COLLOQUIUM MATHEMATICUM

VOL. 141 2015 NO. 2

SINGULARITY CATEGORIES OF SKEWED-GENTLE ALGEBRAS

BY

XINHONG CHEN and MING LU (Chengdu)

Abstract. Let K be an algebraically closed field. Let (Q, Sp, I) be a skewed-gentle triple, and let (Q^{sg}, I^{sg}) and (Q^g, I^g) be the corresponding skewed-gentle pair and the associated gentle pair, respectively. We prove that the skewed-gentle algebra $KQ^{sg}/\langle I^{sg}\rangle$ is singularity equivalent to $KQ/\langle I\rangle$. Moreover, we use (Q, Sp, I) to describe the singularity category of $KQ^g/\langle I^g\rangle$. As a corollary, we find that $\operatorname{gldim} KQ^{sg}/\langle I^{sg}\rangle < \infty$ if and only if $\operatorname{gldim} KQ/\langle I\rangle < \infty$ if and only if $\operatorname{gldim} KQ^g/\langle I^g\rangle < \infty$.

1. Introduction. The singularity category of an algebra is defined to be the Verdier quotient of the bounded derived category with respect to the thick subcategory formed by complexes isomorphic to bounded complexes of finitely generated projective modules ([7], see also [13]). Recently, Orlov's global version [17] attracted a lot of interest in algebraic geometry and theoretical physics. In particular, the singularity category measures the homological singularity of an algebra [13]: the algebra has finite global dimension if and only if its singularity category is trivial.

A fundamental result of Buchweitz [7] and Happel [13] states that for a Gorenstein algebra A, the singularity category is triangle equivalent to the stable category of (maximal) Cohen–Macaulay (also called Gorenstein projective) A-modules, which generalizes Rickard's result [19] on self-injective algebras.

As an important class of Gorenstein algebras [12], gentle algebras were introduced in [2] as an appropriate context for the investigation of algebras derived equivalent to hereditary algebras of type $\tilde{\mathbb{A}}_n$. Moreover, many important algebras are gentle, such as tilted algebras of type $\mathbb{A}_n$, algebras derived equivalent to $\mathbb{A}_n$-configurations of projective lines [8], and also the cluster-tilted algebras of type $\mathbb{A}_n$, $\tilde{\mathbb{A}}_n$ [1]. As a generalization of gentle algebras, skewed-gentle algebras were introduced by Geiß and de la Peña [11], who also proved that a skewed-gentle algebra is Morita equivalent to a skew-group algebra of a gentle algebra (which is called the *associated gentle algebra* in this note) with a group of order two. In this way, skewed-gentle algebras

2010 *Mathematics Subject Classification*: 16G20, 16E35, 18E30.

Key words and phrases: gentle algebras, skewed-gentle algebras, Gorenstein algebras, singularity categories.

DOI: 10.4064/cm141-2-4

and gentle algebras share many common properties, such as the Gorenstein property [12]. Moreover, the indecomposable objects in the derived categories of gentle algebras and skewed-gentle algebras are described in [5, 4]. Kalck [14] determined the singularity categories of gentle algebras in terms of finite products of n-cluster categories of type $\mathbb{A}_1$.

The aim of this note is to describe the singularity categories for skewed-gentle algebras, following Kalck's work [14]. In order to state our main results, we need to introduce some notation. Let (Q, Sp, I) be a skewed-gentle triple, (Q^{sg}, I^{sg}) the corresponding skewed-gentle pair, and (Q^g, I^g) the associated gentle pair. Inspired by [9], which shows that a certain homological epimorphism between two algebras induces a triangle equivalence between their singularity categories, we prove that there is a morphism of this type between the skewed-gentle algebra $KQ^{sg}/\langle I^{sg}\rangle$ and the gentle algebra $KQ/\langle I\rangle$, and so they are singularity equivalent (Theorem 3.5). Moreover, with the help of [14], we also use (Q, Sp, I) to describe the singularity category of the associated gentle algebra $KQ^g/\langle I^g\rangle$, and then get a relation between it and the singularity category of $KQ^{sg}/\langle I^{sg}\rangle$ (Theorem 4.4). As a direct corollary, the global dimension of $KQ^{sg}/\langle I^{sg}\rangle$ is finite if and only if the global dimension of $KQ^g/\langle I^g\rangle$ is finite, if and only if the global dimension of $KQ/\langle I\rangle$ is finite, without any restriction on the characteristic of the field K (Corollary 4.5).

2. Preliminaries. Throughout this note, we always assume that K is an algebraically closed field. For any finite set S, we denote by $\sharp(S)$ the number of elements in S. For any algebra A, we denote by $\operatorname{gldim} A$ its global dimension.

Let Q be a quiver and $\langle I\rangle$ an admissible ideal in the path algebra KQ which is generated by a set I of relations. Denote by (Q, I) the *associated bound quiver*. For any arrow α in Q, we denote by $s(\alpha)$ its starting vertex and by $t(\alpha)$ its ending vertex. An *oriented path* (or path for short) p in Q is a sequence $p = \alpha_1 \dots \alpha_r$ of arrows α_i such that $t(\alpha_i) = s(\alpha_{i-1})$ for all $i = 2, \dots, r$. For any two paths p_1, p_2 in (Q, I), we write $p_1 \sim p_2$ if $p_1 - p_2 \in \langle I\rangle$.

2.1. Gentle algebras. We first recall the definition of special biserial algebras and of gentle algebras.

DEFINITION 2.1 ([20]). The pair (Q, I) is called *special biserial* if:

- Each vertex of Q is the starting point of at most two arrows, and the end point of at most two arrows.
- For each arrow α in Q there is at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.

Definition 2.2 ([2]). The pair (Q, I) is called *gentle* if it is special biserial and:

- The set I is generated by zero relations of length 2.
- For each arrow α in Q there is at most one arrow β with $t(\beta) = s(\alpha)$ such that $\alpha\beta \in I$, and at most one arrow γ with $s(\gamma) = t(\alpha)$ such that $\gamma\alpha \in I$.

A finite-dimensional algebra A is called *special biserial* (resp. *gentle*) if it has a presentation $A = KQ/\langle I\rangle$ where (Q, I) is special biserial (resp. gentle).

2.2. Skewed-gentle algebras. Skewed-gentle algebras were introduced in [11]; here we mostly follow [4].

Let (Q, I) be a gentle pair. Let Sp be a subset of vertices of the quiver Q whose elements are called *special vertices*; the remaining vertices are called *ordinary.*

For a triple (Q, Sp, I), set $Q_0^{sp} := Q_0$, $Q_1^{sp} := Q_1 \cup \{\alpha_i \mid i \in Sp,\, s(\alpha_i) = t(\alpha_i) = i\}$ and $I^{sp} := I \cup \{\alpha_i^2 \mid i \in Sp\}$.

Definition 2.3. A triple (Q, Sp, I) as above is called *skewed-gentle* if the pair (Q^{sp}, I^{sp}) is gentle.

For any vertex in a quiver Q, its *valency* is defined as the number of arrows attached to it, i.e. the number of incoming arrows plus the number of outgoing arrows (note that in particular any loop contributes twice to the valency).

In fact, Bessenrodt and Holm [6] pointed out that the admissibility of the set Sp of special vertices is both a local and a global condition. Let v be a vertex in the gentle quiver (Q, I); then we can only add a loop at v if v is of valency 1 or 0, or if it is of valency 2 with a zero relation, but not one coming from a loop. Furthermore, for the choice of an admissible set of special vertices we also have to take care of the global condition that after adding all loops, the pair (Q^{sp}, I^{sp}) still does not have paths of arbitrary lengths.

Example 2.4. (a) Let (Q, I) be the bound quiver as in the diagram below. Then (Q, I) is gentle. In order to have (Q, Sp, I) skewed-gentle, the set Sp can only be $\{1\}$, $\{2\}$ or the empty set.

$$\begin{tikzcd} \circ^1 \arrow[r, shift left, "\alpha"] & \circ^2 \arrow[l, shift left, "\beta"] \end{tikzcd} \qquad I = \{\alpha\beta, \beta\alpha\}$$

(b) Let (Q, I) be the bound quiver as in the diagram below. Then (Q, I) is gentle. In order to have (Q, Sp, I) skewed-gentle, Sp can only be $\{1\}$, $\{2\}$,

$\{3\}$, $\{1,2\}$, $\{2,3\}$, $\{1,3\}$ or the empty set.

$$\begin{tikzcd} & \circ^3 \arrow[dl, "\gamma"'] & \\ \circ^1 \arrow[rr, "\alpha"'] & & \circ^2 \arrow[ul, "\beta"'] \end{tikzcd} \qquad I = \{\alpha\gamma, \beta\alpha, \gamma\beta\}$$

Let (Q, Sp, I) be a skewed-gentle triple. We associate to each vertex $i \in Q_0$ a set, denoted by $Q_0(i)$, in the following way: If i is an ordinary vertex then $Q_0(i) = \{i\}$; if i is special then $Q_0(i) = \{i^-, i^+\}$. We denote by (Q^{sg}, I^{sg}) the pair defined as follows:

$$Q_0^{sg} := \bigcup_{i \in Q_0} Q_0(i),$$

$$Q_1^{sg}[a,b] := \{(a, \alpha, b) \mid \alpha \in Q_1,\ a \in Q_0(s(\alpha)),\ b \in Q_0(t(\alpha))\},$$

$$I^{sg} := \Big\{ \sum_{b \in Q_0(s(\alpha))} \lambda_b (b, \alpha, c)(a, \beta, b) \;\Big|\; \alpha\beta \in I,\ a \in Q_0(s(\beta)),\ c \in Q_0(t(\alpha)) \Big\},$$

where $\lambda_b = -1$ if $b = i^-$ for some $i \in Q_0$, and $\lambda_b = 1$ otherwise.

Note that the relations in I^{sg} are zero relations or commutativity relations.

Definition 2.5 ([11]). A K-algebra A is called *skewed-gentle* if it is Morita equivalent to a factor algebra $KQ^{sg}/\langle I^{sg} \rangle$, where the triple (Q, Sp, I) is skewed-gentle. The corresponding pair (Q^{sg}, I^{sg}) is also said to be skewed-gentle.

Example 2.6.

(a) In the notation of Example 2.4(a), if $Sp = \{2\}$ then (Q^{sg}, I^{sg}) is as shown below.

$$\begin{tikzcd} \circ^{2^-} \arrow[r, shift left, "\beta^-"] & \circ^1 \arrow[l, shift left, "\alpha^-"] \arrow[r, shift left, "\alpha^+"] & \circ^{2^+} \arrow[l, shift left, "\beta^+"] \end{tikzcd} \qquad I^{sg} = \{\alpha^\pm\beta^\pm, \beta^+\alpha^+ - \beta^-\alpha^-\}$$

(b) In the notation of Example 2.4(b), if $Sp = \{3\}$ then (Q^{sg}, I^{sg}) is as shown below.

$$\begin{tikzcd} & \circ^{3^+} \arrow[dl, "\gamma^+"'] & \\ \circ^1 \arrow[rr, "\alpha"'] & & \circ^2 \arrow[ul, "\beta^+"'] \arrow[dl, "\beta^-"] \\ & \circ^{3^-} \arrow[ul, "\gamma^-"] & \end{tikzcd} \qquad I^{sg} = \{\alpha\gamma^\pm, \beta^\pm\alpha, \gamma^+\beta^+ - \gamma^-\beta^-\}$$

2.3. Skew-group algebras. It follows from Geiß and de la Peña [11] that for any skewed-gentle triple (Q, Sp, I), the corresponding skewed-gentle algebra $KQ^{sg}/\langle I^{sg}\rangle$ is Morita equivalent to a skew-group algebra BG in the case of $\operatorname{char} K \neq 2$, where B is a gentle algebra and G is a finite group (see also [4]). We now give some details.

Let A be a K-algebra, and G a finite group acting on A via K-linear automorphisms. The *skew-group algebra* AG is the vector space $\bigoplus_{g\in G} A[g]$ with multiplication induced by

$$a[g]b[h] := ag(b)[gh].$$

Let (Q, Sp, I) be a skewed-gentle triple. For a given special (resp. ordinary) vertex i, denote by $Q_0[i]$ the set $\{i\}$ (resp. $\{i^-, i^+\}$). Consider the pair (Q^g, I^g), where $Q_0^g := \bigcup_{i\in Q_0} Q_0[i]$, $Q_1^g := \{\alpha^+, \alpha^- \mid \alpha \in Q_1\}$,

$$s(\alpha^\pm) := \begin{cases} s(\alpha)^\pm & \text{if } s(\alpha) \notin Sp, \\ s(\alpha) & \text{if } s(\alpha) \in Sp, \end{cases} \qquad t(\alpha^\pm) := \begin{cases} t(\alpha)^\pm & \text{if } t(\alpha) \notin Sp, \\ t(\alpha) & \text{if } t(\alpha) \in Sp, \end{cases}$$

and

$$I^g := \{\beta^+\alpha^+, \beta^-\alpha^- \mid \beta\alpha \in I,\ t(\alpha) \notin Sp\} \cup \{\beta^+\alpha^-, \beta^-\alpha^+ \mid \beta\alpha \in I,\ t(\alpha) \in Sp\}.$$

It follows from [11] that the algebra $B := KQ^g/\langle I^g\rangle$ is gentle. We call (Q^g, I^g) (resp. $KQ^g/\langle I^g\rangle$) the *associated gentle pair* (resp. *associated gentle algebra*) of (Q, Sp, I) or $KQ^{sg}/\langle I^{sg}\rangle$.

Consider the group $G = \{e, g \mid g^2 = e\}$ which acts on B by the rule

$$g(i^\pm) := i^\mp, \quad g(j) := j, \quad g(\alpha^\pm) := \alpha^\mp,$$

for all $i \in Q_0 \setminus Sp$, $j \in Sp$ and $\alpha \in Q_1$. Then we get the skew-group algebra BG.

Example 2.7. (a) In the notation of Example 2.6(a), if $Sp = \{2\}$ then (Q^g, I^g) is as shown below.

$$\begin{tikzcd} \circ^{1^+} \arrow[r, shift left, "\alpha^+"] & \circ^{2} \arrow[l, shift left, "\beta^+"] \arrow[r, shift left, "\beta^-"] & \circ^{1^-} \arrow[l, shift left, "\alpha^-"] \end{tikzcd} \qquad I^g = \{\alpha^+\beta^+, \alpha^-\beta^-, \beta^+\alpha^-, \beta^-\alpha^+\}$$

(b) In the notation of Example 2.6(b), if $Sp = \{3\}$ then (Q^g, I^g) is as shown below.

$$\begin{tikzcd} \circ^{1^-} \arrow[rr, "\alpha^-"] & & \circ^{2^-} \arrow[dl, "\beta^-"] \\ & \circ^{3} \arrow[ul, "\gamma^-"'] \arrow[dl, "\gamma^+"'] & \\ \circ^{1^+} \arrow[rr, "\alpha^+"'] & & \circ^{2^+} \arrow[ul, "\beta^+"'] \end{tikzcd} \qquad I^g = \{\alpha^+\gamma^+, \alpha^-\gamma^-, \beta^+\alpha^+, \beta^-\alpha^-, \gamma^-\beta^+, \gamma^+\beta^-\}$$

2.4. Singularity categories and Gorenstein algebras. Let Γ be a finite-dimensional K-algebra. Let $\operatorname{mod}\Gamma$ be the category of finitely generated left Γ-modules. We denote by $D = \operatorname{Hom}_K(-,K)$ the standard duality with respect to the ground field. Then ${}_\Gamma D(\Gamma_\Gamma)$ is an injective cogenerator for $\operatorname{mod}\Gamma$. For an arbitrary Γ-module ${}_\Gamma X$, we denote by proj.dim ${}_\Gamma X$ (resp. inj.dim ${}_\Gamma X$) the projective dimension (resp. the injective dimension) of the module ${}_\Gamma X$. A Γ-module G is *Gorenstein projective* [10] if there is an exact sequence

$$P^\bullet : \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to \cdots$$

of projective Γ-modules which stays exact after taking $\operatorname{Hom}_\Gamma(-,\Gamma)$, and $G \cong \operatorname{Ker} d^0$. We denote by $\operatorname{Gproj}(\Gamma)$ the subcategory of Gorenstein projective Γ-modules.

Definition 2.8 ([3, 13]). A finite-dimensional algebra Γ is called a *Gorenstein algebra* if Γ satisfies proj.dim ${}_\Gamma D(\Gamma_\Gamma)<\infty$ and inj.dim ${}_\Gamma\Gamma<\infty$.

For an algebra Γ, the *singularity category* of Γ is defined to be the quotient category $D^b_{sg}(\Gamma) := D^b(\Gamma)/K^b(\operatorname{proj}\Gamma)$ [7, 13, 17]. Note that $D^b_{sg}(\Gamma)$ is zero if and only if $\operatorname{gldim}\Gamma < \infty$ [13]. For any two algebras, if their singularity categories are equivalent, then we call them *singularity equivalent.*

Theorem 2.9 ([7, 13]). *Let Γ be a Gorenstein algebra. Then* $\operatorname{Gproj}(\Gamma)$ *is a Frobenius category with the projective modules as the projective-injective objects. The stable category* $\underline{\operatorname{Gproj}}(\Gamma)$ *is triangle equivalent to the singularity category $D^b_{sg}(\Gamma)$ of Γ.*

Geiß and Reiten [12] have shown that gentle algebras are Gorenstein algebras. Since the property of being Gorenstein is also preserved under the skew-group ring construction with a finite group whose order is invertible in K (see [18, 3]), Geiß and Reiten [12] also pointed out that skewed-gentle algebras are Gorenstein algebras in the case of $\operatorname{char} K \neq 2$.

3. The first main theorem. In order to prove the first main result, we describe a construction of matrix algebras given by X.-W. Chen [9] (see also [16, Section 4]). Let A be a finite-dimensional algebra over a field K. Let ${}_AM$ and N_A be a left and a right A-module, respectively. Then $M\otimes_K N$ becomes an A-A-bimodule. Consider an A-A-bimodule monomorphism $\phi : M\otimes_K N \to A$ such that $\operatorname{Im}\phi$ vanishes on both M and N. Note that $\operatorname{Im}\phi \subseteq A$ is an ideal. Then the matrix $\Gamma = \left(\begin{smallmatrix} A & M \\ N & K\end{smallmatrix}\right)$ becomes an associative algebra via the following multiplication:

$$\begin{pmatrix} a & m \\ n & \lambda \end{pmatrix}\begin{pmatrix} a' & m' \\ n' & \lambda' \end{pmatrix} = \begin{pmatrix} aa' + \phi(m\otimes n) & am' + \lambda' m \\ na' + \lambda n' & \lambda\lambda' \end{pmatrix}.$$

PROPOSITION 3.1 ([9]). *Under the notation and assumptions as above, there is a triangle equivalence* $D^b_{sg}(\Gamma) \simeq D^b_{sg}(A/\operatorname{Im}\phi)$.

Note that the above construction contains as special cases the one-point extension and one-point coextension of algebras, where M or N is zero.

The following two lemmas are crucial to the proof of our first main theorem.

LEMMA 3.2. *Let* (Q, Sp, I) *be a skewed-gentle triple and* (Q^{sg}, I^{sg}) *its corresponding skewed-gentle pair. Then for any paths* $p_1 = \alpha_1 \dots \alpha_n$ *and* $p_2 = \alpha_{n+1} \dots \alpha_{n+m}$ *with* $p_1, p_2, \alpha_n\alpha_{n+1} \notin \langle I^{sg}\rangle$, *we have* $p_1p_2 \notin \langle I^{sg}\rangle$.

Proof. The following diagram shows the path p_1p_2:

$$\circ^{a_1} \xleftarrow{\alpha_1} \circ^{a_2} \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_n} \circ^{a_{n+1}} \xleftarrow{\alpha_{n+1}} \circ^{a_{n+2}} \xleftarrow{\alpha_{n+2}} \cdots \xleftarrow{\alpha_{n+m}} \circ^{a_{n+m+1}}.$$

Note that $p_1, p_2, \alpha_n\alpha_{n+1} \notin \langle I^{sg}\rangle$. Moreover, $\alpha_i\alpha_{i+1} \notin I^{sg}$ for any $1 \le i \le n+m-1$. We define an operator Φ on p_1p_2 as follows: For any $2 \le s \le n+m$, if a_s comes from a special vertex $b_s \in Sp$, then there exist two arrows β_1, β_2 with $s(\beta_1) = t(\beta_2)$ such that $a_s \in \{b_s^+, b_s^-\}$ and $\{\alpha_{s-1}, \alpha'_{s-1}\} = \{(a_s, \beta_1, a_{s-1}), (a'_s, \beta_1, a_{s-1})\}$, $\{\alpha_s, \alpha'_s\} = \{(a_{s+1}, \beta_2, a_s), (a_{s+1}, \beta_2, a'_s)\}$, where a'_s is the vertex such that $\{a_s, a'_s\} = \{b_s^+, b_s^-\}$, by the definition of I^{sg}. Set $p'_1 = \alpha'_1 \dots \alpha'_n$ and $p'_2 = \alpha'_{n+1} \dots \alpha'_{n+m}$, where $\alpha'_i = \alpha_i$ for $i \notin \{s-1, s\}$. Then $p_1p_2 \sim p'_1p'_2$ and we define $\Phi(p_1p_2) := p'_1p'_2$.

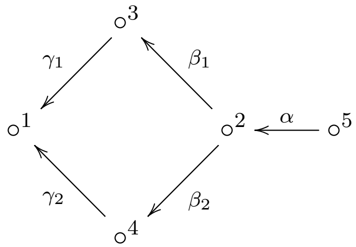
If $\alpha'_i\alpha'_{i+1} \in I^{sg}$ for some $1 \le i \le n+m-1$, then $i = s-2$, $s-1$ or s, since $\alpha'_i\alpha'_{i+1} = \alpha_i\alpha_{i+1} \notin I^{sg}$ for any $i \ne s-2, s-1, s$. However, it is easy to see that $\alpha'_{s-1}\alpha'_s \notin I^{sg}$ from $\alpha'_{s-1}\alpha'_s \sim \alpha_{s-1}\alpha_s$. So $i = s-2$ or s, and we have $\alpha_{s-2}\alpha'_{s-1} \in I^{sg}$ or $\alpha'_s\alpha_{s+1} \in I^{sg}$, by the definition of (Q^{sg}, I^{sg}), which implies that $\alpha_{s-2}\alpha_{s-1} \in I^{sg}$ or $\alpha_s\alpha_{s+1} \in I^{sg}$, respectively, a contradiction. So $\alpha'_i\alpha'_{i+1} \notin I^{sg}$ for any $1 \le i \le n+m-1$.

Suppose $p_1p_2 \in \langle I^{sg}\rangle$. Since $p_1, p_2, \alpha_n\alpha_{n+1} \notin \langle I^{sg}\rangle$ and the relations in I^{sg} are either zero relations or commutativity relations, there must be a finite sequence of operations

$$p_1p_2 \xrightarrow{\Phi_1} p'_1p'_2 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_r} p_1^{(r)}p_2^{(r)},$$

where $p_1^{(i)} = \alpha_1^{(i)} \dots \alpha_n^{(i)}$ is a path of length n and $p_2^{(i)} = \alpha_{n+1}^{(i)} \dots \alpha_{n+m}^{(i)}$ is a path of length m for any $1 \le i \le r$, and Φ_j is the operator defined for some special vertex in $p_1^{(j-1)}p_2^{(j-1)}$ for any $1 \le j \le r$, such that $\alpha_k^{(r)}\alpha_{k+1}^{(r)} \in I^{sg}$ for some $1 \le k \le n+m-1$. By the property of Φ_1 and since $\alpha_i\alpha_{i+1} \notin I^{sg}$ for any $1 \le i \le n+m-1$, we know that $\alpha'_i\alpha'_{i+1} \notin I^{sg}$ for any $1 \le i \le n+m-1$, and discussing Φ_j recursively, we find that $\alpha_i^{(j)}\alpha_{i+1}^{(j)} \notin I^{sg}$ for any $1 \le j \le r$ and $1 \le i \le n+m-1$, which contradicts $\alpha_k^{(r)}\alpha_{k+1}^{(r)} \in I^{sg}$ for some $1 \le k \le n+m-1$. So $p_1p_2 \notin \langle I^{sg}\rangle$. ■

REMARK 3.3. The above lemma is not true for any finite-dimensional algebra $KQ/\langle I\rangle$ with the relations in I being zero relations or commutativity relations. Here is an example. Let (Q, I) be as in the following diagram:



$$I = \{\gamma_1\beta_1 - \gamma_2\beta_2, \beta_2\alpha\}$$

Then $\gamma_1\beta_1, \beta_1\alpha \notin \langle I\rangle$, but $\gamma_1\beta_1\alpha \sim \gamma_2\beta_2\alpha \in \langle I\rangle$.

LEMMA 3.4. *Let (Q, Sp, I) be a skewed-gentle triple and (Q^{sg}, I^{sg}) the corresponding skewed-gentle pair. For any oriented cycle $\alpha_2\alpha_3\ldots\alpha_n\alpha_1$ in Q^{sg}, we have either $\alpha_1\alpha_2 \in \langle I^{sg}\rangle$, or $\alpha_2\alpha_3\ldots\alpha_n\alpha_1 \in \langle I^{sg}\rangle$.*

Proof. Suppose $\alpha_1\alpha_2 \notin \langle I^{sg}\rangle$ and $\alpha_2\alpha_3\ldots\alpha_n\alpha_1 \notin \langle I^{sg}\rangle$. Then Lemma 3.2 implies

$$(\alpha_2\alpha_3\ldots\alpha_n\alpha_1)^n \notin \langle I^{sg}\rangle$$

for any $n > 0$, and so $KQ^{sg}/\langle I^{sg}\rangle$ is infinite-dimensional, a contradiction. ∎

Let Q be any quiver. For any path $c = \alpha_1\ldots\alpha_n$ in Q, say that *c passes through a vertex b* if $b = t(\alpha_j)$ for some $2 \le j \le n$.

The following theorem is the first main result of this paper.

THEOREM 3.5. *Let (Q, Sp, I) be a skewed-gentle triple. Then the corresponding skewed-gentle algebra $KQ^{sg}/\langle I^{sg}\rangle$ is singularity equivalent to the gentle algebra $KQ/\langle I\rangle$.*

Proof. For any vertex $a \in Sp$, there are two vertices a^+, a^- in Q^{sg}. We denote by e_{a^-} the primitive idempotent corresponding to a^-. Set $\Gamma = KQ^{sg}/\langle I^{sg}\rangle$ and $\Gamma' = \Gamma/\Gamma e_{a^-}\Gamma$. The quiver of Γ' is obtained from Q^{sg} by removing the vertex a^- and the adjacent arrows α^+, α^-. Then Γ' is a skewed-gentle algebra, in fact, it is the skewed-gentle algebra corresponding to the skewed-gentle triple $(Q, Sp' = Sp \setminus \{a\}, I)$. We consider the following three cases.

CASE (a). If the valency of a is 0, then $KQ^{sg}/\langle I^{sg}\rangle = \Gamma' \oplus K$, and it is easy to see that $D^b_{sg}(KQ^{sg}/\langle I^{sg}\rangle) \simeq D^b_{sg}(\Gamma')$.

CASE (b). If the valency of a is 1, then $KQ^{sg}/\langle I^{sg}\rangle$ is a one-point extension or a one-point coextension of Γ', so $D^b_{sg}(KQ^{sg}/\langle I^{sg}\rangle) \simeq D^b_{sg}(\Gamma')$ by Proposition 3.1.

Case (c). If the valency of a is 2, then there exist only two arrows α_1, α_2 in Q such that $s(\alpha_1) = a = t(\alpha_2)$, and $\alpha_1\alpha_2 \in I$. Set $b = s(\alpha_2)$ and $c = t(\alpha_1)$. We do not exclude $b = c$. Then there are four subcases.

Case (c1). If $b, c \notin Sp$, then the quiver of Q^{sg} is as in Case (c1) of Figure 1. Then $\alpha_1^+\alpha_2^+ - \alpha_1^-\alpha_2^- \in I^{sg}$.

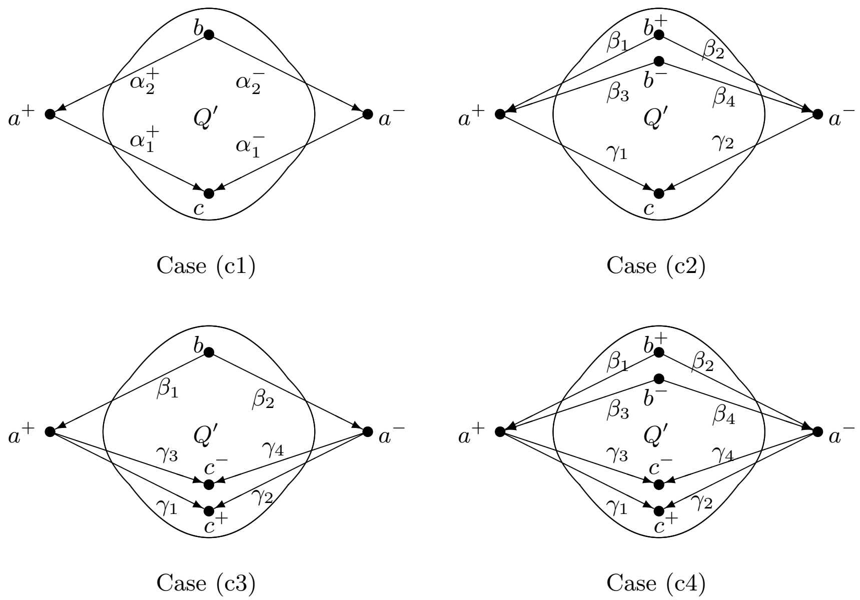


Fig. 1. The quiver Q^{sg} in Case (c)

We fix some notation. Let $A = (1 - e_{a^-})\Gamma(1 - e_{a^-})$,

$$M := \mathrm{Span}_K\{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = p_1\alpha_1^- \text{ for some path } p_1\},$$

$$N := \mathrm{Span}_K\{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = \alpha_2^- p_2 \text{ for some path } p_2\}.$$

Then M is naturally a left A-module, and N is a right A-module. Note that M and N are finite-dimensional vector spaces since Γ is finite-dimensional. Furthermore, we define an A-A-bimodule morphism $\phi : M \otimes_K N \to A$ by

$$\phi(p_1\alpha_1^- \otimes \alpha_2^- p_2) = p_1\alpha_1^-\alpha_2^- p_2.$$

We claim that ϕ is injective. We define

$$T_1 := \{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = p_1\alpha_1^- \text{ for some path } p_1 \text{ which passes through } e^+ \text{ for no special vertex } e\},$$

$$T_2 := \{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = \alpha_2^- p_2 \text{ for some path } p_2 \text{ which passes through } e^+ \text{ for no special vertex } e\},$$

where $\alpha_1^+ = (a^+, \alpha_1, c)$ and $\alpha_2^+ = (b, \alpha_2, a^+)$. By the definition of (Q^{sg}, I^{sg}), it is easy to see that $M = \mathrm{Span}_K T_1$ and $N = \mathrm{Span}_K T_2$.

Let $I_1 = \{\beta\alpha \in I \mid t(\alpha) \notin Sp\}$ and $\Lambda = KQ/\langle I_1\rangle$. We also define

$$\begin{aligned} S_1 &:= \{p \text{ a path in } (Q, I_1) \mid p = p_1\alpha_1 \text{ for some path } p_1\},\\ S_2 &:= \{p \text{ a path in } (Q, I_1) \mid p = \alpha_2 p_2 \text{ for some path } p_2\},\\ M' &:= \mathrm{Span}_K S_1, \qquad N' := \mathrm{Span}_K S_2. \end{aligned}$$

In fact, M' (resp. N') is a left (resp. right) Λ-module which is isomorphic to the radical of the indecomposable left (resp. right) projective module P_a corresponding to the vertex a. Note that Λ is the algebra $KQ/\langle I_1\rangle$ with the ideal generated by zero relations of length two. So S_1 (resp. S_2) is a basis of M' (resp. N'). Set $S_1 = \{u_1, \dots, u_m\}$, $S_2 = \{v_1, \dots, v_n\}$. By the definition of (Q^{sg}, I^{sg}), we get

$$\begin{aligned} \dim_K M &= \dim_K M' + \sharp\{u_i \mid t(u_i) \in Sp\} = \sharp(T_1),\\ \dim_K N &= \dim_K N' + \sharp\{v_i \mid s(v_i) \in Sp\} = \sharp(T_2), \end{aligned}$$

which implies that T_1 and T_2 are bases of the linear spaces M and N over K, respectively.

Similarly, the set

$$T_A := \{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p \text{ passes through } e^+ \text{ for no special vertex } e, \text{ and } s(p), t(p) \neq a^-\}$$

is a basis of A over K. Note that

$$\phi(p_1\alpha_1^- \otimes \alpha_2^- p_2) = p_1\alpha_1^-\alpha_2^- p_2$$

for any $p_1\alpha_1^- \in T_1$ and $\alpha_2^- p_2 \in T_2$. Set $T := \{u_i \otimes v_j \mid u_i \in T_1,\ v_j \in T_2\}$. Then T is a basis of $M \otimes_K N$. Lemma 3.2 implies that $p_1\alpha_1^-\alpha_2^- p_2$ is nonzero in A, which is in T_A. It is easy to see that ϕ induces an injective map from T to T_A, which means that ϕ itself is injective.

It follows from Lemma 3.4 that $e_{a^-}\Gamma e_{a^-}$ is isomorphic to K. We identify Γ with $\left(\begin{smallmatrix} A & M\\ N & K\end{smallmatrix}\right)$, where the K is identified with $e_{a^-}\Gamma e_{a^-}$. Note that $A/\mathrm{Im}\,\phi = \Gamma'$. Then Proposition 3.1 yields a triangle equivalence

$$D_{sg}^b(\Gamma) \simeq D_{sg}^b(\Gamma').$$

Case (c2). If $b \in Sp$, $c \notin Sp$, then the quiver Q^{sg} is as in Case (c2) of Figure 1. Then $\gamma_1\beta_1 - \gamma_2\beta_2, \gamma_1\beta_3 - \gamma_2\beta_4 \in I^{sg}$. Let $A = (1 - e_{a^-})\Gamma(1 - e_{a^-})$,

$$\begin{aligned} M &:= \mathrm{Span}_K\{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = p_1\gamma_2 \text{ for some path } p_1\},\\ N &:= \mathrm{Span}_K\{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = \beta_2 p_2 \text{ or } \beta_4 p_2 \text{ for some path } p_2\}. \end{aligned}$$

The remaining situation is similar to Case (c1); we omit the proof.

Case (c3). If $b \notin Sp$ and $c \in Sp$, then the quiver Q^{sg} is as in Case (c3) of Figure 1. This case is similar to Case (c2); we omit the proof.

CASE (c4). If $b, c \in Sp$, then the quiver Q^{sg} is as in Case (c4) of Figure 1. Then $\gamma_1\beta_1 - \gamma_2\beta_2, \gamma_1\beta_3 - \gamma_2\beta_4, \gamma_3\beta_1 - \gamma_4\beta_2, \gamma_3\beta_3 - \gamma_4\beta_4 \in I^{sg}$. Let $A := (1 - e_{a^-})\Gamma(1 - e_{a^-})$,

$$M := \mathrm{Span}_K\{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = p_1\gamma_2 \text{ or } p_1\gamma_4 \text{ for some path } p_1\},$$
$$N := \mathrm{Span}_K\{p \text{ a path in } (Q^{sg}, I^{sg}) \mid p = \beta_2 p_2 \text{ or } \beta_4 p_2 \text{ for some path } p_2\}.$$

The remaining situation is similar to Case (c1); we omit the proof.

To sum up, we see that $D^b_{sg}(\Gamma) \simeq D^b_{sg}(\Gamma')$. Since Γ' is also skewed-gentle, we replace Γ in the above with Γ', and discuss it recursively. After $\sharp(Sp)$ steps, we conclude that $D^b_{sg}(\Gamma) \simeq D^b_{sg}(KQ/\langle I\rangle)$. ∎

COROLLARY 3.6. *Let (Q, Sp, I) be a skewed-gentle triple. Then the following statements are equivalent:*

(i) $\mathrm{gldim}\, KQ^{sg}/\langle I^{sg}\rangle < \infty$.
(ii) $\mathrm{gldim}\, KQ/\langle I\rangle < \infty$.

Proof. This follows from Theorem 3.5 which states that $KQ^{sg}/\langle I^{sg}\rangle$ and $KQ/\langle I\rangle$ are singularity equivalent. ∎

4. The second main theorem. We first recall the singularity category of a gentle algebra from [14]. For a gentle algebra $\Lambda = KQ/\langle I\rangle$, we denote by $\mathcal{C}(\Lambda)$ the set of equivalence classes (with respect to cyclic permutation) of *repetition-free* cyclic paths $\alpha_1 \ldots \alpha_n$ in Q such that $\alpha_i\alpha_{i+1} \in I$ for all i, where we set $n + 1 = 1$. For convenience, we call any element c in $\mathcal{C}(\Lambda)$ *full repetition-free cyclic*. For every arrow $\alpha \in Q_1$, there is at most one cycle $c \in \mathcal{C}(\Lambda)$ containing it. Moreover, let $l(c)$ denote the *length* of a cycle $c \in \mathcal{C}(\Lambda)$, i.e. $l(\alpha_1 \ldots \alpha_n) = n$.

THEOREM 4.1 ([14]). *There is an equivalence of triangulated categories*

$$D^b_{sg}(\Lambda) \simeq \prod_{c\in\mathcal{C}(\Lambda)} \frac{D^b(K)}{[l(c)]},$$

where $D^b(K)/[l(c)]$ denotes the triangulated orbit category (see Keller [15]).

Let (Q, Sp, I) be a skewed-gentle triple. For any $c = \alpha_1 \ldots \alpha_n \in \mathcal{C}(\Lambda)$, if there are an even (resp. odd) number of special vertices lying on c, then we call c an *even* (resp. *odd*) *repetition-free cyclic path*. We denote by $\mathcal{C}^{\mathrm{even}}(\Lambda)$ (resp. $\mathcal{C}^{\mathrm{odd}}(\Lambda)$) the set of even (resp. odd) repetition-free cyclic paths. Recall that we say that a cyclic path $c = \alpha_1 \ldots \alpha_n$ passes through a vertex b if $b = t(\alpha_j)$ for some $2 \le j \le n$.

For any path $c = \alpha_1 \ldots \alpha_n$ in Q and $2 \le i \le n$, we set

$$\sigma_i(c) := \begin{cases} + & \text{if } \alpha_1 \ldots \alpha_i \text{ passes through an even number of special vertices,} \\ - & \text{if } \alpha_1 \ldots \alpha_i \text{ passes through an odd number of special vertices.} \end{cases}$$

Sometimes, we write just σ_i if c is obvious. Similarly, for any $2 \le i \le n$, we set

$$\tau_i(c) := \begin{cases} + & \text{if } \alpha_1 \dots \alpha_i \text{ passes through an odd number of special vertices,} \\ - & \text{if } \alpha_1 \dots \alpha_i \text{ passes through an even number of special vertices.} \end{cases}$$

LEMMA 4.2. *Let (Q, Sp, I) be a skewed-gentle triple. For any oriented path $c = \alpha_1 \dots \alpha_n$ in Q, $\alpha_1^+ \alpha_2^{\sigma_2} \dots \alpha_n^{\sigma_n}$ and $\alpha_1^- \alpha_2^{\tau_2} \dots \alpha_n^{\tau_n}$ are oriented paths in Q^g. In particular, if $\alpha_{i-1}\alpha_i \in I$ for some $2 \le i \le n$, then $\alpha_{i-1}^{\sigma_{i-1}} \alpha_i^{\sigma_i}$ and $\alpha_{i-1}^{\tau_{i-1}} \alpha_i^{\tau_i}$ are in I^g.*

Proof. We prove this for $\alpha_1^+ \alpha_2^{\sigma_2} \dots \alpha_n^{\sigma_n}$; the other is similar. Set $\sigma_1 = +$. For any $2 \le i \le n$, if $t(\alpha_i)$ is special, then $s(\alpha_{i-1}^{\pm}) = s(\alpha_i) = t(\alpha_i^{\pm})$. On the other hand, if $t(\alpha_i)$ is ordinary, then $\sigma_{i-1} = \sigma_i$, so $s(\alpha_{i-1}^{\sigma_{i-1}}) = s(\alpha_i)^{\sigma_{i-1}} = t(\alpha_i^{\sigma_i})$. To sum up, $s(\alpha_{i-1}^{\sigma_{i-1}}) = t(\alpha_i^{\sigma_i})$ for any $2 \le i \le n$, so $\alpha_1^+ \alpha_2^{\sigma_2} \dots \alpha_n^{\sigma_n}$ is a path in Q^g.

For the last statement, if $t(\alpha_i)$ is special, then $\sigma_{i-1} \ne \sigma_i$, which implies that $\alpha_{i-1}^{\sigma_{i-1}} \alpha_i^{\sigma_i} \in I^g$ by the definition of I^g; if $t(\alpha_i)$ is ordinary, then $\sigma_{i-1} = \sigma_i$, which also implies that $\alpha_{i-1}^{\sigma_{i-1}} \alpha_i^{\sigma_i} \in I^g$. ■

LEMMA 4.3. *Let (Q, Sp, I) be a skewed-gentle triple. Then for any arrow α_1 in Q, the following statements are equivalent:*

(i) *There is a full repetition-free cyclic path in Q containing α_1.*
(ii) *There is a full repetition-free cyclic path in Q^g containing α_1^+.*
(iii) *There is a full repetition-free cyclic path in Q^g containing α_1^-.*

Proof. Recall that $Q_0^g := \bigcup_{i \in Q_0} Q_0[i]$ where $Q_0[i]$ is the set $\{i\}$ (resp. $\{i^-, i^+\}$) if the vertex i is special (resp. ordinary), $Q_1^g := \{\alpha^+, \alpha^- \mid \alpha \in Q_1\}$,

$$s(\alpha^{\pm}) := \begin{cases} s(\alpha)^{\pm} & \text{if } s(\alpha) \notin Sp, \\ s(\alpha) & \text{if } s(\alpha) \in Sp, \end{cases} \qquad t(\alpha^{\pm}) := \begin{cases} t(\alpha)^{\pm} & \text{if } t(\alpha) \notin Sp, \\ t(\alpha) & \text{if } t(\alpha) \in Sp, \end{cases}$$

and

$$\begin{aligned} I^g := \{&\beta^+\alpha^+, \beta^-\alpha^- \mid \beta\alpha \in I,\, t(\alpha) \notin Sp\} \\ &\cup \{\beta^+\alpha^-, \beta^-\alpha^+ \mid \beta\alpha \in I,\, t(\alpha) \in Sp\}. \end{aligned}$$

For (i)⇒(ii), (iii), let $c = \alpha_1 \dots \alpha_n$ be a full repetition-free cyclic path in Q containing α_1. We consider two cases.

(a) If $c \in \mathcal{C}^{\text{even}}(\Lambda)$, then we claim that $\alpha_1^+ \alpha_2^{\sigma_2} \dots \alpha_n^{\sigma_n}$ and $\alpha_1^- \alpha_2^{\tau_2} \dots \alpha_n^{\tau_n}$ are full repetition-free cyclic paths.

We prove this for $\alpha_1^+ \alpha_2^{\sigma_2} \dots \alpha_n^{\sigma_n}$; the other is similar. In fact, Lemma 4.2 implies that it is an oriented path in Q^g.

If $t(\alpha_1) = s(\alpha_n)$ is special, then $t(\alpha_1^{\pm}) = s(\alpha_n^{\pm}) = t(\alpha_1)$, so $\alpha_1^+ \alpha_2^{\sigma_2} \dots \alpha_n^{\sigma_n}$ is a cyclic path. Furthermore, $\alpha_1 \dots \alpha_n$ passes through an odd number of special vertices, so $\sigma_n = -$, which implies $\alpha_n^{\sigma_n} \alpha_1^+ \in I^g$. Lemma 4.2 also shows that

$$\alpha_1^+\alpha_2^{\sigma_2},\dots,\alpha_{i-1}^{\sigma_{i-1}}\alpha_i^{\sigma_i},\dots,\alpha_{n-1}^{\sigma_{n-1}}\alpha_n^{\sigma_n}\in I^g,$$

and then together with the above, we see that $\alpha_1^+\alpha_2^{\sigma_2}\dots\alpha_n^{\sigma_n}$ is full repetition-free.

If $t(\alpha)=s(\alpha_n)$ is ordinary, then $t(\alpha^+)=t(\alpha)^+=s(\alpha_n)^+$ and $t(\alpha^-)=t(\alpha)^-=s(\alpha_n)^-$. In particular, $\alpha_1\dots\alpha_n$ passes through an even number of special vertices, so $\sigma_n=+$, which implies $\alpha_1^+\alpha_2^{\sigma_2}\dots\alpha_n^{\sigma_n}$ is cyclic and $\alpha_n^{\sigma_n}\alpha_1^+\in I^g$. Similarly, we conclude that it is also full repetition-free.

(b) If $c\in\mathcal{C}^{\mathrm{odd}}(\Lambda)$, set $l=\alpha_1\dots\alpha_n\alpha_1\dots\alpha_n$. Then we claim that

$$p=\alpha_1^+\alpha_2^{\sigma_2(l)}\dots\alpha_n^{\sigma_n(l)}\alpha_1^{\sigma_{n+1}(l)}\alpha_2^{\sigma_{n+2}(l)}\dots\alpha_n^{\sigma_{2n}(l)}$$

is a full repetition-free cyclic path. From $c\in\mathcal{C}^{\mathrm{odd}}(\Lambda)$, it is easy to see that $\sigma_{n+1}(l)=-$ and $\sigma_{n+i}(l)=\tau_i(c)$ for any $2\le i\le n$. Thus

$$p=\alpha_1^+\alpha_2^{\sigma_2(c)}\dots\alpha_n^{\sigma_n(c)}\alpha_1^-\alpha_2^{\tau_2(c)}\dots\alpha_n^{\tau_n(c)}.$$

Similarly to (a), we need only check that $s(\alpha_n^{\sigma_{2n}(l)})=t(\alpha_1^+)$ and $\alpha_n^{\sigma_{2n}(l)}\alpha_1^+$ is in I^g. If $s(\alpha_n)$ is special, then l passes through an odd number of special vertices, and so $\sigma_{2n}(l)=-$. In this case, we have $s(\alpha_n^\pm)=t(\alpha_1^\pm)$, and so $\alpha_n^{\sigma_{2n}(l)}\alpha_1^+\in I^g$ by the definition of I^g. If $s(\alpha_n)$ is ordinary, then l passes through an even number of special vertices, and so $\sigma_{2n}(l)=+$. In this case, we also have $s(\alpha_n^+)=s(\alpha_n)^+=t(\alpha_1^+)$, and so $\alpha_n^{\sigma_{2n}(l)}\alpha_1^+\in I^g$ by the definition of I^g.

For (ii)⇒(i) and (iii)⇒(i), let $\alpha_1^{\delta_1}\dots\alpha_n^{\delta_n}$ be any full repetition-free cyclic path in Q^g, where $\delta_i=+$ or $-$, and α_i is an arrow in Q for any $1\le i\le n$. Since $s(\alpha_i^{\delta_i})=t(\alpha_{i+1}^{\delta_{i+1}})$ for any $1\le i\le n$, where we set $n+1=1$, it is easy to see that $s(\alpha_i)=t(\alpha_{i+1})$. So $\alpha_1\dots\alpha_n$ is cyclic. On the other hand, by the definition of I^g, every zero relation in I^g comes from a zero relation in I, and it is easy to see that $\alpha_i\alpha_{i+1}\in I$ since $\alpha_i^{\delta_i}\alpha_{i+1}^{\delta_{i+1}}\in I^g$ for any $1\le i\le n$. If $\alpha_i\ne\alpha_j$ for any $i\ne j$, then $\alpha_1\dots\alpha_n$ is a repetition-free cyclic path.

Otherwise, without losing generality, we assume that $\alpha_{m+1}=\alpha_1$ for some $1<m<n$. It is easy to see that $\delta_{m+1}\ne\delta_1$ since $\alpha_1^{\delta_1}\dots\alpha_n^{\delta_n}$ is repetition-free, which also implies that m is unique. In fact, for any $1\le i\le n$, there exists at most one j satisfying $1\le j\ne i\le n$ and $\alpha_i=\alpha_j$. We claim that $\alpha_1\dots\alpha_m$ is a full repetition-free cyclic path. Since (Q,I) is gentle, for α_1, there exists at most one arrow β with $t(\beta)=s(\alpha_1)$ such that $\alpha_1\beta\in I$. However, we know that $\alpha_1\alpha_2,\alpha_{m+1}\alpha_{m+2}\in I$, so $\alpha_{m+2}=\alpha_2$. Inductively, we get $\alpha_{m+i}=\alpha_i$ for all $1\le i\le m$. Since $\alpha_n\alpha_1\in I$ and $\alpha_m\alpha_{m+1}\in I$, we also get $\alpha_n=\alpha_m=\alpha_{2m}$, which implies that $\alpha_1\dots\alpha_m$ is cyclic and $\alpha_1\dots\alpha_m=\alpha_{m+1}\alpha_{m+2}\dots\alpha_{2m}$. In fact, we also know that $n=2m$. For any $1\le i\ne j\le m$, if $\alpha_i=\alpha_j$, then $\alpha_{m+i}=\alpha_i=\alpha_j=\alpha_{m+j}$, which contradicts

$\alpha_1^{\delta_1}\dots\alpha_n^{\delta_n}$ being repetition-free. So $\alpha_1\dots\alpha_m$ is repetition-free. To sum up, $\alpha_1\dots\alpha_m$ is a full repetition-free cyclic path. ∎

The following theorem is the second main result of this paper.

THEOREM 4.4. *Let (Q, Sp, I) be a skewed-gentle triple. There is an equivalence of triangulated categories*

$$D^b_{sg}(KQ^g/\langle I^g\rangle) \simeq \Big(\prod_{c\in\mathcal{C}^{\mathrm{even}}(\Lambda)} \frac{D^b(K)}{[l(c)]}\Big) \times \Big(\prod_{c\in\mathcal{C}^{\mathrm{even}}(\Lambda)} \frac{D^b(K)}{[l(c)]}\Big) \times \Big(\prod_{c\in\mathcal{C}^{\mathrm{odd}}(\Lambda)} \frac{D^b(K)}{[2l(c)]}\Big),$$

where $\Lambda = KQ/\langle I\rangle$ and $D^b(K)/[l(c)]$ denotes the triangulated orbit category.

Proof. Since (Q^g, I^g) is a gentle pair, for any arrow $\alpha^\delta \in Q^g$ where $\delta = +$ or $-$, there is at most one full repetition-free cyclic path containing α^δ. It follows from Lemma 4.3 and its proof that

$$\begin{aligned}\mathcal{C}(KQ^g/\langle I^g\rangle) = {}&\{\alpha_1^+\alpha_2^{\sigma_2}\dots\alpha_n^{\sigma_n}, \alpha_1^-\alpha_2^{\tau_2}\dots\alpha_n^{\tau_n} \mid c = \alpha_1\dots\alpha_n \in \mathcal{C}^{\mathrm{even}}(\Lambda)\}\\ &\cup \{\alpha_1^+\alpha_2^{\sigma_2(c)}\dots\alpha_n^{\sigma_n(c)}\alpha_1^-\alpha_2^{\tau_2(c)}\dots\alpha_n^{\tau_n(c)} \mid c = \alpha_1\dots\alpha_n \in \mathcal{C}^{\mathrm{odd}}(\Lambda)\}.\end{aligned}$$

Then Theorem 4.1 yields the result immediately. ∎

In general, we do not have $D^b_{sg}(KQ^{sg}/\langle I^{sg}\rangle) \simeq D^b_{sg}(KQ^g/I^g)$. In fact, Theorem 4.4 shows that $D^b_{sg}(KQ^{sg}/\langle I^{sg}\rangle) \simeq D^b_{sg}(KQ^g/I^g)$ if and only if they are zero. So we have the following direct corollary.

COROLLARY 4.5. *Let (Q, Sp, I) be a skewed-gentle triple. Then the following statements are equivalent:*

(i) $\operatorname{gldim} KQ^{sg}/\langle I^{sg}\rangle < \infty$.
(ii) $\operatorname{gldim} KQ/\langle I\rangle < \infty$.
(iii) $\operatorname{gldim} KQ^g/\langle I^g\rangle < \infty$.

Proof. By Corollary 3.6, we need only prove that (ii)⇔(iii). Theorem 4.4 shows that $D^b_{sg}(KQ/\langle I\rangle) \simeq 0$ if and only if $D^b_{sg}(KQ^g/\langle I^g\rangle) \simeq 0$, which implies that $\operatorname{gldim} KQ/\langle I\rangle < \infty$ if and only if $\operatorname{gldim} KQ^g/\langle I^g\rangle < \infty$. ∎

REMARK 4.6. Note that we do not assume $\operatorname{char} K \neq 2$ in Corollary 4.5. When $\operatorname{char} K \neq 2$, we know that $KQ^{sg}/\langle I^{sg}\rangle$ is Morita equivalent to a skew-group algebra $(KQ^g/\langle I^g\rangle)G$ defined in Section 2.3. In this case, [18, Theorem 1.3] shows that $\operatorname{gldim} KQ^{sg}/\langle I^{sg}\rangle < \infty$ if and only if $\operatorname{gldim} KQ^g/\langle I^g\rangle < \infty$.

In the following, we denote by S_n the self-injective Nakayama algebra of a cyclic quiver with n vertices modulo the ideal generated by paths of length 2.

Example 4.7. (a) In the notation of Examples 2.6(a) and 2.7(a), we have $D^b_{sg}(KQ^{sg}/\langle I^{sg}\rangle) \simeq \underline{\mathrm{mod}}(KQ/\langle I\rangle) \simeq \underline{\mathrm{mod}}\, S_2$, while $D^b_{sg}(KQ^{g}/\langle I^{g}\rangle) \simeq \underline{\mathrm{mod}}\, S_4$.

(b) In the notation of Examples 2.6(b) and 2.7(b), we have $D^b_{sg}(KQ^{sg}/\langle I^{sg}\rangle) \simeq \underline{\mathrm{mod}}(KQ/\langle I\rangle) \simeq \underline{\mathrm{mod}}\, S_3$, while $D^b_{sg}(KQ^{g}/\langle I^{g}\rangle) \simeq \underline{\mathrm{mod}}\, S_6$.

Acknowledgements. This work is inspired by some discussions with Changjian Fu. X. Chen was supported by the Fundamental Research Funds for the Central Universities A0920502051411-45. M. Lu was supported by the National Natural Science Foundation of China (No. 11401401).

REFERENCES

[1] I. Assem, T. Brüstle, G. Charbonneau-Jodoin and P. Plamondon, *Gentle algebras arising from surface triangulations*, Algebra Number Theory 4 (2010), 201–229.

[2] I. Assem and A. Skowroński, *Iterated tilted algebras of type* $\tilde{\mathbb{A}}_n$, Math. Z. 195 (1987), 269–290.

[3] M. Auslander and I. Reiten, *Cohen–Macaulay and Gorenstein artin algebras*, in: Representation Theory of Finite Groups and Finite-Dimensional Algebras, Progr. Math. 95, Birkhäuser, Basel, 1991, 221–245.

[4] V. Bekkert, E. N. Marcos and H. A. Merklen, *Indecomposables in derived categories of skewed-gentle algebras*, Comm. Algebra 31 (2003), 2615–2654.

[5] V. Bekkert and H. A. Merklen, *Indecomposables in derived categories of gentle algebras*, Algebr. Represent. Theor. 6 (2003), 285–302.

[6] C. Bessenrodt and T. Holm, *q-Cartan matrices and combinatorial invariants of derived categories fro skewed-gentle algebras*, Pacific J. Math. 229 (2007), 25–48.

[7] R. Buchweitz, *Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings*, unpublished manuscript, 1987; http://hdl.handle.net/1807/16682.

[8] I. Burban, *Derived categories of coherent sheaves on rational singular curves*, in: Representations of Finite Dimensional Algebras and Related Topics in Lie Theory and Geometry, Fields Inst. Comm. 40, Amer. Math. Soc., Providence, RI, 2004, 173–188.

[9] X.-W. Chen, *Singular equivalences induced by homological epimorphisms*, Proc. Amer. Math. Soc. 142 (2014), 2633–2640.

[10] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. 220 (1995), 611–633.

[11] C. Geiß and J. A. de la Peña, *Auslander–Reiten components for clans*, Bol. Soc. Mat. Mexicana 5 (1999), 307–326.

[12] C. Geiß and I. Reiten, *Gentle algebras are Gorenstein*, in: Representations of Algebras and Related Topics, Fields Inst. Comm. 45, Amer. Math. Soc., Providence, RI, 2005, 129–133.

[13] D. Happel, *On Gorenstein algebras*, in: Representation Theory of Finite Groups and Finite-Dimensional Algebras, Progr. Math. 95, Birkhäuser, Basel, 1991, 389–404.

[14] M. Kalck, *Singularity categories of gentle algebras*, Bull. London Math. Soc. 47 (2015), 65–74.

[15] B. Keller, *On triangulated orbit categories*, Doc. Math. 10 (2005), 551–581.

[16] S. Koenig and H. Nagase, *Hochschild cohomology and stratifying ideals*, J. Pure Appl. Algebra 213 (2009), 886–891.
[17] D. Orlov, *Triangulated categories of singularities and D-branes in Landau–Ginzburg models*, Proc. Steklov Inst. Math. 246 (2004), 227–248.
[18] I. Reiten and Ch. Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra 92 (1985), 224–282.
[19] J. Rickard, *Derived categories and stable equivalences*, J. Pure Appl. Algebra 61 (1989), 303–317.
[20] A. Skowroński and J. Waschbüsch, *Representation-finite biserial algebras*, J. Reine Angew. Math. 345 (1983), 172–181.

Xinhong Chen
Department of Mathematics
Southwest Jiaotong University
Chengdu 610031, P.R. China
E-mail: chenxinhong@swjtu.edu.cn

Ming Lu (corresponding author)
Department of Mathematics
Sichuan University
Chengdu 610064, P.R. China
E-mail: luming@scu.edu.cn

Received 25 September 2014;
revised 10 February 2015 (6379)