COLLOQUIUM MATHEMATICUM

VOL. 141 2015 NO. 2

ON THE EXTENT OF SEPARABLE, LOCALLY COMPACT, SELECTIVELY (a)-SPACES

BY

SAMUEL G. DA SILVA (Salvador)

Dedicated to Prof. Richard G. Wilson on the occasion of his 70th birthday

Abstract. The author has recently shown (2014) that separable, selectively (a)-spaces cannot include closed discrete subsets of size $\mathfrak{c}$. It follows that, assuming **CH**, separable selectively (a)-spaces necessarily have countable extent. However, in the same paper it is shown that the weaker hypothesis "$2^{\aleph_0} < 2^{\aleph_1}$" is not enough to ensure the countability of all closed discrete subsets of such spaces. In this paper we show that if one adds the hypothesis of local compactness, a specific effective (i.e., Borel) parametrized weak diamond principle implies countable extent in this context.

1. Introduction. Throughout this paper, all spaces are supposed to be T_1 topological spaces. The *extent* of a topological space is the supremum of the cardinalities of all closed discrete subsets of the space, provided this is an infinite cardinal, and $\aleph_0$ otherwise. The *dominating number*, denoted by $\mathfrak{d}$, is the cofinality of the pre-order of functions from ω into ω with eventual domination (also known as the *mod finite order* $\langle {}^{\omega}\omega, \leqslant^* \rangle$, where $f \leqslant^* g$ means that $\{n < \omega : g(n) < f(n)\}$ is a finite set). It is well-known that $\mathfrak{d}$ is also the cofinality of the stricter order $\langle {}^{\omega}\omega, < \rangle$ defined pointwise—see [3], where the reader will also find the definition of other small cardinals, for instance the *almost disjointness number*, denoted by $\mathfrak{a}$.

In what follows, we investigate a *star selection principle*—i.e., the *selective version* of a *star covering property*. For background information on star covering properties we refer to [4] and [9]; for selection principles and topology we refer to [16] and [7].

Property (a) was introduced by Matveev [8], and its selective version by Caserta, Di Maio and Kočinac [2].

2010 *Mathematics Subject Classification*: Primary 54D45, 54A25, 03E05; Secondary 54A35, 03E65, 03E17.

Key words and phrases: locally compact spaces, star covering properties, property (a), selection principles, selectively (a), parametrized weak diamond principles.

DOI: 10.4064/cm141-2-5

Definition 1.1 ([8]). A topological space X satisfies *property* (a) (or is said to be an (a)*-space*) if for every open cover $\mathcal{U}$ of X and every dense set $D \subseteq X$ there is a set $F \subseteq D$ which is closed and discrete in X and such that $\mathrm{St}(F,\mathcal{U}) = X$ (where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$).

Definition 1.2 ([2]). A topological space X is said to be a *selectively* (a)*-space* if for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers and every dense set $D \subseteq X$ there is a sequence $\langle A_n : n < \omega \rangle$ of subsets of D which are closed and discrete in X and such that $\{\mathrm{St}(A_n,\mathcal{U}_n) : n < \omega\}$ covers X.

Notice that (a) implies selectively (a).

In [18] and [13], Morgan and the present author established a number of results on the extent of spaces within a certain class of separable, locally compact, selectively (a)-spaces: namely, *Mrówka–Isbell spaces from almost disjoint families* which are selectively (a). We assume the reader is familiar with the definition of these spaces; nevertheless, both of the above mentioned papers include detailed descriptions of such constructions. In what follows, $\mathcal{A}$ will always denote an almost disjoint family of infinite sets of ω, and $\Psi(\mathcal{A})$ denotes the corresponding Ψ-space. For a given topological property $\mathcal{P}$, we will say that "$\mathcal{A}$ is $\mathcal{P}$" if $\Psi(\mathcal{A})$ satisfies $\mathcal{P}$. It is well-known that $\Psi(\mathcal{A}) = \mathcal{A} \cup \omega$ is always locally compact, and includes ω as a dense set of isolated points and $\mathcal{A}$ as a closed discrete subset, so any statement on the cardinality of $\mathcal{A}$ is a statement on the extent of the separable space $\Psi(\mathcal{A})$.

Typical results of [18] and [13] include the following:

- Separable, selectively (a) spaces cannot include a closed discrete set of size $\mathfrak{c}$—and therefore, if $\Psi(\mathcal{A})$ is selectively (a) then $|\mathcal{A}| < \mathfrak{c}$ [18].
- It follows that under the Continuum Hypothesis, selectively (a) almost disjoint families are necessarily countable.
- If $|\mathcal{A}| < \mathfrak{d}$, then $\Psi(\mathcal{A})$ is selectively (a) [18].
- Suppose $\mathcal{A}$ is maximal. Then $\Psi(\mathcal{A})$ is selectively (a) if, and only if, $|\mathcal{A}| < \mathfrak{d}$ [18].
- Considering the relative consistency with **ZFC** of

  $$\text{“}\aleph_1 = \mathfrak{a} < \mathfrak{d} = \mathfrak{c}\text{”} \;+\; \text{“}2^{\aleph_0} < 2^{\aleph_1}\text{”}$$

  (details in [18]), we conclude that, in comparison/contrast with the Continuum Hypothesis, the consistently weaker hypothesis given by "$2^{\aleph_0} < 2^{\aleph_1}$" *is not enough* to ensure countability of the almost disjoint families which are selectively (a).
- However, an effective (here meaning Borel) parametrized weak diamond principle is enough to ensure countability of the almost disjoint families in this context [13].

In this paper, we generalize the main result of [13] by showing that the very same Borel parametrized weak diamond principle used in that paper—namely, $\diamondsuit({}^{\omega}\omega, <)$—implies countable extent for *all* separable, locally compact, selectively (a)-spaces. As a corollary (using a number of results—already noted in [13]—on the deductive strength of such principles) we establish that the statement "All separable, locally compact, selectively (a)-spaces have countable extent" is consistent with the negation of the Continuum Hypothesis.

Our set-theoretical terminology and notation are standard; let us describe them. Throughout this paper, $\omega = \aleph_0$ denotes the set of all natural numbers, which is also the least limit ordinal and the least infinite cardinal. The first uncountable cardinal is denoted by $\omega_1 = \aleph_1$. For a given set X, $|X|$ denotes the cardinality of X. **CH** denotes the *Continuum Hypothesis*, which is the statement "$\mathfrak{c} = \aleph_1$", where $\mathfrak{c}$ is the cardinality of the continuum, i.e., $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0}$. The *Generalized Continuum Hypothesis* (denoted by **GCH**) is the statement "$\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ for every ordinal α". A *stationary* subset of ω_1 is a subset of ω_1 which intersects all *club* (closed unbounded) subsets of ω_1 (where "closed" means "closed in the order topology"). *Jensen's diamond*, denoted by $\diamondsuit$, is the combinatorial guessing principle asserting the existence of a $\diamondsuit$-*sequence*, which is a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ such that (i) $A_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$; and (ii) for any given set $A \subseteq \omega_1$, the $\diamondsuit$-sequence "guesses" A stationarily many times, meaning that $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary. It is easy to see that $\diamondsuit \to \mathbf{CH} \to 2^{\aleph_0} < 2^{\aleph_1}$.

2. Effective, parametrized weak diamond principles. The combinatorial principle used in the proof of our main theorem is one of the so-called *parametrized weak diamond principles* introduced by Moore, Hrušák and Džamonja [10]. More precisely, we use an *effective*—here meaning Borel—principle from this collection. The family of parameters for the weak diamond principles of [10] is given by the category $\mathcal{PV}$. This category (named after de Paiva [14] and Vojtáš [20], its introducers) is a small subcategory of the dual of the simplest example of a Dialectica category, $\mathbf{Dial}_2(\mathbf{Sets})$ [15]. The *objects* of $\mathcal{PV}$ are triples $o = (A, B, E)$ consisting of sets A and B, both of size not larger than $\mathfrak{c}$, and a relation $E \subseteq A \times B$ such that

$$\forall a \in A\ \exists b \in B \quad a\,E\,b \quad \text{and} \quad \forall b \in B\ \exists a \in A \quad \neg a\,E\,b.$$

(ϕ, ψ) is a *morphism* from $o_2 = (A_2, B_2, E_2)$ to $o_1 = (A_1, B_1, E_1)$ if $\phi : A_1 \to A_2$, $\psi : B_2 \to B_1$ and

$$\forall a \in A_1\ \forall b \in B_2 \quad \phi(a)\,E_2\,b \ \to\ a\,E_1\,\psi(b).$$

If (A, B, E) is an object of $\mathcal{PV}$, one can associate to it a parametrized weak diamond principle $\Phi(A, B, E)$, the following combinatorial statement:

"For every function F with values in A, defined on the complete binary tree of height ω_1, there is a function $g : \omega_1 \to B$ such that g "guesses" every branch of the tree, meaning that for all $f \in {}^{\omega_1}2$ the set $\{\alpha < \omega_1 : F(f{\restriction}\alpha)\, E\, g(\alpha)\}$ is stationary."

The function g is sometimes called an *oracle* for F, given by the principle $\Phi(A, B, E)$. In the case of $A = B$, $\Phi(A, B, E)$ is usually written as $\Phi(A, E)$. For every object $o \in \mathcal{PV}$ the following implications hold (see [10]):

$$\diamondsuit \;\to\; \Phi(o) \;\to\; 2^{\aleph_0} < 2^{\aleph_1}.$$

The first implication justifes the term "weak diamond" for these guessing principles. The second justifies why the cardinal inequality $2^{\aleph_0} < 2^{\aleph_1}$ is often understood as being the weakest diamond of all.

Effective parametrized weak diamond principles are obtained by restricting the usual definitions to Borel subsets (and functions) in the context of *Polish spaces* (i.e., separable and completely metrizable topological spaces). A subset of a Polish space is *Borel* if it belongs to the smallest σ-algebra containing all open subsets of the space.

Definition 2.1.

(i) An object $o = (A, B, E)$ in $\mathcal{PV}$ is *Borel* if A, B and E are Borel subsets of some Polish space.
(ii) A map $f : X \to Y$ from a Borel subset of a Polish space to a Borel subset of another is itself *Borel* if for every Borel $Z \subseteq Y$ the set $f^{-1}[Z] \subseteq X$ is Borel.
(iii) A map $F : {}^{<\omega_1}2 \to A$ is *Borel* if it is level-by-level Borel, i.e., for each $\alpha < \omega_1$ the map $F{\restriction}{}^{\alpha}2 : {}^{\alpha}2 \to A$ is Borel.
(iv) If o is Borel we define the principle $\diamondsuit(o)$ as in [10]:

$$\forall \text{ Borel } F : {}^{<\omega_1}2 \to A \;\; \exists g \in {}^{\omega_1}B \;\; \forall f \in {}^{\omega_1}2$$
$$\{\alpha < \omega_1 : F(f{\restriction}\alpha)\, E\, g(\alpha)\} \text{ is stationary in } \omega_1.$$

Next, we define the Borel object from $\mathcal{PV}$ which will be used in our main theorem.

Definition 2.2. The object $({}^{\omega}\omega, {}^{\omega}\omega, <)$ is defined in the following way: for every $f, g \in {}^{\omega}\omega$, $f < g$ if and only if $f(n) < g(n)$ for every $n < \omega$.

In [13], we used the Borel principle $\diamondsuit({}^{\omega}\omega, <)$—in fact, we used a principle equivalent to it—to ensure countability of all almost disjoint families which are selectively (a); in this paper, we will apply $\diamondsuit({}^{\omega}\omega, <)$ directly to show that any separable, locally compact, selectively (a)-space necessarily has countable extent under that principle.

Let us briefly discuss the deductive strength of $\diamondsuit({}^{\omega}\omega, <)$; a more detailed discussion may be found in [13]. Previous results due to Morgan and the author [11], [13] show that $\diamondsuit({}^{\omega}\omega,<)$ is independent of **CH**. It is a little harder to show that the same happens the other way round, i.e., **CH** is independent of $\diamondsuit({}^{\omega}\omega,<)$; however, it is possible to exhibit models of both of the following conjunctions: "$\diamondsuit({}^{\omega}\omega,<)$ + $\neg$**CH** + $2^{\aleph_0} < 2^{\aleph_1}$" and "$\diamondsuit({}^{\omega}\omega,<)$ + $\neg$**CH** + $2^{\aleph_0} = 2^{\aleph_1}$". For the former, it suffices (similarly to the analogous result in [11]) to add $\aleph_{\omega_1}$ Cohen reals to a model of **GCH**. For the latter, a very powerful result of [10, Theorem 6.6] is needed, and one concludes that in the countable support iteration of length ω_2 of Sacks forcings one has $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and $\diamondsuit({}^{\omega}\omega,<)$ holds.

For future reference, we write down the outcome of our discussion:

FACT 2.3 ([13]). *$\diamondsuit({}^{\omega}\omega,<)$ is consistent with $\neg$**CH**. Moreover, the statements*

$$\text{“}\diamondsuit({}^{\omega}\omega,<)\ +\ \neg\mathbf{CH}\ +\ 2^{\aleph_0} < 2^{\aleph_1}\text{”}\quad \textit{and}$$
$$\text{“}\diamondsuit({}^{\omega}\omega,<)\ +\ \neg\mathbf{CH}\ +\ 2^{\aleph_0} = 2^{\aleph_1}\text{”}$$

are both consistent. ∎

3. The main theorem. The main result of this paper is:

MAIN THEOREM 3.1. *Separable, locally compact, selectively (a)-spaces have countable extent under $\diamondsuit({}^{\omega}\omega,<)$.*

In fact, we are able to prove a more general result, stated in terms of *relative topological properties* (see [1], [21]). Let X be a topological space and $Y \subseteq X$. The subset Y is said to be *locally compact in* X (or *relatively locally compact* in X) if every $y \in Y$ has a neighbourhood V_y such that $\overline{V_y}$ is a compact subset of X [12].

It is easy to see that the main theorem above is an immediate corollary of the following proposition.

PROPOSITION 3.2. *Assume $\diamondsuit({}^{\omega}\omega,<)$ and suppose X is a separable space with an uncountable closed discrete subset H which is locally compact in X. Then X is not a selectively (a)-space.*

Proof. Let D be a countable dense subset of X, and H an uncountable closed discrete subset of X, as in the statement. We may suppose, without loss of generality, that H and D are disjoint, and moreover we assume $D=\omega$ and $H=\omega_1\backslash\omega$. For every $\beta\in\omega_1\backslash\omega$, fix an open neighbourhood of β, say O_β, satisfying $O_\beta\cap(\omega_1\setminus\omega)=\{\beta\}$. As $\omega_1\setminus\omega$ is relatively locally compact in X, we may take for every $\beta\in\omega_1\setminus\omega$ a compact set K_β with non-empty interior and an open neighbourhood U_β of β with $U_\beta\subseteq O_\beta$ and $\overline{U_\beta}\subseteq K_\beta$. Let $A_\beta=U_\beta\cap\omega$ for every $\beta\in\omega_1\setminus\omega$.

As ${}^{\omega}(\mathcal{P}(\omega))$ is topologically the same as ${}^{\omega}({}^{\omega}2)$, and the latter is homeomorphic to the Cantor set ${}^{\omega}2$, we are allowed to consider an enumeration

$\{X_f : f \in {}^{\omega}2\}$ of all sequences of subsets of ω such that the bijection $f \mapsto X_f$ is Borel (in fact, a homeomorphism). For every such sequence X_f we write $X_f = \langle X_{f,n} : n < \omega\rangle$.

For every $h \in {}^{<\omega_1}2$ with infinite domain and $n < \omega$, let $Y_{h,n}$ be the subset of ω given by $Y_{h,n} = A_{\mathrm{dom}(h)} \cap X_{h\restriction\omega,n}$.

Let $F : {}^{<\omega_1}2 \to {}^{\omega}\omega$ be such that, for every $n < \omega$,

$$F(h)(n) = \begin{cases} \sup(Y_{h,n}) + 1 & \text{if } |h| = \aleph_0 \text{ and } Y_{h,n} \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that F is Borel ([1]). Let $g : \omega_1 \to {}^{\omega}\omega$ be the oracle for F, given by $\diamondsuit({}^{\omega}\omega, <)$. Let $\langle \mathcal{U}_n : n < \omega\rangle$ be the sequence of open covers of X defined in the following way: for every $n < \omega$, let $\mathcal{U}_n$ be given by

$$\mathcal{U}_n = \{X \setminus (\omega_1 \setminus \omega)\} \cup \big\{U_\beta \setminus g(\beta)(n) : \beta \in \omega_1 \setminus \omega\big\}.$$

As we are assuming that X is a T_1 space, indeed $U_\beta \setminus g(\beta)(n)$ is an open neighbourhood of $\beta \in \omega_1 \setminus \omega$, and, by construction, the open set $U_\beta \setminus g(\beta)(n)$ is the only element of $\mathcal{U}_n$ which contains β.

We claim that this sequence of open covers witnesses that X is not selectively (a). To see this, let $P = \langle P_n : n < \omega\rangle$ be an arbitrary sequence of subsets of ω which are closed and discrete in X. Fix any $f : \omega_1 \to 2$ such that $X_{f\restriction\omega} = P$. Applying $\diamondsuit({}^{\omega}\omega, <)$ for this f, we find that the subset of ω_1 given by $S = \{\beta < \omega_1 : F(f\restriction\beta) < g(\beta)\}$ is stationary, and the same holds for

$$S' = S \cap [\omega, \omega_1) = S \setminus \omega = \{\beta \in \omega_1 \setminus \omega : F(f\restriction\beta) < g(\beta)\}.$$

Notice that, for every $\beta \in \omega_1 \setminus \omega$ and $n < \omega$, $A_\beta \cap P_n$ is a finite set, since it is contained in $P_n \subseteq K_\beta \cap P_n$ and the latter is a closed discrete subset of the compact set K_β. It follows that for every $\beta \in S'$ and $n < \omega$,

$$g(\beta)(n) > F(f\restriction\beta)(n) > \sup(A_\beta \cap P_n),$$

and therefore $(U_\beta \setminus g(\beta)(n)) \cap P_n = (A_\beta \cap P_n) \setminus g(\beta)(n) = \emptyset$, thus $\beta \notin \mathrm{St}(\mathcal{U}_n, P_n)$ for all $n < \omega$. It follows that $S' \cap \bigcup\{\mathrm{St}(A_n, \mathcal{U}_n) : n < \omega\} = \emptyset$. As the sequence P was arbitrary, X is not a selectively (a)-space, as desired. ∎

And now it is immediate from the main theorem and Fact 2.3 that one has the following:

COROLLARY 3.3. *The statement "All separable, locally compact, selectively (a)-spaces have countable extent" is consistent with* $\neg$**CH**—*regardless of the validity of the weak diamond* $2^{\aleph_0} < 2^{\aleph_1}$.

([1]) For readers without previous interest in descriptive set theory, we point out that we included in [13] a number of examples of how to check that a given function (or a given object in $\mathcal{PV}$) is Borel.

Notice that, by inspecting the proof of Proposition 3.2, one can conclude that for every sequence of closed and discrete subsets of the countable dense set we are allowed to exhibit a somehow (and at least) "medium sized" set of counterexample points—i.e., points of the uncountable closed discrete set that are not covered by the (clear within the context) countable union of stars. Our claim is justified by the fact that such a set will be indexed by a *stationary* subset of ω_1.

4. Notes and questions. The main result of this paper generalizes previous results on spaces from almost disjoint families. Indeed, it is quite natural to look at such spaces when investigating issues related to the extent of separable, locally compact spaces. It turns out that if we want to pose questions on the extent of spaces in this class, we should start by considering these questions for Ψ-spaces. So we take this opportunity to give some publicity to a number of related questions previously posed in the literature, as well as present some new questions and problems.

There are several cardinal invariants related to the size of Ψ-spaces satisfying (or not) certain specific topological properties. These cardinals are well-defined because they are related to topological properties which cannot be satisfied by $\Psi(\mathcal{A})$ whenever $\mathcal{A}$ has the size of the continuum. For any given $\mathcal{P}$ of this kind, one can define the *non-$\mathcal{P}$ number* as the minimum size of an almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ does not satisfy $\mathcal{P}$, and the *never-$\mathcal{P}$ number* as the least cardinal κ such that there are no almost disjoint families $\mathcal{A}$ of size κ such that $\Psi(\mathcal{A})$ satisfies $\mathcal{P}$.

The properties for which these cardinals have been defined are, as far as our knowledge goes, the following: normality; countable paracompactness; property (a) and its selective version ([2]). The "non-(a) number" ($\mathfrak{nsa}$) was introduzed by Szeptycki [19]; the "non-countably-paracompact number" ($\mathfrak{ncp}$) was introduced by the author [17], but proved to be $\aleph_1$ in [5]; the "never-(a) number" and the "never-countably-paracompact number" (resp. $\mathfrak{vsa}$ and $\mathfrak{vcp}$) were introduced by Morgan and the author [11]; the "non-selectively-(a) number" and the "never-selectively-(a) number" (resp. $\mathfrak{nssa}$ and $\mathfrak{vssa}$) were introduced by the Morgan and the author [13]. The "non-normal number" is $\aleph_1$, because of Luzin gaps (indeed, Luzin gaps were also used in [5] to prove that $\mathfrak{ncp} = \aleph_1$).

The following inequalities hold in **ZFC** (references and details are in [13]; the definitions of $\mathfrak{p}$ and $\mathfrak{b}$ may be found in [3]):

([2]) It is not known whether normal Ψ-spaces have to be (a) or selectively (a) (resp. [19] and [18]), and the same questions were posed for countably paracompact Ψ-spaces (resp. [17] and [18]).

(i) $\mathfrak{p} \leqslant \mathfrak{nsa} \leqslant \mathfrak{b} \leqslant \mathfrak{a}$;
(ii) $\mathfrak{b} \leqslant \mathfrak{d} \leqslant \mathfrak{nssa} \leqslant \mathfrak{vssa}$;
(iii) $\mathfrak{vsa} \leqslant \mathfrak{vssa}$;
(iv) $\mathfrak{vn} \leqslant \mathfrak{vcp}$.

The cardinal inequality $2^{\aleph_0} < 2^{\aleph_1}$ has huge influence on $\mathfrak{vn}$: under the weakest diamond, one necessarily has the equality $\mathfrak{vn} = \aleph_1$, because of the well-known Jones Lemma [6]. In [11] we asked if similar results hold for $\mathfrak{vsa}$ and $\mathfrak{vcp}$.

QUESTION 4.1 ([11]). *Does* $2^{\aleph_0} < 2^{\aleph_1}$ *alone imply* $\mathfrak{vsa} = \aleph_1$ *or* $\mathfrak{vcp} = \aleph_1$*?*

In [11] it is proved that $\diamondsuit(\omega, <)$ (a Borel diamond principle weaker than the one used in the present paper) implies $\mathfrak{vsa} = \mathfrak{vcp} = \aleph_1$. As already commented, in [18] it was established that the statement "There is an uncountable selectively (a) Ψ-space" is consistent with "$2^{\aleph_0} < 2^{\aleph_1}$" and "$\aleph_1 < \mathfrak{d}$"—and therefore $2^{\aleph_0} < 2^{\aleph_1}$ alone does not imply $\mathfrak{vssa} = \aleph_1$. However, $\diamondsuit({}^{\omega}\omega, <)$ does imply $\mathfrak{vssa} = \aleph_1$: this is the main result of [13], generalized in the present paper to all separable, locally compact, selectively (a)-spaces.

It is remarkable that

$$\mathrm{Con}(\mathbf{ZFC} + \text{“}2^{\aleph_0} < 2^{\aleph_1}\text{”} + \text{“}2^{\aleph_0} \text{ is regular”} + \text{“}\aleph_1 < \max\{\mathfrak{vsa}, \mathfrak{vcp}\}\text{”})$$

implies the existence of inner models with measurable cardinals (because of *small dominating families*—see comments and references in [11]).

While on one hand all almost disjoint families of size less than $\mathfrak{d}$ are selectively (a) [18], on the other hand it is not known if it is even consistent that there is an a.d. family $\mathcal{A}$ of size $\mathfrak{d}$ such that $\Psi(\mathcal{A})$ is (a) or selectively (a) ([17], [18]). A related question is the following:

QUESTION 4.2 (R. Dias, reported in [13]). *Is there a* **ZFC** *example of an a.d. family of size* $\mathfrak{d}$ *such that* $\Psi(\mathcal{A})$ is not *a selectively* (a)*-space?*

If the answer to the preceding question is yes, then $\mathfrak{nssa} = \mathfrak{d}$. We can "double the bet" and ask:

QUESTION 4.3. *Does* **ZFC** *prove* $\mathfrak{vssa} = \mathfrak{d}$*?*

The author conjectures that the preceding question has a negative answer.

We finish this paper by presenting the following problem:

PROBLEM 4.4. *It is known that* $2^{\aleph_0} < 2^{\aleph_1}$ *alone does not imply* $\mathfrak{vssa} = \aleph_1$. *Can one find reasonable (and consistent) combinatorial/topological hypotheses which on their own do not imply* $\mathfrak{vssa} = \aleph_1$ *either, but which do so when taken together with* $2^{\aleph_0} < 2^{\aleph_1}$*?*

Notice that such a reasonable hypothesis has to be consistent with both "$2^{\aleph_0} < 2^{\aleph_1}$" and "$\mathfrak{d} = \aleph_1$" (more specifically, it has to imply "$\mathfrak{d} = \aleph_1$" under "$2^{\aleph_0} < 2^{\aleph_1}$").

A similar problem—regarding the cardinal function extent—could be posed for selectively (a), separable spaces in general (and adding or not the hypothesis of local compactness): to find a non-trivial condition (e.g., other than normality) which, when taken together with $2^{\aleph_0} < 2^{\aleph_1}$, implies countable extent for spaces in the relevant class. Notice again that, precisely because of the author's already mentioned results on spaces from almost disjoint families, the existence of separable, locally compact, selectively (a)-spaces with uncountable extent is consistent with the conjunction of the statements "$2^{\aleph_0} < 2^{\aleph_1}$" and "$\aleph_1 < \mathfrak{d}$"—so, again, any consistent hypothesis solving the "locally compact extent version" of the previous problem should also imply "$\mathfrak{d} = \aleph_1$" under "$2^{\aleph_0} < 2^{\aleph_1}$".

Acknowledgments. The author wishes to thank the anonymous referee for his/her careful reading of the paper and for providing a number of useful comments and suggestions.

This paper was presented at MICTA 2014—Mexican International Conference on Topology and its Applications (Cocoyoc, Morelos, Mexico, August 2014), celebrating the 70th anniversary of Prof. Richard G. Wilson. The author would like to express his gratitude to all members of the local organizing committee, it was indeed a lovely conference.

The author also thanks Rodrigo Dias for a number of discussions at Cocoyoc and also at São Paulo Topology Seminar.

REFERENCES

[1] A. Arhangel'skii, *Relative topological properties and relative topological spaces*, Topology Appl. 70 (1996), 87–99.

[2] A. Caserta, G. Di Maio and Lj. D. R. Kočinac, *Versions of properties* (a) *and* (pp), Topology Appl. 158 (2011), 1360–1368.

[3] E. K. van Douwen, *The integers and topology*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan (eds.), North-Holland, Amsterdam, 1984, 111–167.

[4] E. K. van Douwen, G. M. Reed, A. W. Roscoe and I. J. Tree, *Star covering properties*, Topology Appl. 39 (1991), 71–103.

[5] M. Hrušák, C. J. G. Morgan and S. G. da Silva, *Luzin gaps are not countably paracompact*, Questions Answers Gen. Topology 30 (2012), 59–66.

[6] F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. 43 (1937), 671–677.

[7] Lj. D. R. Kočinac, *Selected results on selection principles*, in: Proc. 3rd Seminar on Geometry and Topology, Azarb. Univ. Tarbiat Moallem, Tabriz, 2004, 71–104.

[8] M. V. Matveev, *Some questions on property (a)*, Questions Answers Gen. Topology 15 (1997), 103–111.

[9] M. V. Matveev, *A survey on star covering properties*, Topology Atlas, preprint 330, 1998.
[10] J. T. Moore, M. Hrušák and M. Džamonja, *Parametrized $\diamondsuit$ principles*, Trans. Amer. Math. Soc. 356 (2004), 2281–2306.
[11] C. J. G. Morgan and S. G. da Silva, *Almost disjoint families and "never" cardinal invariants*, Comment. Math. Univ. Carolin. 50 (2009), 433–444.
[12] C. J. G. Morgan and S. G. da Silva, *A note on closed discrete subsets of separable (a)-spaces*, Houston J. Math. 38 (2012), 991–997.
[13] C. J. G. Morgan and S. G. da Silva, *Selectively (a)-spaces from almost disjoint families are necessarily countable under a certain parametrized weak diamond principle*, Houston J. Math., to appear.
[14] V. C. V. de Paiva, *A Dialectica-like model of linear logic*, in: Category Theory and Computer Science (Manchester, 1989), Lecture Notes in Comput. Sci. 389, Springer, Berlin, 1989, 341–356; www.cs.bham.ac.uk/∼vdp/publications/CTCS89.pdf.
[15] V. C. V. de Paiva, *Dialectica and Chu constructions: cousins?*, Theory Appl. Categories 17 (2007), 127–152; www.tac.mta.ca/tac/volumes/17/7/17-07.pdf.
[16] M. Scheepers, *Selection principles and covering properties in topology*, Note Mat. 22 (2003/04), 3–41.
[17] S. G. da Silva, *On the presence of countable paracompactness, normality and property (a) in spaces from almost disjoint families*, Questions Answers Gen. Topology 25 (2007), 1–18.
[18] S. G. da Silva, *(a)-spaces and selectively (a)-spaces from almost disjoint families*, Acta Math. Hungar. 142 (2014), 420–432.
[19] P. J. Szeptycki, *Soft almost disjoint families*, Proc. Amer. Math. Soc. 130 (2002), 3713–3717.
[20] P. Vojtáš, *Generalized Galois–Tukey-connections between explicit relations on classical objects of real analysis*, in: Set Theory of the Reals (Ramat Gan, 1991), Israel Math. Conf. Proc. 6, Bar-Ilan Univ., Ramat Gan, 1993, 619–643.
[21] Y. Yasui, *Relative properties*, in: K. P. Hart et al. (eds.), Encyclopedia of General Topology, Elsevier, Amsterdam, 2004, 33–36.

Samuel G. da Silva
Instituto de Matemática
Universidade Federal da Bahia
Av. Adhemar de Barros, S/N, Ondina
CEP 40170-110, Salvador, BA, Brazil
E-mail: samuel@ufba.br

Received 25 September 2014;
revised 27 February 2015 (6381)