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PROBABILITY THEORY AND STOCHASTIC PROCESSES

A Weak-Type Inequality for Submartingales and Itô Processes

by

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Summary. Let $\alpha \in [0, 1]$ be a fixed parameter. We show that for any nonnegative submartingale X and any semimartingale Y which is α-subordinate to X, we have the sharp estimate

$$\|Y\|_W \leq \frac{2(\alpha+1)^2}{2\alpha+1}\|X\|_{L^\infty}.$$

Here W is the weak-L^∞ space introduced by Bennett, DeVore and Sharpley. The inequality is already sharp in the context of α-subordinate Itô processes.

1. Introduction. Our goal is to provide a sharp weak-type estimate for a certain class of Itô processes and, more generally, for the class of semimartingales satisfying the so-called α-subordination relation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. As usual, we assume that the filtration is also complete, i.e., $\mathcal{F}_0$ contains all the sets A satisfying $\mathbb{P}(A) = 0$. Suppose that $B = (B_t)_{t\geq 0}$ is an adapted Brownian motion starting from 0, and let $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ be Itô processes with respect to B (cf. Ikeda and Watanabe [9]): for $t \geq 0$,

$$X_t = X_0 + \int_{0+}^{t} \phi_s\,dB_s + \int_{0+}^{t} \psi_s\,ds, \qquad Y_t = Y_0 + \int_{0+}^{t} \zeta_s\,dB_s + \int_{0+}^{t} \xi_s\,ds. \tag{1.1}$$

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Here $(\phi_t)_{t\geq 0}$, $(\psi_t)_{t\geq 0}$, $(\zeta_t)_{t\geq 0}$, $(\xi_t)_{t\geq 0}$ are predictable processes such that

$$\mathbb{P}\Big(\int_{0+}^{t} |\phi_s|^2\,ds < \infty \ \text{ and } \int_{0+}^{t} |\psi_s|\,ds < \infty \text{ for all } t > 0\Big) = 1,$$

$$\mathbb{P}\Big(\int_{0+}^{t} |\zeta_s|^2\,ds < \infty \ \text{ and } \int_{0+}^{t} |\xi_s|\,ds < \infty \text{ for all } t > 0\Big) = 1.$$

The problem of comparing the sizes of X and Y under some structural assumptions on ϕ, ψ, ζ and ξ has been investigated quite intensively in the literature; e.g. the whole class of so-called comparison theorems falls within the scope of this subject: see Yamada [15], Ikeda and Watanabe [8], [9], Le Gall [10] and references therein. Our result is closely related to the problem which was studied for the first time in Burkholder's paper [3]. He showed that if X is a nonnegative submartingale and we have the domination $X_0 \geq |Y_0|$, $|\phi_s| \geq |\zeta_s|$ and $\psi_s \geq |\xi_s|$ for all s, then

$$\lambda\mathbb{P}\Big(\sup_{t\geq 0}|Y_t| \geq \lambda\Big) \leq 3\|X\|_1, \qquad \lambda > 0,$$

and

$$\|Y\|_p \leq \max\{(p-1)^{-1}, 2p-1\}\|X\|_p, \qquad 1 < p < \infty.$$

Here we have used the notation $\|X\|_p = \sup_t \|X_t\|_p$ for the pth moment of X, $p \geq 1$. Furthermore, Burkholder proved that both inequalities above are sharp. These results were generalized by C. S. Choi [5], [6], who showed that if $\alpha \in [0,1]$ is a fixed number, X is a nonnegative submartingale and, in addition,

$$X_0 \geq |Y_0| \quad \text{and} \quad |\phi_s| \geq |\zeta_s|, \quad \alpha\psi_s \geq |\xi_s| \quad \text{for all } s, \tag{1.2}$$

then we have the weak-type bound

$$\lambda\mathbb{P}\Big(\sup_{t\geq 0}|Y_t| \geq \lambda\Big) \leq (\alpha+2)\|X\|_1, \qquad \lambda > 0, \tag{1.3}$$

and the moment estimate

$$\|Y\|_p \leq \max\{(p-1)^{-1}, (\alpha+1)p-1\}\|X\|_p, \qquad 1 < p < \infty. \tag{1.4}$$

Again, the constants $\alpha+2$ and $\max\{(p-1)^{-1}, (\alpha+1)p-1\}$ are optimal.

In fact, one can study the above results in a much wider setting. For any real-valued semimartingales X and Y, we say that Y is *differentially subordinate* to X if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t (see Bañuelos and Wang [1] or Wang [14]). Here $[X,X]$ denotes the quadratic variance process of X (see e.g. Dellacherie and Meyer [7]). This type of domination implies many interesting inequalities if X and Y are martingales or local martingales (see [14]). In the semimartingale setting, one strengthens the domination and imposes some control on the finite variation parts. In what follows, we will work under the assumption of *α-strong differential subordination* (α-subordination for short), which was introduced by Wang [14] in the case $\alpha = 1$, and generalized by the author [11] to other values of α. Let us recall the definition.

Suppose that X is an adapted submartingale, Y is an adapted semimartingale and write Doob–Meyer decompositions

$$X = X_0 + M + A, \qquad Y = Y_0 + N + B, \tag{1.5}$$

where M, N are local martingale parts, and A, B are finite variation processes (M, N, A and B are assumed to vanish at 0). In general, the decompositions may not be unique; however, we assume that A is predictable and this determines the first of them. Let α be a fixed nonnegative number. We say that Y is *α-subordinate* to X if Y is differentially subordinate to X and there is a decomposition (1.5) for Y such that the process $(\alpha A_t - |B|_t)_{t\geq 0}$ is nondecreasing as a function of t. Here $|B|_t$ denotes the total variation of B on the interval $[0,t]$. In the setting of Itô processes described in (1.1), if $|\phi_s| \geq |\zeta_s|$ and $\alpha\psi_s \geq |\xi_s|$ for all s, then obviously Y is α-subordinate to X, so the setup introduced above is indeed more general.

Let us turn our attention to the results studied in this paper. The inequality (1.3) can be regarded as an endpoint version of (1.4) as $p \to 1$. There is a natural question about the weak-type substitute for (1.4) for $p \to \infty$, and our purpose is to provide an appropriate counterpart. To state the result, we need more notation. For a given random variable ξ defined on a nonatomic probability space, we define ξ^*, the decreasing rearrangement of ξ, by

$$\xi^*(t) = \inf\{\lambda \geq 0 : \mathbb{P}(|\xi| > \lambda) \leq t\}.$$

Then $\xi^{**} : (0,1] \to [0,\infty)$, the *maximal function* of ξ^*, is given by

$$\xi^{**}(t) = \frac{1}{t}\int_0^t \xi^*(s)\,ds, \qquad t \in (0,1].$$

One easily verifies that ξ^{**} can alternatively be defined by the formula

$$\xi^{**}(t) = \frac{1}{t}\sup\Big\{\int_E |\xi|\,d\mathbb{P} : E \in \mathcal{F},\, \mathbb{P}(E) = t\Big\}.$$

Now, following Bennett, DeVore and Sharpley [2], we let

$$\|\xi\|_{W(\Omega)} = \sup_{t\in(0,1]} (\xi^{**}(t) - \xi^*(t))$$

and define

$$W(\Omega) = \{\xi : \|\xi\|_{W(\Omega)} < \infty\}.$$

To describe the motivation behind this definition, note that for each $1 \leq p < \infty$, the usual weak space $L^{p,\infty}$ properly contains L^p; on the contrary, for $p = \infty$, the two spaces coincide. Thus, there is no Marcinkiewicz interpolation theorem between L^1 and L^∞ for operators which are unbounded on L^∞. The space W was invented to fill this gap. It contains L^∞, can be understood as an appropriate limit of $L^{p,\infty}$ as $p \to \infty$, and has the appropriate interpolation property: if an operator T is bounded from L^1 to $L^{1,\infty}$ and from L^∞ to W, then it can be extended to a bounded operator on all L^p spaces, $1 < p < \infty$. See [2] for details.

In analogy to the previous notation, the weak-L^∞ norm of a process X is given by $\|X\|_{W(\Omega)} = \sup_{t\geq 0} \|X_t\|_{W(\Omega)}$. We will prove the following.

THEOREM 1.1. *Let $\alpha \in [0,1]$ be fixed. Suppose that X is a nonnegative submartingale and Y is α-subordinate to X. Then*

$$\|Y\|_W \leq \frac{2(\alpha+1)^2}{2\alpha+1}\|X\|_{L^\infty} \tag{1.6}$$

and the constant $2(\alpha+1)^2/(2\alpha+1)$ is the best possible. It is already the best in the context of Itô processes (1.1).

A few words about the organization of the paper are in order. In the next section we provide the proof of the inequality (1.6). We will rewrite the estimate in a slightly different form and study it with the use of Burkholder's technique: the inequality will be extracted from the existence of a certain special function. Section 3 is devoted to the sharpness of (1.6): we will construct appropriate examples.

2. Proof of Theorem 1.1. The bound (1.6) will be deduced from the following auxiliary fact.

THEOREM 2.1. *Let $\alpha \in [0,1]$. Suppose that X is a nonnegative submartingale satisfying $\|X\|_\infty \leq 1$, and let Y be a semimartingale which is α-subordinate to X. Then for any $\lambda \geq 0$ and $t \geq 0$,*

$$\mathbb{E}\bigg(|Y_t| - \lambda - \frac{2(\alpha+1)^2}{2\alpha+1}\bigg)1_{(\lambda,\infty)}(|Y_t|) \leq 0. \tag{2.1}$$

The proof of this inequality will use Burkholder's method. Let S denote the strip $[0,1]\times\mathbb{R}$, and consider the following four subsets:

$$\begin{aligned}
D_1 &= \{(x,y) \in S : x + |y| \leq 1\},\\
D_2 &= \{(x,y) \in S : x + (2\alpha+1)^{-1} \geq |y| > -x + 1\},\\
D_3 &= \{(x,y) \in S : x + |y| \leq 1,\ x \leq \alpha(2\alpha+1)^{-1}\},\\
D_4 &= S \setminus (D_1 \cup D_2 \cup D_3).
\end{aligned}$$

Introduce the function $U : S \to \mathbb{R}$ by the following formulas. If $(x, y) \in D_1$, set $U(x, y) = 0$. If $(x, y) \in D_2$, let

$$U(x, y) = x + |y| - 1 - \frac{2(\alpha+1)^2}{2\alpha+1} \cdot \frac{x + |y| - 1}{-x + |y| + 1}.$$

On the sets D_3 and D_4, the definition is a little more complicated. For $(x, y) \in D_3$, we let $U(x, y)$ be

$$|y| - \alpha x - \frac{3\alpha+2}{2\alpha+1} + (\alpha+1)\exp\left[-\frac{2\alpha+1}{\alpha+1}(x + |y| - 1)\right]\left(x + \frac{1}{2\alpha+1}\right).$$

Finally, on D_4, $U(x, y)$ is given by

$$|y| - \alpha x - \frac{3\alpha+2}{2\alpha+1} + (\alpha+1)\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x + |y| - \frac{1}{2\alpha+1}\right)\right](1 - x).$$

It is not difficult to check that U is continuous on $S \setminus \{(1, 0)\}$; it is even of class C^1 in the interior of $S \setminus \{(x, y) : x + |y| = 1\}$. The key property of U is studied in the following lemma.

LEMMA 2.2. *The function U is concave along any line segment of slope $k \in [-1, 1]$, contained in S.*

Proof. For any fixed $x \in [0, 1]$ and $y \in \mathbb{R}$, consider the function $G = G_{x,y,k} : t \mapsto U(x + t, y + tk)$, defined on $[-x, 1 - x]$. We need to show that G is concave. To accomplish this, we will first check that $G''(t) \le 0$ for those t for which $(x+t, y+tk)$ belongs to the interior of D_1, D_2, D_3 or D_4. Note that $G''_{x,y,k}(t) = G''_{x+t,y+tk,k}(0)$, so we may assume $t = 0$ in this desired inequality. If (x, y) belongs to D_1^o, the interior of D_1, then $G''(0) = 0$. If $(x, y) \in D_2^o$, then

$$G''(0) = -\frac{8(\alpha+1)^2(1-k)}{(2\alpha+1)(-x + |y| + 1)^3}(|y| + (1 - x)k),$$

which is nonpositive. This follows from the fact that $|k| \le 1$ and $|y| > 1 - |x|$. If $(x, y) \in D_3^o$, a little calculation shows that

$$\begin{aligned} G''(0) = (2\alpha+1)&\exp\left[-\frac{2\alpha+1}{\alpha+1}(x+|y|-1)\right] \\ &\times (1+k)\left\{\left[\frac{2\alpha+1}{\alpha+1}\left(x + \frac{1}{2\alpha+1}\right) - 2\right] + \frac{2\alpha+1}{\alpha+1}\left(x + \frac{1}{2\alpha+1}\right)k\right\}. \end{aligned}$$

This expression is again nonpositive, because

$$|k| \le 1, \qquad \frac{2\alpha+1}{\alpha+1}\left(x + \frac{1}{2\alpha+1}\right) - 2 \le -1, \qquad \frac{2\alpha+1}{\alpha+1}\left(x + \frac{1}{2\alpha+1}\right) \le 1.$$

Finally, if (x,y) belongs to the interior of D_4, then we derive that

$$G''(0) = (2\alpha+1)\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|-\frac{1}{2\alpha+1}\right)\right] \\ \times (1-k)\left\{\left[\frac{2\alpha+1}{\alpha+1}(1-x)-2\right]-\frac{2\alpha+1}{\alpha+1}(1-x)k\right\} \leq 0,$$

since

$$|k| \leq 1, \qquad \frac{2\alpha+1}{\alpha+1}(1-x)-2 \leq -1, \qquad \frac{2\alpha+1}{\alpha+1}(1-x) \leq 1.$$

Recall that U is of class C^1 in the interior of $S \setminus \{(x,y) : x+|y|=1\}$. So, if the segment $I = \{(x+t, y+tk) : t \in [-x, 1-x]\}$ is entirely contained in $D_2 \cup D_3 \cup D_4$ or entirely contained in D_1, then G is concave. Suppose that this segment has nonempty intersection with both sets D_1 and $D_2 \cup D_3 \cup D_4$. Then there is t_0 such that

$$I \cap D_1 = \{(x+t, y+tk) : t \in [-x, t_0]\}, \\ I \cap (D_2 \cup D_3 \cup D_4) = \{(x+t, y+tk) : t \in (t_0, 1-x]\},$$

so $x + t_0 + |y + t_0 k| = 1$; by symmetry, we may assume that $y + t_0 k > 0$. The function U vanishes on $I \cap D_1$ and is concave on $I \cap (D_2 \cup D_3 \cup D_4)$. Thus, we will be done if we show that $G'(t_0+) \leq 0$. We derive directly that

$$G'(t_0+) = \begin{cases} -(1+k)(2\alpha+1)(x+t_0) & \text{if } x+t_0 \leq \alpha(2\alpha+1)^{-1}, \\ \dfrac{1+k}{1-(x+t_0)}\left(-(x+t_0)-\dfrac{\alpha^2}{2\alpha+1}\right) & \text{if } x+t_0 > \alpha(2\alpha+1)^{-1}, \end{cases}$$

and this is clearly nonpositive. ■

The next property we will need is the monotonicity along line segments of slope $\pm\alpha$. It is convenient to formulate it in the language of the functions $G_{x,y,k}$ defined above.

LEMMA 2.3. *For any $y \in \mathbb{R}$, the functions $G_{0,y,\pm\alpha}$ are nonincreasing.*

Proof. It suffices to focus on the function $G_{0,y,\alpha}$, since $G_{0,y,-\alpha} = G_{0,-y,\alpha}$ for all y and α. Since $\alpha \in [0,1]$, the function G is concave, as we have shown above, and hence it is enough to check that the right-hand derivative $G'_{0,y,\alpha}(0+)$ is nonpositive. A direct calculation shows that

$$G'_{0,y,\alpha}(0+) = \begin{cases} 0 & \text{if } y \geq -1, \\ -2\alpha\left[1-\exp\left(-\dfrac{2\alpha+1}{\alpha+1}(-y-1)\right)\right] & \text{if } y < -1 \end{cases}$$

is nonpositive. This yields the assertion. ■

In our considerations below, we will actually need to work with "stretched" versions of U. Define a family $(U^{(\lambda)})_{\lambda\geq 0}$ of functions on S by

$$U^{(\lambda)}(x,y)=\begin{cases} U(x,|y|-\lambda) & \text{if } |y|\geq\lambda,\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that for each $\lambda\geq 0$, the function $U^{(\lambda)}$ inherits the properties studied in the above two lemmas: it is concave along line segments of slope belonging to $[-1,1]$, and nonincreasing along the line segments of slope $\pm\alpha$. We will require the following majorizations.

Lemma 2.4. *Let $\lambda\geq 0$ be a fixed parameter.*

(i) *For any $(x,y)\in S$ satisfying $|y|\leq x$ we have*

$$U^{(\lambda)}(x,y)\leq 0. \tag{2.2}$$

(ii) *For any $(x,y)\in S$ we have*

$$U^{(\lambda)}(x,y)\geq\left(|y|-\lambda-\frac{2(\alpha+1)^2}{2\alpha+1}\right)\chi_{(\lambda,\infty)}(|y|). \tag{2.3}$$

(iii) *There is a constant C depending only on α and λ such that*

$$|U^{(\lambda)}(x,y)|\leq|y|+C\quad\textit{for all } (x,y)\in S. \tag{2.4}$$

Proof. (i) We know from the proof of Lemma 2.2 that the function $t\mapsto U^{(\lambda)}(tx,ty)$ is concave. It remains to note that this function vanishes for small values of t.

(ii) The dependence of both sides on y is through $|y|-\lambda$, so we may assume that $\lambda=0$ (and hence $U^{(\lambda)}=U$). For any $y\in\mathbb{R}$, the function $x\mapsto U(x,y)$ is concave. Therefore, it is enough to check the majorization for $x=0$ and $x=1$ only. Furthermore, since $U(x,y)=U(x,-y)$, we may assume that $y\geq 0$. If $x=0$ and $y\leq 1$, then $U(x,y)=0$, while the right-hand side of (2.3) is nonpositive. If $x=0$ and $y\in(1,2(\alpha+1)^2/(2\alpha+1))$, then $U(x,y)>0$ and the right-hand side of (2.3) is negative. If $x=0$ and $y\geq 2(\alpha+1)^2/(2\alpha+1)$, then the majorization is equivalent to

$$\alpha+\frac{\alpha+1}{2\alpha+1}\exp\left[-\frac{2\alpha+1}{\alpha+1}(y-1)\right]\geq 0,$$

which is obvious. If $x=1$ and $y\geq 0$, then (2.3) is an equality.

(iii) This is evident. ∎

Recall the following well-known fact (cf. [7]). For any semimartingale X there is a unique continuous local martingale part X^c satisfying

$$[X,X]_t=|X_0|^2+[X^c,X^c]_t+\sum_{0<s\leq t}|\Delta X_s|^2$$

for all $t \geq 0$ (here $\Delta X_s = X_s - X_{s-}$ is the jump of X at time $s > 0$). Moreover, $[X^c, X^c] = [X, X]^c$ is the pathwise continuous part of $[X, X]$. We will also need Lemma 1 from [14], which is as follows.

LEMMA 2.5. *If X and Y are semimartingales, then Y is differentially subordinate to X if and only if Y^c is differentially subordinate to X^c and for any $s > 0$ we have*

$$|\Delta Y_s| \leq |\Delta X_s|.$$

Proof of Theorem 2.1. We split the reasoning into three separate parts.

STEP 1: *A mollification argument.* Let $\varepsilon \in (0, 1/2)$ and $\delta \in (0, \varepsilon]$ be fixed numbers, and suppose that $g : \mathbb{R}^2 \to [0, \infty)$ is a C^∞ function, supported on the unit ball of $\mathbb{R}^2$ and satisfying $\int_{\mathbb{R}^2} g = 1$. We introduce the function $U^{\delta,\lambda} : S \to \mathbb{R}$ by the convolution

$$U^{\delta,\lambda}(x, y) = \int_{[-1,1]^2} U^{(\lambda)}\big(\varepsilon + \delta u + (1 - 2\varepsilon)x, (1 - 2\varepsilon)y + \delta v\big) g(u, v)\, du\, dv.$$

Of course, this function is of class C^∞ and inherits the properties of $U^{(\lambda)}$: it is nonincreasing along the lines of slope $\pm\alpha$:

$$U_x^{\delta,\lambda}(x, y) + \alpha |U_y^{\delta,\lambda}(x, y)| \leq 0, \qquad (x, y) \in S^o, \tag{2.5}$$

and concave along the lines of slope $k \in [-1, 1]$:

$$U_{xx}^{\delta,\lambda}(x, y) \pm 2U_{xy}^{\delta,\lambda}(x, y)k + U_{yy}^{\delta,\lambda}(x, y)k^2 \leq 0, \qquad (x, y) \in S^o. \tag{2.6}$$

STEP 2: *An application of Itô's formula.* Let M, N, A, B be the local martingale and finite variation parts of X and Y, coming from the Doob–Meyer decompositions (1.5). It follows from the general theory of stochastic integration that the process

$$\Big(\int_{0+}^{t} U_x^{\delta,\lambda}(X_{s-}, Y_{s-})\, dM_s + \int_{0+}^{t} U_y^{\delta,\lambda}(X_{s-}, Y_{s-})\, dN_s \Big)_{t \geq 0}$$

is a local martingale. Let $(\sigma_n)_{n \geq 0}$ denote the corresponding localizing sequence of stopping times. Since the function $U^{\delta,\lambda}$ is of class C^∞, we are allowed to apply Itô's formula to $(U^{\delta,\lambda}(X_{\sigma_n \wedge t}, Y_{\sigma_n \wedge t}))_{t \geq 0}$:

$$U^{\delta,\lambda}(X_{\sigma_n \wedge t}, Y_{\sigma_n \wedge t}) = U^{\delta,\lambda}(x, y) + I_1 + I_2 + I_3/2 + I_4, \tag{2.7}$$

where

$$I_1 = \int_{0+}^{\sigma_n \wedge t} U_x^{\delta,\lambda}(X_{s-}, Y_{s-})\, dM_s + \int_{0+}^{\sigma_n \wedge t} U_y^{\delta,\lambda}(X_{s-}, Y_{s-})\, dN_s,$$

$$I_2 = \int_{0+}^{\sigma_n \wedge t} U_x^{\delta,\lambda}(X_{s-}, Y_{s-})\, dA_s + \int_{0+}^{\sigma_n \wedge t} U_y^{\delta,\lambda}(X_{s-}, Y_{s-})\, dB_s,$$

$$I_3 = \int_{0+}^{\sigma_n\wedge t} U_{xx}^{\delta,\lambda}(X_{s-},Y_{s-})\,d[X,X]_s^c$$
$$+\,2\int_{0+}^{\sigma_n\wedge t} U_{xy}^{\delta,\lambda}(X_{s-},Y_{s-})\,d[X,Y]_s^c + \int_{0+}^{\sigma_n\wedge t} U_{yy}^{\delta,\lambda}(X_{s-},Y_{s-})\,d[Y,Y]_s^c,$$
$$I_4 = \sum_{0<s\le\sigma_n\wedge t}\big[U^{\delta,\lambda}(X_s,Y_s) - U^{\delta,\lambda}(X_{s-},Y_{s-})$$
$$-\,\langle\nabla U^{\delta,\lambda}(X_{s-},Y_{s-}),(\Delta X_s,\Delta Y_s)\rangle\big].$$

The term I_1 is a martingale (as a function of t), so $\mathbb{E}I_1 = 0$. By the α-subordination of Y to X and (2.5), we have

$$I_2 \le \int_{0+}^{\sigma_n\wedge t} U_x^{\delta,\lambda}(X_{s-},Y_{s-})\,dA_s + \int_{0+}^{\sigma_n\wedge t} |U_y^{\delta,\lambda}(X_{s-},Y_{s-})|\,d|B|_s$$
$$\le \int_{0+}^{\sigma_n\wedge t} [U_x^{\delta,\lambda}(X_{s-},Y_{s-}) + \alpha|U_y^{\delta,\lambda}(X_{s-},Y_{s-})|]\,dA_s \le 0.$$

The term I_3 is also nonpositive, which is due to (2.6) and the Kunita–Watanabe inequality (cf. [9]). Indeed, for $0\le s_0<s_1\le t$ we have

$$[X^c,Y^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1} \le ([X^c,X^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1})^{1/2}([Y^c,Y^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1})^{1/2}$$

and, by the differential subordination of Y^c to X^c (see Lemma 2.5), we get $[Y^c,Y^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1} \le [X^c,X^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1}$. Combining these two observations with (2.6) gives

$$U_{xx}^{\delta,\lambda}(X_{s_0-},Y_{s_0-})[X^c,X^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1} + 2U_{xy}^{\delta,\lambda}(X_{s_0-},Y_{s_0-})[X^c,Y^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1}$$
$$+\,U_{yy}^{\delta,\lambda}(X_{s_0-},Y_{s_0-})[Y^c,Y^c]_{\sigma_n\wedge s_0}^{\sigma_n\wedge s_1} \le 0,$$

which implies $I_3\le 0$, by the approximation of integrals by Riemann sums. Finally, $I_4\le 0$ because of the concavity of the function $U^{\delta,\lambda}$ along the lines of slope $k\in[-1,1]$ and the fact that $|\Delta Y_s|\le|\Delta X_s|$, in virtue of differential subordination. Consequently, combining all the above facts with (2.7) and taking the expectation of both sides yields

$$\mathbb{E}U^{\delta,\lambda}(X_{\sigma_n\wedge t},Y_{\sigma_n\wedge t}) \le \mathbb{E}U^{\delta,\lambda}(x,y) = U^{\delta,\lambda}(x,y).$$

STEP 3: *Limiting arguments.* Let $\delta\to 0$ and $n\to\infty$. The function $U^{\delta,\lambda}$ is continuous, so $U^{\delta,\lambda}(x,y)\to U^{(\lambda)}(\varepsilon+(1-2\varepsilon)x,(1-2\varepsilon)y)$ and

$$U^{\delta,\lambda}(X_{\sigma_n\wedge t},Y_{\sigma_n\wedge t}) \to U^{(\lambda)}(\varepsilon+(1-2\varepsilon)X_t,(1-2\varepsilon)Y_t)$$

almost surely. Next, note that (2.4) yields

$$|U^{\delta,\lambda}(X_{\sigma_n\wedge t},Y_{\sigma_n\wedge t})| \le (1-2\varepsilon)|Y_{\sigma_n\wedge t}| + \delta + C \le \sup_{0\le s\le t}|Y_s| + C + 1.$$

The random variable $\sup_{0\le s\le t}|Y_s|$ is integrable (cf. [12]), so by Lebesgue's dominated convergence theorem, we obtain

$$\mathbb{E}U^{(\lambda)}(\varepsilon+(1-2\varepsilon)X_t,(1-2\varepsilon)Y_t)\le U^{(\lambda)}(\varepsilon+(1-2\varepsilon)x,(1-2\varepsilon)y).$$

This, by (2.3), gives

$$\mathbb{E}\bigg(|(1-2\varepsilon)Y_t|-\lambda-\frac{2(\alpha+1)^2}{2\alpha+1}\bigg)\chi_{(\lambda,\infty)}((1-2\varepsilon)|Y_t|)\\ \le U^{(\lambda)}(\varepsilon+(1-2\varepsilon)x,(1-2\varepsilon)y).$$

Now we let $\varepsilon\to 0$ and apply Lebesgue's dominated convergence theorem again to get

$$\mathbb{E}\bigg(|Y_t|-\lambda-\frac{2(\alpha+1)^2}{2\alpha+1}\bigg)\chi_{(\lambda,\infty)}(|Y_t|)\le U^{(\lambda)}(x,y)\le 0,$$

where the last inequality is due to (2.2). This is precisely the claim. ■

Now we will show how to deduce the inequality (1.6). Fix α, X, Y as in the statement of Theorem 1.1 and let $t\ge 0$, $s\in(0,1]$. We have

$$Y_t^{**}(s)=\sup\bigg\{\frac{1}{s}\int_A|Y_t|\,d\mathbb{P}:A\in\mathcal{F},\,\mathbb{P}(A)=s\bigg\}.$$

Let $\lambda\ge 0$ be the smallest number such that $\mathbb{P}(|Y_t|>\lambda)\le s\le\mathbb{P}(|Y_t|\ge\lambda)$. Clearly, the above supremum is attained for A satisfying $\{|Y_t|>\lambda\}\subseteq A\subseteq\{|Y_t|\ge\lambda\}$ and the required condition $\mathbb{P}(A)=s$. By the definition of a nonincreasing rearrangement, we get $Y_t^*(s)=\lambda$. So,

$$\begin{aligned}Y_t^{**}(s)-Y_t^*(s)&=\frac{1}{s}\int_A(|Y_t|-\lambda)\,d\mathbb{P}\\&\le\frac{1}{\mathbb{P}(|Y_t|>\lambda)}\mathbb{E}(|Y_t|-\lambda)\chi_{(\lambda,\infty)}(|Y_t|)\le\frac{2(\alpha+1)^2}{2\alpha+1},\end{aligned}$$

where the latter inequality is due to (2.1). Since s was arbitrary, the estimate (1.6) follows.

3. Sharpness. Now we will show that the constant $2(\alpha+1)^2/(2\alpha+1)$ is the best possible even for the class of Itô processes (1.1). It will be convenient for us to work with discrete-time processes. Suppose that the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is equipped with a filtration $(\mathcal{F}_n)_{n=0,1,\ldots}$. Let $f=(f_n)_{n\ge0}$ be an adapted, nonnegative submartingale with the corresponding difference sequence $df=(df_n)_{n\ge0}$ defined by $df_0=f_0$ and $df_n=f_n-f_{n-1}$, $n=1,2,\ldots$. Let $g=(g_n)_{n\ge0}$ be an adapted sequence of integrable random variables. If we treat f and g as continuous-time processes (via $f_t=f_{\lfloor t\rfloor}$, $g_t=g_{\lfloor t\rfloor}$, $t\ge0$), then it is easy to check that g is α-subordinate to f if and only if for any $n\ge0$ we have

$$|dg_n|\le|df_n|,\qquad \mathbb{E}(|dg_{n+1}|\,|\mathcal{F}_n)\le\alpha\mathbb{E}(df_{n+1}|\mathcal{F}_n).$$

We will show that the constant $2(\alpha+1)^2/(2\alpha+1)$ is optimal even for processes induced by the discrete-time setting. Then the passage to Itô processes is done by appropriate embedding into Brownian motion (see [13] for details).

Let $\alpha \in (0,1]$, $\delta \in (0, \alpha/(2\alpha+1))$ be fixed. Consider the two-dimensional Markov family (f,g) with the transition function uniquely determined by the following conditions:

(i) The state $(1/2,1/2)$ leads to $(1,0)$ or to $(0,1)$ with probabilities $1/2$.
(ii) The state $(\alpha/(2\alpha+1),(\alpha+1)/(2\alpha+1))$ leads to $(1,0)$ or to $(0,1)$ with probabilities $\alpha/(2\alpha+1)$ and $(\alpha+1)/(2\alpha+1)$, respectively.
(iii) For any $y \geq (\alpha+1)/(2\alpha+1)+2\delta$, the state $(\alpha/(2\alpha+1), y)$ leads to $(\alpha/(2\alpha+1)-\delta, y-\delta)$ or to $(1, y+(\alpha+1)/(2\alpha+1))$ with probabilities
$$\frac{\alpha+1}{\alpha+1+(2\alpha+1)\delta} \quad \text{and} \quad \frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta},$$
respectively.
(iv) For any $y \geq (\alpha+1)/(2\alpha+1)+2\delta$, the state $(\alpha/(2\alpha+1)-\delta, y-\delta)$ leads to $(0, y+\alpha/(2\alpha+1)-2\delta)$ or to $(\alpha/(2\alpha+1), y-2\delta)$ with probabilities $(2\alpha+1)\delta/\alpha$ and $1-(2\alpha+1)\delta/\alpha$, respectively.
(v) For any $y \geq 1$, the state $(0,y)$ leads to $(2\delta/(\alpha+1), y+2\alpha\delta/(\alpha+1))$.
(vi) For any $y \geq 1$, the state $(2\delta/(\alpha+1), y+2\alpha\delta/(\alpha+1))$ leads to $(0, y+2\delta)$ or to $(\alpha/(2\alpha+1), y+2\delta-\alpha/(2\alpha+1))$ with probabilities $2\delta/\alpha$ and $1-2\delta/\alpha$, respectively.
(vii) All the remaining states are absorbing.

It is easy to check that if the process (f,g) starts from $(1/2,1/2)$, then f and g are nonnegative submartingales such that g is α-subordinate to f. By the escape bounds of Burkholder [4], g converges almost surely to a limit g_∞; it is easy to see that this random variable takes values in $\{0\} \cup [(2\alpha+2)/(2\alpha+1), \infty)$. The further analysis splits into three parts.

STEP 1. We will first derive the probability that (f,g) ever visits the point $(1,0)$. To use the above Markov description, we extend this problem to an arbitrary starting point: for any $(x,y) \in \mathbb{R}^2$, define
$$P(x,y) = \mathbb{P}((f,g) \text{ ever visits } (1,0) \mid (f_0,g_0)=(x,y)).$$
For notational convenience, introduce the functions $A(y) = P(\alpha/(2\alpha+1), y)$ and $B(y) = P(0, y+\alpha/(2\alpha+1))$. By (iii), we derive that
$$\begin{aligned} A(y) &= \frac{\alpha+1}{\alpha+1+(2\alpha+1)\delta} P\Big(\frac{\alpha}{2\alpha+1}-\delta, y-\delta\Big) \\ &\quad + \frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta} P\Big(1, y+\frac{\alpha+1}{2\alpha+1}\Big). \end{aligned} \tag{3.1}$$

But the state $(1, y+(\alpha+1)/(2\alpha+1))$ is absorbing, so the second term on the right is equal to zero. To handle the first term, we exploit the condition (iv):

$$P\left(\frac{\alpha}{2\alpha+1}-\delta, y-\delta\right)=\frac{(2\alpha+1)\delta}{\alpha}B(y-2\delta)+\left(1-\frac{(2\alpha+1)\delta}{\alpha}\right)A(y-2\delta).$$

This, combined with the preceding equality, yields

$$\begin{aligned}(3.2)\qquad A(y)&=\frac{(\alpha+1)(\alpha-\delta(2\alpha+1))}{\alpha(\alpha+1+(2\alpha+1)\delta)}A(y-2\delta)\\&\quad+\frac{(\alpha+1)(2\alpha+1)\delta}{\alpha(\alpha+1+(2\alpha+1)\delta)}B(y-2\delta).\end{aligned}$$

Next, we use (v) and (vi) to obtain

$$\begin{aligned}(3.3)\qquad B(y)&=P\left(\frac{2\delta}{\alpha+1}, y+\frac{\alpha}{2\alpha+1}+\frac{2\alpha\delta}{\alpha+1}\right)\\&=\frac{2\delta(2\alpha+1)}{\alpha(\alpha+1)}A(y)+\frac{\alpha(\alpha+1)-2\delta(2\alpha+1)}{\alpha(\alpha+1)}B(y).\end{aligned}$$

Now, multiply the equation (3.3) by

$$\lambda=\frac{(\alpha+1)(2\alpha+3)+(\alpha+1)\sqrt{(2\alpha+1)^2-8(2\alpha+1)\delta}}{4(\alpha+1+(2\alpha+1)\delta)}$$

and add it to (3.2). After some calculations, one obtains the identity

$$(3.4)\qquad \gamma_1 A(y-2\delta)-\gamma_2 B(y-2\delta)=r(\gamma_1 A(y)-\gamma_2 B(y))$$

with

$$\gamma_1=\frac{(\alpha+1)(\alpha-\delta(2\alpha+1))}{\alpha(\alpha+1+(2\alpha+1)\delta)},\qquad \gamma_2=\lambda-\frac{(\alpha+1)(2\alpha+1)\delta}{\alpha(\alpha+1+(2\alpha+1)\delta)}$$

and

$$\begin{aligned}r&=\frac{(\alpha(\alpha+1)-2\delta(2\alpha+1)\lambda)(\alpha+1+(2\alpha+1)\delta)}{(\alpha+1)^2(\alpha-\delta(2\alpha+1))}\\&=1+\frac{\delta(2\alpha+1)(-2\lambda+2\alpha+1)+O(\delta^2)}{(\alpha+1)(\alpha-\delta(2\alpha+1)}.\end{aligned}$$

The reason for the above complicated choice for λ is that it guarantees that on both sides of (3.4) we have the same coefficients of the function A and the same coefficients of B. This fact enables the use of induction: we deduce that for N,

$$\gamma_1 A(y-2\delta)-\gamma_2 B(y-2\delta)=r^N[\gamma_1 A(y+2\delta(N-1))-\gamma_2 B(y+2\delta(N-1))].$$

However, when $\delta\to 0$, then $\lambda\to\alpha+1$; hence r is smaller than 1 if δ is sufficiently close to 0. Furthermore, directly from the definition it can be seen that the functions A and B take values in $[0,1]$. Thus, letting

$N \to \infty$ yields $\gamma_1 A(y-2\delta) - \gamma_2 B(y-2\delta) = 0$ and hence, in particular,

$$\gamma_1 A\left(\frac{\alpha+1}{2\alpha+1}\right) = \gamma_2 B\left(\frac{\alpha+1}{2\alpha+1}\right).$$

On the other hand, condition (ii) implies

$$\begin{aligned} A\left(\frac{\alpha+1}{2\alpha+1}\right) &= \frac{\alpha}{2\alpha+1}P(1,0) + \frac{\alpha+1}{2\alpha+1}B\left(\frac{\alpha+1}{2\alpha+1}\right) \\ &= \frac{\alpha}{2\alpha+1} + \frac{\alpha+1}{2\alpha+1}B\left(\frac{\alpha+1}{2\alpha+1}\right). \end{aligned}$$

Combining the latter two equations gives

$$B\left(\frac{\alpha+1}{2\alpha+1}\right) = \frac{\alpha}{2\alpha+1}\left(\frac{\gamma_2}{\gamma_1} - \frac{\alpha+1}{2\alpha+1}\right)^{-1}$$

and hence, by (i),

$$\begin{aligned} P\left(\frac{1}{2},\frac{1}{2}\right) &= \frac{1}{2}P(1,0) + \frac{1}{2}B\left(\frac{\alpha+1}{2\alpha+1}\right) \\ &= \frac{1}{2} + \frac{\alpha}{2(2\alpha+1)}\left(\frac{\gamma_2}{\gamma_1} - \frac{\alpha+1}{2\alpha+1}\right)^{-1}. \end{aligned} \tag{3.5}$$

STEP 2. The second part of the analysis concerns the first norm of the sequence g, i.e., the value of $\mathbb{E}g_\infty$, where g_∞ is the pointwise limit of g. As previously, we consider the more general setting in which the process (f,g) starts from an arbitrary point (x,y) in $\mathbb{R}^2$ and define

$$E(x,y) = \mathbb{E}[g_\infty \mid (f_0,g_0) = (x,y)].$$

With a slight abuse of notation (but, hopefully, for the convenience of the reader), set $A(y) = E(\alpha/(2\alpha+1), y)$ and $B(y) = E(0, y+\alpha/(2\alpha+1))$, in analogy to the above considerations. Then all the above calculations remain valid, with a small change: the term $P\big(1, y+\frac{\alpha+1}{2\alpha+1}\big)$ vanished in (3.1), now the value of $E\big(1, y+\frac{\alpha+1}{2\alpha+1}\big)$ is equal to $y+\frac{\alpha+1}{2\alpha+1}$. So, the analogue of (3.2) is

$$\begin{aligned} A(y) = {} & \frac{(\alpha+1)(\alpha-\delta(2\alpha+1))}{\alpha(\alpha+1+(2\alpha+1)\delta)}A(y-2\delta) \\ &+ \frac{(\alpha+1)(2\alpha+1)\delta}{\alpha(\alpha+1+(2\alpha+1)\delta)}B(y-2\delta) \\ &+ \frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta}\left(y+\frac{\alpha}{2\alpha+1}\right), \end{aligned} \tag{3.6}$$

and the equality (3.3) holds true. Multiply (3.3) by λ (the same as above) and add it to (3.6). After some computations, one obtains

$$\begin{aligned}\gamma_1 A(y-2\delta) &- \gamma_2 B(y-2\delta)\\ &= r(\gamma_1 A(y)-\gamma_2 B(y)) - \frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta}\left(y+\frac{\alpha+1}{2\alpha+1}\right),\end{aligned}$$

where γ_1, γ_2, r are as above. Hence, by induction,

$$\begin{aligned}\gamma_1 A(y-2\delta) &- \gamma_2 B(y-2\delta)\\ &= r^N\big(\gamma_1 A(y+2(N-1)\delta)-\gamma_2 B(y+2(N-1)\delta)\big)\\ &\quad - \frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta}\sum_{k=0}^{N-1} r^k\left(y+2k\delta+\frac{\alpha+1}{2\alpha+1}\right).\end{aligned}$$

Now, if δ is sufficiently close to zero, then r is smaller than 1. Since A and B have linear growth at infinity, letting $N\to\infty$ above yields

$$\begin{aligned}\gamma_1 A(y-2\delta) &- \gamma_2 B(y-2\delta)\\ &= -\frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta}\sum_{k=0}^{\infty} r^k\left(y+2k\delta+\frac{\alpha+1}{2\alpha+1}\right)\\ &= -\frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta}\left[\left(y+\frac{\alpha+1}{2\alpha+1}\right)\frac{1}{1-r}+\frac{2\delta}{(1-r)^2}\right],\end{aligned}$$

and hence in particular

$$\begin{aligned}\gamma_1 A\left(\frac{\alpha+1}{2\alpha+1}\right) &- \gamma_2 B\left(\frac{\alpha+1}{2\alpha+1}\right)\\ &= -\frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta}\left[\left(2\delta+\frac{2\alpha+2}{2\alpha+1}\right)\frac{1}{1-r}+\frac{2\delta}{(1-r)^2}\right].\end{aligned}$$

On the other hand, condition (ii) gives

$$\begin{aligned}A\left(\frac{\alpha+1}{2\alpha+1}\right) &= \frac{\alpha}{2\alpha+1}E(1,0)+\frac{\alpha+1}{2\alpha+1}B\left(\frac{\alpha+1}{2\alpha+1}\right)\\ &= \frac{\alpha+1}{2\alpha+1}B\left(\frac{\alpha+1}{2\alpha+1}\right),\end{aligned}$$

which combined with the preceding identity yields

$$\begin{aligned}B\left(\frac{\alpha+1}{2\alpha+1}\right) &= -\frac{(2\alpha+1)\delta}{\alpha+1+(2\alpha+1)\delta}\left(\frac{\alpha+1}{2\alpha+1}\gamma_1-\gamma_2\right)^{-1}\\ &\quad\times\left[\left(2\delta+\frac{2\alpha+2}{2\alpha+1}\right)\frac{1}{1-r}+\frac{2\delta}{(1-r)^2}\right].\end{aligned}$$

Hence by (i),

$$E\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}B(1,0)+\frac{1}{2}B\left(\frac{\alpha+1}{2\alpha+1}\right) \tag{3.7}$$
$$=-\frac{(2\alpha+1)\delta}{2(\alpha+1+(2\alpha+1)\delta)}\left(\frac{\alpha+1}{2\alpha+1}\gamma_1-\gamma_2\right)^{-1}$$
$$\times\left[\left(2\delta+\frac{2\alpha+2}{2\alpha+1}\right)\frac{1}{1-r}+\frac{2\delta}{(1-r)^2}\right].$$

STEP 3. Now we carry out the final limiting procedure: we let $\delta\to 0$. It is not difficult to check that then the expressions on the right of (3.5) and (3.7) simplify considerably: we obtain

$$P\left(\frac{1}{2},\frac{1}{2}\right)\to\frac{1}{2}+\frac{1}{4(\alpha+1)},\qquad E\left(\frac{1}{2},\frac{1}{2}\right)\to\frac{\alpha+1}{2}.$$

So, if δ is sufficiently small and n is sufficiently large, then $\mathbb{E}g_n$ can be made arbitrarily close to $(\alpha+1)/2$, and the probability that g_n is positive can be made arbitrarily close to $1-(1/2+1/(4(\alpha+1)))=(2\alpha+1)/(4\alpha+4)$. Now take $t>(2\alpha+1)/(4\alpha+4)$ and $\varepsilon>0$. By the very definition of g_n^*, the preceding analysis gives $g_n^*(t)<\varepsilon$ provided δ is sufficiently small. On the other hand,

$$g_n^{**}(t)=\sup\left\{\frac{1}{t}\int_A g_n\,d\mathbb{P}: A\in\mathcal{F},\,\mathbb{P}(A)=t\right\}.$$

But if δ is small and n is large, then g_n *vanishes* on a set of probability larger than $1-t$, so $g_n^{**}(t)=t^{-1}\mathbb{E}g_n$. Thus, the norm $\|g_n\|_W$ can be made arbitrarily close to

$$\frac{4\alpha+4}{2\alpha+1}\cdot\frac{\alpha+1}{2}=\frac{2(\alpha+1)^2}{2\alpha+1}.$$

This proves the desired optimality of the constant.

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