), then there is a free ultrafilter on $\bigcup \mathcal{A}$".

(b) Since $\mathbf{UF}(\omega)$ holds in every Fraenkel–Mostowski permutation model (see [11]) of **ZFA** (i.e. **ZF** with the Axiom of Extensionality modified in order to allow the existence of atoms), it follows (by Theorem 3) that the statement "there exists a free ultrafilter on every Russell-set" holds in each such model.

(c) The statement "there exists a free ultrafilter on every Russell-set" is relatively consistent with $\mathbf{ZF} + \neg\mathbf{BPI}(\omega)$. This follows from Theorems 1 and 3.

THEOREM 4. *The statement "there exists a free ultrafilter on every Russell-set" is not provable in* $\mathbf{ZF}$.

Proof. We shall use Blass' forcing model N from [1] (this is the model $\mathcal{M}15$ in [11]): Start with a countable transitive model M of $\mathbf{ZF} + V = L$. By forcing with finite partial functions from $\omega \times \omega$ into 2 (i.e. the partially ordered set of forcing conditions is $\mathrm{Fn}(\omega \times \omega, 2)$ equipped with reverse inclusion), obtain a model $M[G]$ with a sequence $(a_n)_{n\in\omega}$ of generic subsets of ω (for $n \in \omega$, $a_n = \{m \in \omega : \exists p \in G,\ p(n,m) = 1\}$, hence (by standard density arguments) $\omega \setminus a_n = \{m \in \omega : \exists p \in G,\ p(n,m) = 0\}$; further, $\dot{a_n} = \{(\check{m}, p) : m \in \omega,\ p \in \mathrm{Fn}(\omega \times \omega, 2),\ p(n,m) = 1\}$ and $\dot{\omega \setminus a_n} = \{(\check{m}, p) : m \in \omega,\ p \in \mathrm{Fn}(\omega \times \omega, 2),\ p(n,m) = 0\}$ are names for a_n and $\omega \setminus a_n$, respectively). For any $x \subseteq \omega$, let $\delta(x)$ be the set of reals whose symmetric difference with x is finite. Let $f \in M[G]$ be the function defined by $f(n) = \{\delta(a_n), \delta(\omega \setminus a_n)\}$, $n \in \omega$, and let

$$S = \bigcup\{\delta(a_n) \cup \delta(\omega \setminus a_n) : n \in \omega\} \cup \{f\}.$$

Let N be the submodel of $M[G]$ consisting of all sets in $M[G]$ which are hereditarily ordinal definable from S and finitely many members of S, i.e. $N = \mathrm{HOD}(S)$, or N is HOD over S in the terminology and notation of [13]. As Blass points out in [1, p. 329], N can also be described as consisting of the sets that are HOD from (finitely many) members of S—but not S itself—in $M[G]$, since S is certainly definable from f. (We note that Blass' model N is similar to Feferman's model constructed in [2] (and labeled as model $\mathcal{M}2$ in [11]), however N differs from Feferman's model in that f belongs to N.)

In [1], it is shown that all ultrafilters in N are principal. Now, by the definition of N, it follows that $\mathcal{R} = \{f(n) : n \in \omega\} = \{\{\delta(a_n), \delta(\omega \setminus a_n)\} : n \in \omega\}$ is a countable family of pairs in N (because $f \in N$), and we assert that $\mathcal{R}$ has no partial choice function in the model, thus $\bigcup \mathcal{R}$ is a Russell-set in N. To see this, assume, for contradiction, that there is an infinite subset $K \subseteq \omega$ and a bijection $g : \omega \to D$, where $D = \bigcup\{f(k) : k \in K\} = \bigcup\{\{\delta(a_k), \delta(\omega \setminus a_k)\} : k \in K\}$ (note that our assumption that there exists an infinite subfamily of $\mathcal{R}$ with a choice function is equivalent to "$\bigcup \mathcal{R}$ is a Dedekind-infinite set", i.e. $\bigcup \mathcal{R}$ has a countably infinite subset).

Since g is HOD from finitely many elements of S, there is a formula $\phi(x, w, z_1, \ldots, z_n, y_1, \ldots, y_m)$ in the language of set theory and elements

$s_1, \ldots, s_n$ of $S \setminus \{f\}$ and ordinals $\alpha_1, \ldots, \alpha_m$ such that

$$M[G] \models \forall x, (\phi(x, f, s_1, \ldots, s_n, \alpha_1, \ldots, \alpha_m) \Leftrightarrow (x = g)). \tag{1}$$

(We may as well assume that f is a parameter.) For each $k \in \omega$, $\delta(a_k)$ (and each element of $\delta(a_k)$) and $\delta(\omega \setminus a_k)$ (and each element of $\delta(\omega \setminus a_k)$) have canonical names, for example,

$$\delta(\dot{a_k}) = \{(a_k \dot{\triangle} x, \emptyset) : x \in [\omega]^{<\omega}\},$$

where

$$a_k \dot{\triangle} x = \{(\check{m}, p) : (m \in (\omega \setminus x) \wedge p(k, m) = 1) \vee (m \in x \wedge p(k, m) = 0)\},$$

and similarly for $\delta(\omega \setminus a_k)$. Therefore, each s_i has a canonical name $\dot{s_i} = a_{m_i} \dot{\triangle} x_i$, or $\dot{s_i} = (\omega \setminus \dot{a_{m_i}}) \triangle x_i$ (similar to $a_{m_i} \dot{\triangle} x_i$), where $m_i \in \omega$ and x_i is a finite subset of ω. Further, f has a canonical name

$$\dot{f} = \{(\mathrm{op}(\check{n}, \mathrm{up}(\delta(\dot{a_n}), \delta(\omega \dot{\setminus} a_n))), \emptyset) : n \in \omega\},$$

where op and up are the functions from Kunen [16, p. 191] (if $\bar{a}$ is a name for a and $\bar{b}$ is a name for b, then $\mathrm{op}(\bar{a}, \bar{b})$ and $\mathrm{up}(\bar{a}, \bar{b})$ are names for (a, b) and $\{a, b\}$, respectively).

Since K is an infinite set, there must be an element $k \in K$ such that $k \notin \{m_i : i = 1, \ldots, n\}$. This implies that the parameters $s_1, \ldots, s_n$ of ϕ do not belong to $\delta(a_k) \cup \delta(\omega \setminus a_k)$. Recall here that the generic subsets of ω, a_i, $i \in \omega$, form an independent family of subsets of ω, which implies that if r, s are distinct natural numbers, then $|a_r \triangle a_s| = \aleph_0$. As g is a function from ω onto $D = \bigcup\{\{\delta(a_m), \delta(\omega \setminus a_m)\} : m \in K\}$, there exists an $n^* \in \omega$ such that $g(n^*) = \delta(a_k)$. Then

$$M[G] \models \exists x, (\phi(x, f, s_1, \ldots, s_n, \alpha_1, \ldots, \alpha_m) \wedge (x(n^*) = \delta(a_k))). \tag{2}$$

Thus, there is a forcing condition $p \in G$ such that

$$p \Vdash \exists x, (\phi(x, \dot{f}, \dot{s_1}, \ldots, \dot{s_n}, \check{\alpha_1}, \ldots, \check{\alpha_m}) \wedge (x(\check{n^*}) = \delta(\dot{a_k}))). \tag{3}$$

Since p is finite, let $m_0 \in \omega$ be such that

$$\forall m \geq m_0,\ (k, m) \notin \mathrm{dom}(p), \tag{4}$$

and let

$$X = \{(k, m) \in \omega \times \omega : m \geq m_0\}.$$

Define $\pi_X : \mathrm{Fn}(\omega \times \omega, 2) \to \mathrm{Fn}(\omega \times \omega, 2)$ by

$$\pi_X(s)(u, v) = \begin{cases} s(u, v) & \text{if } (u, v) \notin X, \\ 1 - s(u, v) & \text{if } (u, v) \in X, \end{cases}$$

where s is any forcing condition and the equations above are interpreted to mean that if either side is defined then so is the other and they are equal. It is not hard to verify that π_X is an order automorphism of the partially ordered set $(\mathrm{Fn}(\omega \times \omega, 2), \supseteq)$. From (4) and the definitions of X (which is a subset of $\{k\} \times \omega$) and π_X, we conclude that:

1. $\pi_X(p) = p$.
2. π_X fixes the canonical names $\dot{\delta(a_r)}$ and $\dot{\delta(\omega \setminus a_r)}$ of $\delta(a_r)$ and $\delta(\omega \setminus a_r)$ for $r \in \omega \setminus \{k\}$, pointwise. Since $s_1, \ldots, s_n \in \bigcup\{\delta(a_r) \cup \delta(\omega \setminus a_r) : r \in \omega \setminus \{k\}\}$, we infer that $\pi_X(\dot{s_i}) = \dot{s_i}$ for all $i = 1, \ldots, n$.
3. π_X interchanges the names $\dot{\delta(a_k)}$ and $\dot{\delta(\omega \setminus a_k)}$.
4. $\pi_X(\dot{f}) = \dot{f}$ (due to items 2 and 3).

Applying π_X to (3) gives

(5) $\quad p \Vdash \exists x, (\phi(x, \dot{f}, \dot{s_1}, \ldots, \dot{s_n}, \check{\alpha_1}, \ldots, \check{\alpha_m}) \wedge (x(\check{n^*}) = \dot{\delta(\omega \setminus a_k)}))$.

So since $p \in G$, we conclude that

(6) $\quad M[G] \models \exists x, (\phi(x, f, s_1, \ldots, s_n, \alpha_1, \ldots, \alpha_m) \wedge (x(n^*) = \delta(\omega \setminus a_k)))$.

Then relations (1), (2) and (6) contradict the fact that g is a function. Thus, $\bigcup \mathcal{R}$ is a Russell-set in the model N as asserted.

By [1, Theorem, p. 331], there is no free ultrafilter on $\bigcup \mathcal{R}$ in N, finishing the proof of the theorem. ∎

THEOREM 5. *If there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A, then* **UF**(ω) *holds. The latter implication is not reversible in* **ZF**.

Proof. Assume that there exists a Russell-set $A = \bigcup\{A_n : n \in \omega\}$ and a free ultrafilter $\mathcal{F}$ on A. Let $f : A \to \omega$ be defined by $f(a) = n$ if and only if $a \in A_n$. Define

$$\mathcal{G} = \{M \in \mathcal{P}(\omega) : f^{-1}(M) \in \mathcal{F}\}.$$

Then $\mathcal{G}$ is free, that is, $\bigcap \mathcal{G} = \emptyset$, since $\mathcal{G}$ contains all the cofinite subsets of ω. Indeed, let M be a cofinite subset of ω. Since for all $K \in [\omega]^{<\omega}$, $\bigcup\{A_n : n \in K\}$ is finite and $\mathcal{F}$ is a free ultrafilter on A, it follows that $\bigcup\{A_n : n \in \omega \setminus K\} \in \mathcal{F}$ for all $K \in [\omega]^{<\omega}$. Since M is cofinite in ω, we thus have $f^{-1}(M) = \bigcup\{A_m : m \in M\} \in \mathcal{F}$, hence $M \in \mathcal{G}$. (Using the same argument, we see that $\mathcal{G}$ cannot contain any finite subset of ω.) Now, $\mathcal{F}$ being a filter on A easily implies that $\mathcal{G}$ is a filter on ω. Further, if N is an (infinite) subset of ω, then since $\mathcal{F}$ is an ultrafilter on A, it follows that either $W = \bigcup\{A_n : n \in N\} \in \mathcal{F}$ or $A \setminus W = \bigcup\{A_n : n \in \omega \setminus N\} \in \mathcal{F}$. By the definition of $\mathcal{G}$, this means that either $N \in \mathcal{G}$ or $\omega \setminus N \in \mathcal{G}$. Therefore, $\mathcal{G}$ is an ultrafilter on ω.

The second assertion of the theorem follows from the fact (see [11]) that **UF**(ω) holds in the basic Cohen model $\mathcal{M}1$, whereas there are no Russell-sets in $\mathcal{M}1$ (since the Boolean prime ideal theorem, that every non-trivial Boolean algebra has a prime ideal, and hence the axiom of choice for families of non-empty finite sets, are true in that model (see [11])). ∎

In view of Theorems 3 and 5, we immediately obtain the following result.

Theorem 6. *If there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A, then there exists a free ultrafilter on every Russell-set. The implication is not reversible in* **ZF**.

Remark 2. We point out that the fact that $\mathbf{UF}(\omega)$ lies in strength between the statements "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A" and "there exists a free ultrafilter on every Russell-set" (due to Theorems 3 and 5) was *crucial* to obtaining the result of Theorem 6. Indeed, in [9, Proposition 18], it was established that the Russell-set $X = \bigcup\{X_n : n \in \omega\}$ in the second Cohen model (model $\mathcal{M}7$ in [11]) satisfies: for any two subsets M and K of ω with infinite differences $M \setminus K$ and $K \setminus M$, the cardinalities $|\bigcup\{X_m : m \in M\}|$ and $|\bigcup\{X_k : k \in K\}|$ are *incomparable* in that model (i.e. there is neither a one-to-one function $f : \bigcup\{X_m : m \in M\} \to \bigcup\{X_k : k \in K\}$ nor a one-to-one function $g : \bigcup\{X_k : k \in K\} \to \bigcup\{X_m : m \in M\}$). It is also worth recalling here that in the second Cohen model, there exist $2^{\aleph_0}$ Russell-sets with pairwise incomparable cardinalities (see [9, Proposition 20]). Yet, we recall that if there exists a Russell-set, then there exist many Russell-sets; for example, if X is a Russell-set, then so are the sets $n \times X$ with $n \in \omega \setminus \{0\}$ (see [9, Proposition 7]).

Finally, taking also into account the previous results, we deduce the subsequent two results.

Theorem 7. *The statements* $\mathbf{BPI}(\omega)$ *and "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A" are independent of each other in* **ZF**.

Proof. In the proof of [3, Theorem 5.2], a symmetric model N was constructed in which $\mathbf{UF}(\omega)$ holds, whereas $\mathbf{BPI}(\omega)$ fails. Furthermore, in [3], it was shown that, in N, there is a disjoint countably infinite family $T = \{T_i : i \in \omega\}$ of pairs of sets of reals, having no partial choice function in N. Thus, $\bigcup T$ is a Russell-set in N. Since N satisfies $\mathbf{UF}(\omega)$, by Theorem 3 there is a free ultrafilter on $\bigcup T$. Hence, the statement "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A" holds in N.

The second assertion of the theorem follows from the fact that in the basic Cohen model $\mathcal{M}1$, $\mathbf{BPI}(\omega)$ is true, whereas there are no Russell-sets. ∎

Theorem 8. *The statement "there exists a Russell-set and there exists a free ultrafilter on every Russell-set" is, in* **ZF**, *equivalent to "there exists a Russell-set A and a free ultrafilter $\mathcal{F}$ on A". Thus, "there exists a Russell-set and there exists a free ultrafilter on every Russell-set" is also relatively consistent with* **ZF**.

Problem 1. Does the statement "there exists a Russell-set A and a free ultrafilter on A", or $\mathbf{UF}(\omega)$, hold in the second Cohen model $\mathcal{M}7$? We conjecture that the answer to this question is in the affirmative.

3. Summary of results. In the diagram below, we summarize the results of the paper.

$$\boxed{\text{There exists a Russell-set and there exists a free ultrafilter on every Russell-set}}$$

$$\updownarrow$$

$$\boxed{\text{There exists a Russell-set } A \text{ and a free ultrafilter } \mathcal{F} \text{ on } A}$$

$$\not\uparrow\ \not\downarrow \qquad\qquad \downarrow$$

$$\boxed{\mathbf{BPI}(\omega)} \rightarrow \boxed{\mathbf{UF}(\omega)}$$

$$\downarrow\ \not\uparrow$$

$$\boxed{\text{There exists a free ultrafilter on every Russell-set}}$$

$$\boxed{\mathbf{ZF} \nvdash \text{there exists a free ultrafilter on every Russell-set}}$$

Positive results and independence results for the principles studied in the paper

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