BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 63, No. 1, 2015

*NUMBER THEORY*

# $q$-Stern Polynomials as Numerators of Continued Fractions

by

Toufik MANSOUR

*Presented by Andrzej SCHINZEL*

**Summary.** We present a $q$-analogue for the fact that the $n$th Stern polynomial $B_n(t)$ in the sense of Klavžar, Milutinović and Petr [Adv. Appl. Math. 39 (2007)] is the numerator of a continued fraction of $n$ terms. Moreover, we give a combinatorial interpretation for our $q$-analogue.

**1. Introduction.** The *diatomic sequence* $b_n$ defined by the recurrence relation

$$b_1 = 1, \quad b_{2n} = b_n, \quad b_{2n+1} = b_n + b_{n+1}, \qquad n \geq 1,$$

has received a lot of attention in recent years (for example, see [2, 8–10] and references therein). In particular, Graham, Knuth and Patashnik [5, Exer. 6.50] have proved that if $n$ has binary representation

$$n = 1^{a_1} 0^{a_2} \cdots 1^{a_k} \qquad (a_1, \ldots, a_k > 0), \tag{1}$$

then $b_n$ is the numerator of the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}.$$

The sequence $b_n$ has been generalized to polynomials in a few different ways (see [4, 6]). For example, Klavžar, Milutinović and Petr [6] defined

2010 *Mathematics Subject Classification*: 11B37, 11B75, 11B83.
*Key words and phrases*: Stern polynomial, $q$-analogue, hyperbinary expansion, continued fraction.

DOI: 10.4064/ba63-1-2

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the polynomials

$$B_{2n}(t) = tB_n(t), \qquad B_{2n+1}(t) = B_n(t) + B_{n+1}(t)$$

with $B_0(t) = 0$ and $B_1(t) = 1$. Recently, Schinzel [8] showed that if (1) holds then the polynomial $B_n(t)$ is the numerator of the continued fraction

$$[a_1] + \cfrac{t^{a_1}}{[a_2] + \cfrac{t^{a_2}}{\ddots + \cfrac{t^{a_{k-1}}}{[a_k]}}}, \quad \text{where} \quad [a] = \frac{1-t^a}{1-t}.$$

Dilcher and Stolarsky [4] (also, see [2]) defined the polynomials

$$F_{2n}(q) = F_n(q), \qquad F_{2n+1}(q) = qF_n(q) + F_{n+1}(q)$$

with $F_0(q) = F_1(q) = 1$. Bates and Mansour [2] used these polynomials to define the $q$-analogue of the Calkin–Wilf [3] tree and the $q$-analogue of the hyperbinary expansion.

In this paper, we define a $q$-analogue of the polynomials $B_n(t)$ by

$$B_{2n}(q,t) = tB_n(q,t), \qquad B_{2n+1}(q,t) = qB_n(q,t) + B_{n+1}(q,t) \tag{2}$$

with $B_0(q,t) = 0$ and $B_1(q,t) = 1$. For example, $B_3 = q + t$, $B_4 = t^2$, $B_5 = q+(q+1)t$, $B_6 = qt+t^2$, $B_7 = q^2+qt+t^2$, $B_8 = t^3$, $B_9 = q+(q+1)t+qt^2$, $B_{10} = qt + (q+1)t^2$, $B_{11} = q^2 + q(q+2)t + t^2$, $B_{12} = qt^2 + t^3$, $B_{13} = q^2 + q(1+q)t + (q+1)t^2$, $B_{14} = q^2t + qt^2 + t^3$, $B_{15} = q^3 + q^2t + qt^2 + t^3$ and $B_{16} = t^4$.

We shall prove the following generalization of the result of Schinzel [8].

THEOREM 1.1. *If* (1) *holds then the polynomial* $B_n(q,t)$ *is the numerator of the continued fraction*

$$[a_1]_{q,t} + \cfrac{t^{a_1}}{q[a_2]_{1,t} + \cfrac{q^{a_3}t^{a_2}}{[a_3]_{q,t} + \cfrac{t^{a_3}}{q[a_4]_{1,t} + \cfrac{q^{a_5}t^{a_4}}{\cfrac{\ddots}{[a_k]_{q,t}}}}}}, \quad \textit{where} \quad [a]_{q,t} = \frac{q^a - t^a}{q-t}.$$

Note that by (2) and [8, Theorem 2] we can compute the degree of $B_n(q,t)$ as a polynomial in $t$, and by the main theorem of [2] we can also compute the degree of $B_n(q,t)$ as a polynomial in $q$ (see also [9, Corollary 3.8]).

## 2. Proofs.

We start with the following lemma.

LEMMA 2.1. *For all* $m \geq 0$, $B_{2^m}(q,t) = t^m$ *and* $B_{2^m-1}(q,t) = [m]_{q,t}$.

*Proof.* We proceed by induction on $m \geq 0$. Clearly, $B_1(q,t) = t$ and $B_0(q,t) = 0 = [0]_{q,t}$. Assume that $B_{2^m}(q,t) = t^m$ and $B_{2^m-1}(q,t) = [m]_{q,t}$. Then by (2) we have $B_{2^{m+1}}(q,t) = tB_{2^m}(q,t) = t^{m+1}$ and

$$\begin{aligned}B_{2^{m+1}-1}(q,t) &= B_{2(2^m-1)+1}(q,t) = qB_{2^m-1}(q,t) + B_{2^m} = q[m]_{q,t} + t^m \\ &= [m+1]_{q,t},\end{aligned}$$

which completes the induction. ∎

The next result generalizes Lemma 2.1 and [9, Theorem 2.5].

PROPOSITION 2.2. *For all $d \geq 0$ and $2^m \geq r \geq 0$,*

$$B_{2^m d+r}(q,t) = q^m B_{2^m-r}(1/q, t/q) B_d(q,t) + B_r(q,t) B_{d+1}(q,t).$$

*Proof.* We proceed by induction on $m \geq 0$. Since

$$\begin{aligned}B_d(q,t) &= q^0 B_{2^0-0}(1/q,t/q)B_d(q,t) + B_0(q,t)B_{d+1}(q,t), \\ B_{d+1}(q,t) &= q^0 B_{2^0-1}(1/q,t/q)B_d(q,t) + B_1(q,t)B_{d+1}(q,t),\end{aligned}$$

we find that the claim holds for $m = 0$. Assume that it holds for $m$ and let us prove it for $m = m'+1$ by induction on $r$. By Lemma 2.1,

$$\begin{aligned}B_{2^{m'+1}d+0}(q,t) &= t^{m'+1}B_d(q,t) \\ &= q^{m'+1}B_{2^{m'+1}-0}(1/q,t/q)B_d(q,t) + B_0(q,t)B_{d+1}(q,t),\end{aligned}$$

which proves the claim for $m = m'+1$ and $r = 0$. Assume that the claim holds for $m = m'+1$ and $r \geq 1$, and let us prove it for $m = m'+1$ and either $r = 2r'$ or $r = 2r'+1$. By (2) and the induction hypothesis,

$$\begin{aligned}B_{2^m d+r}(q,t) &= B_{2^{m'+1}d+2r'}(q,t) = tB_{2^{m'}d+r'}(q,t) \\ &= q^{m'}tB_{2^{m'}-r'}(1/q,t/q)B_d(q,t) + tB_{r'}(q,t)B_{d+1}(q,t) \\ &= q^{m'+1}B_{2^{m'+1}-2r'}(1/q,t/q)B_d(q,t) + B_{2r'}(q,t)B_{d+1}(q,t) \\ &= q^m B_{2^m-r}(1/q,t/q)B_d(q,t) + B_r(q,t)B_{d+1}(q,t)\end{aligned}$$

and

$$\begin{aligned}B_{2^m d+r}(q,t) &= B_{2^{m'+1}d+2r'+1}(q,t) = qB_{2^{m'}d+r'}(q,t) + B_{2^{m'}d+r'+1}(q,t) \\ &= q\big(q^{m'}B_{2^{m'}-r'}(1/q,t/q)B_d(q,t) + B_{r'}(q,t)B_{d+1}(q,t)\big) \\ &\quad + q^{m'}B_{2^{m'}-r'-1}(1/q,t/q)B_d(q,t) + B_{r'+1}(q,t)B_{d+1}(q,t) \\ &= q^{m'}\big(qB_{2^{m'}-r'}(1/q,t/q) + B_{2^{m'}-r'-1}(1/q,t/q)\big)B_d(q,t) \\ &\quad + (qB_{r'}(q,t) + B_{r'+1}(q,t))B_{d+1}(q,t) \\ &= q^{m'+1}\left(B_{2^{m'}-r'}(1/q,t/q) + \frac{1}{q}B_{2^{m'}-r'-1}(1/q,t/q)\right)B_d(q,t) \\ &\quad + B_{2r'+1}(q,t)B_{d+1}(q,t)\end{aligned}$$

$$= q^{m'+1}B_{2(2^{m'}-r'-1)+1}(1/q,t/q)B_d(q,t) + B_{2r'+1}(q,t)B_{d+1}(q,t)$$
$$= q^{m}B_{2^m-r}(1/q,t/q)B_d(q,t) + B_r(q,t)B_{d+1}(q,t),$$

which completes the induction on $r$ and $m$. ∎

COROLLARY 2.3. *For all $m \geq m' \geq 0$,*

$$B_{2^m-2^{m'}+1}(q,t) = q[m']_{1,t}[m-m']_{q,t} + t^{m-m'}.$$

*Proof.* Proposition 2.2 gives

$$\begin{aligned} &B_{2^m-2^{m'}+1}(q,t) \\ &\quad = q^{m'}B_{2^{m'}-1}(1/q,t/q)B_{2^{m-m'}-1}(q,t) + B_1(q,t)B_{2^{m-m'}-1+1}(q,t). \end{aligned}$$

Hence, by Lemma 2.1, we complete the proof. ∎

Let $x_1,\dots,x_k$ be positive integers. We define $K_k(x_1,\dots,x_k,;q,t)$ recursively by

$$\begin{aligned} K_{2k+1}(x_1,\dots,x_{2k+1};q,t) &= [x_{2k+1}]_{q,t}K_{2k}(x_1,\dots,x_{2k};q,t) \\ &\quad + q^{x_{2k+1}}t^{x_{2k}}K_{2k-1}(x_1,\dots,x_{2k-1};q,t), \\ K_{2k}(x_1,\dots,x_{2k};q,t) &= q[x_{2k}]_{1,t}K_{2k-1}(x_1,\dots,x_{2k-1};q,t) \\ &\quad + t^{x_{2k-1}}K_{2k-2}(x_1,\dots,x_{2k-2};q,t) \end{aligned} \tag{3}$$

with $K_0 = 1$ and $K_1(x_1;q,t) = [x_1]_{q,t}$.

LEMMA 2.4. *Let $k \geq 2$ and $x_1,\dots,x_{k-1}$ be positive integers. Then*

$$K_k(x_1,\dots,x_{k-2},x_{k-1}-1,1;q,t) = K_{k-1}(x_1,\dots,x_{k-1};q,t).$$

*Proof.* We proceed by induction on $k$. For $k=1$, we have $K_1(1;q,t) = 1 = K_0$. For $k=2$,

$$K_2(x_1-1,1;q,t) = q[x_1-1]_{q,t} + t^{x_1-1} = [x_1]_{q,t} = K_1(x_1;q,t).$$

Assume that the claim holds for $2k-2, 2k-1$, and let us prove it for $2k$ and $2k+1$. By induction hypothesis and (3), we have

$$\begin{aligned} &K_{2k}(x_1,\dots,x_{2k-2},x_{2k-1}-1,1;q,t) \\ &= qK_{2k-1}(x_1,\dots,x_{2k-1}-1;q,t) + t^{x_{2k-1}-1}K_{2k-2}(x_1,\dots,x_{2k-2};q,t) \\ &= (q[x_{2k-1}-1]_{q,t} + t^{x_{2k-1}-1})K_{2k-2}(x_1,\dots,x_{2k-2};q,t) \\ &\quad + q^{x_{2k-1}-1+1}t^{x_{2k-2}}K_{2k-3}(x_1,\dots,x_{2k-3};q,t) \\ &= [x_{2k-1}]_{q,t}K_{2k-2}(x_1,\dots,x_{2k-2};q,t) + q^{x_{2k-1}}t^{x_{2k-2}}K_{2k-3}(x_1,\dots,x_{2k-3};q,t) \\ &= K_{2k-1}(x_1,\dots,x_{2k-1};q,t). \end{aligned}$$

Similarly,

$$\begin{aligned}
&K_{2k+1}(x_1,\dots,x_{2k-1},x_{2k}-1,1;q,t)\\
&\qquad = K_{2k}(x_1,\dots,x_{2k}-1;q,t) + qt^{x_{2k}-1}K_{2k-1}(x_1,\dots,x_{2k-1};q,t)\\
&\qquad = q([x_{2k}-1]_{1,t} + t^{x_{2k}-1})K_{2k-1}(x_1,\dots,x_{2k-1};q,t)\\
&\qquad\quad + t^{x_{2k-1}}K_{2k-2}(x_1,\dots,x_{2k-2};q,t)\\
&\qquad = q[x_{2k}]_{1,t}K_{2k-1}(x_1,\dots,x_{2k-1};q,t) + t^{x_{2k-1}}K_{2k-2}(x_1,\dots,x_{2k-2};q,t)\\
&\qquad = K_{2k}(x_1,\dots,x_{2k};q,t),
\end{aligned}$$

which completes the induction. ∎

*Proof Theorem 1.1.* We prove the following general result:

$$B_n(q,t) = K_k(a_1,\dots,a_k;q,t), \tag{4}$$

where $n$ satisfies (1) with $k$ odd, $a_1,\dots,a_{k-2},a_k > 0$ and $a_{k-1} \ge 0$. We proceed by induction on $k$ (odd). For $k=1$, we have

$$B_n(q,t) = K_1(a_1) = [a_1]_{q,t},$$

so (4) holds. Assume $k \ge 3$ is odd, (1) holds and (4) is true for $k-2$. Then

$$n = 2^{a_{k-1}+a_k}d + 2^{a_k} - 1 \quad \text{with} \quad d = 1^{a_1}0^{a_2}\cdots 1^{a_{k-2}}$$

(binary representation). By Proposition 2.2, we have

$$\begin{aligned}
B_n(q,t) = {}& q^{a_{k-1}+a_k}B_{2^{a_{k-1}+a_k}-2^{a_k}+1}(1/q,t/q)B_d(q,t)\\
&+ B_{2^{a_k}-1}(q,t)B_{d+1}(q,t),
\end{aligned}$$

which, by Lemma 2.1 and Corollary 2.3, is equivalent to

$$B_n(q,t) = (q[a_k]_{q,t}[a_{k-1}]_{1,t} + q^{a_k}t^{a_{k-1}})B_d(q,t) + [a_k]_{q,t}t^{a_{k-2}}B_{(d+1)/2^{a_{k-2}}}(q,t).$$

Now, by the induction hypothesis, we have

$$B_d(q,t) = K_{k-2}(a_1,\dots,a_{k-2};q,t),$$

and by Lemma 2.4,

$$\begin{aligned}
B_{(d+1)/2^{a_{k-2}}} &= K_{k-2}(a_1,\dots,a_{k-4},a_{k-3}-1,1)\\
&= K_{k-3}(a_1,\dots,a_{k-3};q,t).
\end{aligned}$$

Hence,

$$\begin{aligned}
B_n(q,t) = {}& (q[a_k]_{q,t}[a_{k-1}]_{1,t} + q^{a_k}t^{a_{k-1}})K_{k-2}(a_1,\dots,a_{k-2};q,t)\\
&+ [a_k]_{q,t}t^{a_{k-2}}K_{k-3}(a_1,\dots,a_{k-3};q,t),
\end{aligned}$$

while by the definition (with $k$ odd)

$$\begin{aligned}
&K_k(a_1,\dots,a_k;q,t)\\
&\quad = q^{a_k-1}[a_k]_{1,t/q}K_{k-1}(a_1,\dots,a_{k-1};q,t) + q^{a_k}t^{a_{k-1}}K_{k-2}(a_1,\dots,a_{k-2};q,t)
\end{aligned}$$

$$\begin{aligned}
&= q^{a_k-1}[a_k]_{1,t/q}\big(q[a_{k-1}]_{1,t}K_{k-2}(a_1,\ldots,a_{k-2};q,t)\\
&\qquad\qquad\qquad\qquad\qquad + t^{a_{k-2}}K_{k-3}(a_1,\ldots,a_{k-3};q,t)\big)\\
&\quad + q^{a_k}t^{a_{k-1}}K_{k-2}(a_1,\ldots,a_{k-2};q,t)\\
&= (q[a_k]_{q,t}[a_{k-1}]_{1,t} + q^{a_k}t^{a_{k-1}})K_{k-2}(a_1,\ldots,a_{k-2};q,t)\\
&\quad + [a_k]_{q,t}t^{a_{k-2}}K_{k-3}(a_1,\ldots,a_{k-3};q,t).
\end{aligned}$$

Therefore, (4) holds for $k$, which completes the induction.

Hence, Theorem 1.1 follows in view of [7, Section 5]. ■

**3. Combinatorial issues.** The *hyperbinary expansion* of a number $n$ is an expansion of $n$ as a sum of powers of 2, each power being used at most twice. We denote the set of all hyperbinary expansions of $n$ by $\mathbb{H}_n$, and the total number of powers that are used exactly twice (resp. once) in the hyperbinary expansion $x \in \mathbb{H}_n$ by $\mathfrak{h}_n(x)$ (resp. $\ell_n(x)$). The $(q,t)$-hyperbinary expansion of $x$ is defined as $q^{\mathfrak{h}_n(x)}t^{\ell_n(x)}$. See [2] in the case $t = 1$. Let $f_n(q,t)$ be the polynomial of the sum of $(q,t)$-hyperbinary expansions of $n$ with $f_0(q,t) = 1$ and $f_{-1}(q,t) = 0$. For example, the hyperbinary expansions of 6 are $4+2$, $4+1+1$ and $2+2+1+1$. Thus $(q,t)$-hyperbinary expansions of 6 are $t^2$, $qt$ and $q^2$. Accordingly, $f_6(q,t) = t^2 + qt + q^2 = [3]_{q,t}$.

Theorem 3.1. *For all $n \ge 0$, $B_n(q,t) = f_{n-1}(q,t)$.*

*Proof.* We proceed by induction on $n$. The conclusion is true for $n = 0,1$. Assume that it holds for $0,1,\ldots,2n$ and let us prove it for $2n+1$ and $2n+2$.

*The case* $2n+1$: By using the proof of the case $f_{2n+1}(1,1) = f_n(1,1)$ with $q = t = 1$ in [1, Theorem 2], there exists a bijection $\alpha : \mathbb{H}_{2n+1} \to \mathbb{H}_n$ such that $\mathfrak{h}_{2n+1}(x) = \mathfrak{h}_n(\alpha(x))$ and $\ell_{2n+1}(x) = \ell_n(\alpha(x)) + 1$. This leads to

$$f_{2n+1}(q,t) = \sum_{x\in\mathbb{H}_{2n+1}} q^{\mathfrak{h}_{2n+1}(x)}t^{\ell_{2n+1}(x)} = t\sum_{y\in\mathbb{H}_n} q^{\mathfrak{h}_n(y)}t^{\ell_n(y)} = tf_n(q,t).$$

By our induction hypothesis and (2), $f_{2n+1}(q,t) = tB_{n+1}(q,t) = B_{2n+2}(q,t)$.

*The case* $2n+2$: From the proof of [1, Theorem 2], it follows that each hyperbinary expansion $x$ in $\mathbb{H}_{2n+2}$ can be mapped to either the hyperbinary expansion $x'$ of $n$ or the hyperbinary expansion $x''$ of $n+1$ such that $\mathfrak{h}_n(x) = \mathfrak{h}_{n+1}(x')+1$, $\ell_n(x) = \ell_{n+1}(x')$, $\mathfrak{h}_n(x) = \mathfrak{h}_{n+1}(x'')$ and $\ell_n(x) = \ell_{n+1}(x'')$. Thus,

$$\begin{aligned}
f_{2n+2}(q,t) &= \sum_{x\in\mathbb{H}_{2n+2}} q^{\mathfrak{h}_{2n+2}(x)}t^{\ell_{2n+2}(x)}\\
&= q\sum_{y\in\mathbb{H}_n} q^{\mathfrak{h}_n(y)}t^{\ell_n(y)} + \sum_{y\in\mathbb{H}_{n+1}} q^{\mathfrak{h}_{n+1}(y)}t^{\ell_{n+1}(y)}\\
&= qf_n(q,t) + f_{n+1}(q,t).
\end{aligned}$$

By our induction hypothesis and (2),

$f_{2n+2}(q,t) = qf_n(q,t) + f_{n+1}(q,t) = qB_{n+1}(q,t) + B_{n+2}(q,t) = B_{2n+3}(q,t)$.

The result follows. ■

We denote the generating function for the $(q,t)$-hyperbinary sequence

$$\{B_n(q,t)\}_{n\geq 0}$$

by $B(z;q,t)$, that is, $B(z;q,t) = \sum_{n\geq 0} B_n(q,t)z^{n+1}$.

THEOREM 3.2. *The generating function $B(z;q,t)$ is given by*

$$\prod_{j\geq 0}(1 + tz^{2^j} + qz^{2^{j+1}}).$$

*Proof.* Let $B(z;q,t) = B_{\mathrm{odd}}(z;q,t) + B_{\mathrm{even}}(z;q,t)$, where

$$B_{\mathrm{odd}}(z;q,t) = \sum_{n\geq 0} B_{2n+1}(q,t)z^{2n+2}, \qquad B_{\mathrm{even}}(z;q,t) = \sum_{n\geq 0} B_{2n}(q,t)z^{2n+1}.$$

By (2), we have

$$\begin{aligned} B_{\mathrm{odd}}(z;q,t) &= qz^2B(z^2;q,t) + B(z^2;q,t),\\ B_{\mathrm{even}}(z;q,t) &= tzB(z^2;q,t). \end{aligned}$$

Hence

$$\begin{aligned} B(z;q,t) &= (1+tz+qz^2)B(z^2;q,t)\\ &= (1+tz+qz^2)(1+tz^2+qz^4)B(z^4;q,t)\\ &= \cdots\\ &= \prod_{j\geq 0}(1 + tz^{2^j} + qz^{2^{j+1}}), \end{aligned}$$

as required. ■

Note that the above result generalizes Theorem 3.1 of [9].

**Acknowledgments.** The author thanks the anonymous referee for pointing out the reference [9].

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Toufik Mansour
Department of Mathematics
University of Haifa
3498838 Haifa, Israel
E-mail: tmansour@univ.haifa.ac.il

*Received October 3, 2014;*
*received in final form February 4, 2015* (7989)