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*ORDINARY DIFFERENTIAL EQUATIONS*

# An Algebraic Approach to Implicit Evolution Equations

by

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**Summary.** A Banach algebra homomorphism on the convolution algebra of integrable functions is the essence of Kisyński's equivalent formulation of the Hille–Yosida theorem for analytic semigroups. For the study of implicit evolution equations the notion of empathy happens to be more appropriate than that of semigroup. This approach is based upon the intertwining of two families of evolution operators and two families of pseudo-resolvents. In this paper we show that the Kisyński approach can be adapted to empathy theory. The adaptation highlights the essential differences between semigroup theory and the theory of empathy.

**1. Background.** The study of *implicit evolution equations*

$$\frac{d}{dt}[Bu(t)] = Au(t), \tag{1.1}$$

motivated by constitutive expressions of continuum mechanics and other applications, has been on-going for some time. In (1.1) the symbols $A$ and $B$ denote unbounded linear operators with a common domain $\mathscr{D}$ in a Banach space $X$ and range in a Banach space $Y$. Among the first mathematical works are [10] (in a Hilbert space setting) and [2]. In these papers the operators $A$ and $B$ were assumed to be closed or closable, and the initial condition consisted of the specification of either the initial state $u(0)$ or $B[u(0)]$. A detailed account can be found in [3].

There are, however, situations pertaining to dynamic boundary conditions in which the operators $A$ and $B$ are not closable, precluding the interchange of $B$ with the time derivative in (1.1) or the limit in the initial

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condition

$$\lim_{t\to 0^+}[Bu(t)] = y \in Y. \tag{1.2}$$

This was pointed out in the early study [7] of the Cauchy problem (1.1), (1.2), where it was also revealed that closedness of the operators $A$ and $B$ was not crucial.

Empathy theory [9, 8] approaches the implicit Cauchy problem by considering two families $\langle S(t), E(t)\rangle$ of bounded evolution operators *intertwined* by the *empathy relation*

$$S(t+s) = S(t)E(s), \qquad s,t > 0, \tag{1.3}$$

in which $S(t) : Y \to X$ and $E(t) : Y \to Y$. The double family of linear operators $\{\langle S(t), E(t)\rangle : t > 0\}$ is called an *empathy.* In [9] it was assumed that $E(t)$ is a semigroup, a hypothesis abandoned in [8]. The basis for further development was the assumption that for $y \in Y$ the Laplace transforms

$$P(\lambda)y = \int_0^\infty \exp\{-\lambda t\}S(t)y\,dt, \qquad R(\lambda)y = \int_0^\infty \exp\{-\lambda t\}E(t)y\,dt$$

exist as Bochner integrals over $(0,\infty)$ for $\lambda > 0$. From this it followed that $E(t)$ is a semigroup, not necessarily of class $C_0$, and that the *entwined pseudo-resolvent equations*

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu), \tag{1.4}$$

$$P(\lambda) - P(\mu) = (\mu - \lambda)P(\lambda)R(\mu) \tag{1.5}$$

hold. The pair $\langle R, P\rangle$ is called an *entwined pseudo-resolvent.*

Under the *invertibility assumption* that for a single $\xi > 0$ the operator $P(\xi)$ is invertible, all $R(\lambda)$ and $P(\lambda)$ are invertible and the linear operators $B = R(\lambda)P^{-1}(\lambda)$ and $A = \lambda B - P^{-1}(\lambda)$ defined on the domain

$$\Delta_X := P(\lambda)[Y] \subset X$$

can be constructed. It turns out that $P(\lambda) = (\lambda B - A)^{-1}$. For $y \in B[\Delta_X] = \Delta_Y := R(\lambda)[Y] \subset Y$, $u(t) = S(t)y$ solves the implicit Cauchy problem (1.1), (1.2). The operator pair $\langle A, B\rangle$ is called the *generator* of the empathy $\langle S(t), E(t)\rangle$.

In [8] a Hille–Yosida theorem giving sufficient conditions for a given operator pair $\langle A, B\rangle$ to be the generator of an empathy is proved. If $A, B : \mathscr{D} \subset X \to Y$ are given linear operators such that for $\lambda > 0$, $P(\lambda) = (\lambda B - A)^{-1} : Y \to \mathscr{D}$ exists and is bounded, let $R(\lambda) = BP(\lambda)$. Then, without any other invertibility assumptions, the pseudo-resolvent equations (1.4), (1.5) are satisfied. Under suitable Widder growth conditions on $R(\lambda)$ and a boundedness condition on $P(\lambda)$ an empathy defined on the closure of $\Delta_Y$ in $Y$ can be constructed. Under the additional assumption that the space $Y$ has the Radon–Nikodým property, it also follows that the constructed empathy has an extension to the whole of $Y$, and that the pseudo-resolvents $P(\lambda)$ and $R(\lambda)$ are the Laplace transforms of the constructed $S(t)$ and $E(t)$.

The results described above came at a price, particularly due to the fact that the Widder inversion theorem [11] holds only in spaces with the Radon–Nikodým property [1]. In this paper we adapt Kisyński's approach [5] to the Hille–Yosida theorem for $C_0$-semigroups to cover a wider assortment of cases. We also show that this approach can be followed to obtain results pertaining to empathy theory and therefore implicit Cauchy problems.

**2. A backdrop to the Kisyński construction.** The Kisyński approach is based upon the space $L^1(\mathbb{R}^+)$ of integrable scalar-valued functions defined on the real line with support in $[0,\infty)$ as a Banach algebra with convolution defined by $f * g(x) = \int_0^\infty f(x-y)g(y)\,dy$ as product. We shall use the notation $Z = \langle L^1(\mathbb{R}^+), *\rangle$ for the algebra. Every $f \in L^1(\mathbb{R}^+)$ (canonically) defines a bounded linear operator $\mathfrak{F}$ by $\mathfrak{F}g = f * g$. The identification $\mathrm{Id} : f \in L^1(\mathbb{R}^+) \mapsto \mathfrak{F} \in \mathcal{L}(L^1(\mathbb{R}^+))$ is a one-to-one Banach algebra representation; $\mathcal{L}(L^1(\mathbb{R}^+))$ is the space of bounded operators on $L^1(\mathbb{R}^+)$.

In particular, the family $\{r_\lambda : \lambda > 0\}$ of exponentials with $r_\lambda(x) = \exp\{-\lambda x\}$ for $x \geq 0$ in this space is important as it defines a canonical pseudo-resolvent of the algebra $Z$ in the sense that

$$r_\lambda - r_\mu = (\mu - \lambda) r_\lambda * r_\mu. \tag{2.1}$$

It is important to note that $\{r_\lambda : \lambda > 0\}$ is a total subset of $L^1(\mathbb{R}^+)$ [1, Chap. 2]. Also of importance is to note that if $f \in L^1(\mathbb{R}^+)$ is represented in the form $f = r_\lambda * \phi_\lambda$, then for $\mu > 0$ there is $\phi_\mu$ such that

$$f = r_\lambda * \phi_\lambda = r_\mu * \phi_\mu, \tag{2.2}$$

as can be seen from (2.1).

Thus the family of bounded linear operators $\mathfrak{R}(\lambda)f = r_\lambda * f$, $\lambda > 0$, satisfies the pseudo-resolvent equation

$$\mathfrak{R}(\lambda) - \mathfrak{R}(\mu) = (\mu - \lambda)\mathfrak{R}(\lambda)\mathfrak{R}(\mu). \tag{2.3}$$

**3. The Kisyński construction.** For an arbitrary Banach space $Y$, let $R(\lambda) : Y \to Y$, $\lambda > 0$, be a pseudo-resolvent. When does the map $T : r_\lambda \mapsto R(\lambda)$ become a unique identification in some sense like Id?

The first step is to construct the *regularity space*

$$\varDelta_K := \Big\{ y \in Y : \lim_{\lambda\to\infty} \|\lambda R(\lambda)y - y\| = 0 \Big\}. \tag{3.1}$$

Under the growth condition $\|\lambda R(\lambda)\| = O(1)$, namely, $\limsup_{\lambda\to\infty} \lambda\|R(\lambda)\| < \infty$, the space $\varDelta_K$ is a closed subspace of $Y$. Under the *strong Widder* *condition* [11]

$$\sup\{\|[\lambda R(\lambda)]^k\| : \lambda > 0,\ k \in \mathbb{N}\} < \infty, \tag{3.2}$$

the following is obtained:

PROPOSITION 3.1 (Kisyński [5]). *If* (3.2) *holds, then the identification* $T : r(\lambda) \mapsto R(\lambda)$ *uniquely extends to all of* $Z$ *as a* (*bounded*) *Banach algebra representation* $T : Z \to \mathcal{L}(Y)$, *the algebra of bounded linear operators. Furthermore,* $T$ *reconstructs the regularity space* $\Delta_K$ *in this way:*

$$\Delta_K = \overline{\Delta}_Y = \bigcup_{\phi \in Z} T(\phi)[Y] \tag{3.3}$$

*with* $\Delta_Y = R(\lambda)[Y]$. *Moreover, convolution in* $Z$ *is mapped to composition, i.e.* $T(f * g) = T(f)T(g)$.

We shall refer to $\Delta_K$ as the *T-regularity space.*

In the final step a semigroup $E(t)$ on $\Delta_K$ is constructed by considering the right-shift operation $E^t f(x) = f(x - t)$ defined for $f \in Z$ (mindful of the fact that $f$ is supported on $[0, \infty)$). The definition of $E(t)$ then is, for $y = T(\phi)y_\phi \in \Delta_K$,

$$E(t)y := [T(E^t(\phi))]y_\phi. \tag{3.4}$$

The expression (3.4) should be interpreted in the light of Proposition 3.1. By letting $\phi$ range over all of $Z$, all $y \in \Delta_K$ are reproduced. This defines a unique $C_0$-semigroup $\{E(t)) : t \geq 0\}$ on $\Delta_K$ [5].

Let $A_E$ denote the generator of $\mathcal{E}$. The operator $A_E$ is constructed from $R$ in the following way: $y \in D(A_E) \subset \Delta_K$ if $\lim_{\lambda \to \infty} \|\lambda(\lambda R(\lambda)y - y)) - y'\| = 0$ exists, and then $A_E y := y'$. Under the growth condition $\|\lambda R(\lambda)\| = O(1)$, the pseudo-resolvent $R$ restricted to the subspace $\Delta_K$ becomes the resolvent $R(\lambda) = (\lambda - A_E)^{-1} = R(\lambda, A_E)$ on $\Delta_K$. Of course $D(A_E)$ is dense in $\Delta_K$.

**4. The empathy construction.** The problem at hand is: Given two linear operators (not necessarily closed) $A, B : \mathscr{D} \subset X \to Y$, find sufficient conditions under which they will be the generator of an empathy. In a preliminary way we proceed, as in the introductory section, to assume that for $\lambda > 0$ the linear operators $P(\lambda) : Y \to \mathscr{D}$, $R(\lambda) : Y \to Y$ defined by

$$P(\lambda) = (\lambda B - A)^{-1}, \tag{4.1}$$

$$R(\lambda) = BP(\lambda) \tag{4.2}$$

exist and are bounded. As mentioned before, the entwined pseudo-resolvent equations (1.4), (1.5) are satisfied. In what follows we shall only use the pseudo-resolvent equations without recourse to (4.1), (4.2), and return to them afterwards. We do, however, assume that $R(\lambda)$ satisfies all the requirements of Section 3, and will use the same notation.

We begin by defining the subspaces $\Delta_K^2 \subset \Delta_K$, $\Delta_X \subset X$ and the operator $C : \Delta_K^2 \to \Delta_X$ by

$$\Delta_K^2 = R(\lambda)[\Delta_K], \tag{4.3}$$

$$\Delta_X = P(\lambda)[\Delta_K], \tag{4.4}$$

$$C = P(\lambda)R^{-1}(\lambda). \tag{4.5}$$

Proposition 4.1. *The definitions* (4.3)–(4.5) *are independent of the choice of* $\lambda$. *Moreover,* $\Delta_K^2$ *is an invariant subspace of* $E(t)$ *and is dense in* $\Delta_K$.

*Proof.* From (1.4) and (1.5) it is seen that the images defined by (4.3) and (4.4) do not depend on the choice of $\lambda$. It also follows from (1.5) that $P(\lambda)R(\mu) = P(\mu)R(\lambda)$, from which it follows that the definition (4.5) is independent of $\lambda$. Since $\Delta_K^2 = D(A_E)$, it is clear that $E(t)[\Delta_K^2] \subset \Delta_K^2$ and $\Delta_K^2$ is dense in $\Delta_K$. ∎

Next we define a representation $T^2 : Z \to \mathcal{L}(\Delta_K^2, X)$ by $T^2(\phi) = CT(\phi)$; $\phi$ has representation $r_\lambda * \phi_\lambda$, and we immediately note that $T^2$ is not an algebra representation and does not have to be continuous or even closed. It does, however, represent the operators $P(\lambda)$ restricted to $\Delta_K$ in the sense that $T^2(r_\lambda) = P(\lambda)$.

Now we are able to construct a family of operators $\{S(t) : t > 0\}$ from $\Delta_K^2$ to $\Delta_X$ by

$$S(t)[R(\lambda)T(\phi)] = T^2(r_\lambda * E^t\phi). \tag{4.6}$$

In a more explicit way the definition (4.6) may be reformulated as follows: Let $y \in \Delta_K^2$ be of the form $R(\lambda)T(\phi_\lambda)y'$ with $\phi_\lambda \in Z$ and $y' \in \Delta_K$. Then

$$S(t)y = P(\lambda)E(t)[T(\phi_\lambda)y']. \tag{4.7}$$

We need to show that the definition (4.7) does not depend on the choice of $\lambda$. By Proposition 3.1 and (2.2),

$$y = R(\lambda)T(\phi_\lambda)y' = R(\mu)T(\phi_\mu)y'. \tag{4.8}$$

Thus the representation of $y$ does not depend on $\lambda$. What we need to show is that the right of (4.7) does not depend on $\lambda$ either. From (4.8) and the commutation rules $E(t)R(\lambda) = R(\lambda)E(t)$, $R(\lambda)R(\mu) = R(\mu)R(\lambda)$, $P(\lambda)R(\mu) = P(\mu)R(\lambda)$, we obtain

$$\begin{aligned} P(\lambda)E(t)[T(\phi_\lambda)y'] &= P(\lambda)E(t)R^{-1}(\lambda)R(\lambda)[T(\phi_\lambda)y'] \\ &= CR(\lambda)E(t)R^{-1}(\lambda)R(\mu)[T(\phi_\mu)y'] \\ &= CE(t)R(\mu)[T(\phi_\mu)y'] \\ &= CR(\mu)E(t)[T(\phi_\mu)y'] \\ &= P(\mu)E(t)[T(\phi_\mu)y']. \end{aligned}$$

Next we verify that $\langle S(t), E(t)\rangle$ satisfies the empathy relation (1.3). This is a (subtle) adaptation of the proof that the family $\{E(t)\}$ is a semigroup. In the same vein, after retracing some of the steps in the calculations immediately above we see that the limit

$$Cy = \lim_{t\to 0^+} S(t)y = P(\lambda)R^{-1}(\lambda)y \tag{4.9}$$

exists for all $y \in \Delta_K^2$. In view of (4.9) we feel justified to refer to $\Delta_K^2$ as the *$T^2$-regularity space.*

THEOREM 4.1. *Let $\langle R(\lambda), P(\lambda)\rangle$ be an entwined pseudo-resolvent, and suppose that $R(\lambda)$ satisfies the strong Widder condition* (3.2). *Then there exists an empathy $\langle S(t), E(t)\rangle$ defined on the $T^2$-regularity space $\Delta_K^2$ for which the limit $Cy$ in* (4.9) *exists and the representation does not depend on $\lambda$.*

COROLLARY 4.1. *If $\|\lambda R(\lambda)\| = O(1)$ then* (1.5) *follows from* (1.4).

The discussion so far did not take the particular forms of $P(\lambda)$ and $R(\lambda)$ as set out in (4.1) and (4.2) into account. Suppose $y \in \Delta_K^2$. By (4.2), the domain $\mathscr{D}$ contains $P(\lambda)[Y] \supset \Delta_X$. From (4.7) we see that

$$BS(t)y = E(t)y, \quad y \in \Delta_K^2. \tag{4.10}$$

This is a restricted form of the *B-evolution property* [7]. We may now return to the implicit Cauchy problem (1.1), (1.2). Let $u(t) = S(t)y$. Then, by (4.10), $Bu(t) = E(t)y$ and

$$\begin{cases} \dfrac{d}{dt}[Bu(t)] = A_E E(t)y = [A_E B]u(t), \\ \lim\limits_{t\to 0^+} Bu(t) = y. \end{cases} \tag{4.11}$$

Without using the full power of the definition (4.1), the following has been proved:

THEOREM 4.2. *Let $\langle P(\lambda), R(\lambda)\rangle$ be an entwined pseudo-resolvent and $B : \mathscr{D} \subset X \to Y$ be a linear operator such that $R(\lambda) = BP(\lambda)$. If $R(\lambda)$ satisfies the strong Widder growth condition* (3.2), *then $u(t)=S(t)y$, $y\in\Delta_K^2$, solves the implicit Cauchy problem* (1.1), (1.2) *with $A = A_E B$. Here $A_E$ is the infinitesimal generator of the* $C_0$*-semigroup associated with $R(\lambda)$.*

REMARK 4.1. If the operators $P(\lambda)$ have the form (4.1), then a direct calculation shows that $B$ is invertible and $C = B^{-1}$. This is at the core of the theory of empathy as expounded in [8].

**5. Essential differences.** The presentation in the previous section is based upon the assumption that the pseudo-resolvents $R(\lambda)$ are associated with a $C_0$-semigroup and therefore invertible. This is in sharp contrast to [8] where the invertibility of at least one of the $P(\lambda)$ took center stage. Also, since the operator $B$ does not have to be closed, it follows that, even in the case where $P(\lambda)$ is invertible, the limit operator $C = B^{-1}$ does not have to be closed.

The notion $S(0) = C$ is tenable, but of little value. Thus empathy theory differs essentially from semigroup theory in the sense that the "initial value" of the family $S(t)$ has no bearing on the initial condition (1.2).

The linear map $T^2$ which represents $P(\lambda)$ on $Z$ and is used to define the family $\{S(t)\}$ is not an algebra representation and is not necessarily closed. In contrast, the bounded map $T$ on $Z$ represents a commutative Banach algebra of bounded linear operators in which the semigroup $E(t)$ on $\Delta_K$ is essentially the image under $T$ of a translation semigroup.

One can identify $\langle P(\lambda), S(t)\rangle$ with $T^2$. Similarly, one can identify $\langle R(\lambda), E(t)\rangle$ with $T$. Therefore a pair $\langle T, T^2\rangle$ is used to generate an empathy $\langle E(t), S(t)\rangle$. One can identify the domains of the empathy $\langle \Delta_K, \Delta_K^2\rangle$ with the pair $\langle T, T^2\rangle$.

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