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COXETER POLYNOMIALS OF SALEM TREES

BY

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Abstract. We compute the Coxeter polynomial of a family of Salem trees, and also the limit of the spectral radii of their Coxeter transformations as the number of their vertices tends to infinity. We also prove that if z is a root of multiplicities $m_1, \ldots, m_k$ for the Coxeter polynomials of the trees $\mathcal{T}_1, \ldots, \mathcal{T}_k$ respectively, then z is a root for the Coxeter polynomial of their join, of multiplicity at least $\min\{m - m_1, \ldots, m - m_k\}$ where $m = m_1 + \cdots + m_k$.

1. Introduction and preliminaries.

In [14], Lakatos determines the limit of the spectral radii of the Coxeter transformations of particular infinite sequences of starlike trees. In the present paper we generalize her result to a wider range of trees. In addition, our idea of proof is different from the one in [14].

We use the same terminology as in [14, 24, 27]. We denote by $\mathbb{N} \subseteq \mathbb{Z}$ the set of positive integers and the ring of integers respectively. The algebra of $n \times n$ integer matrices is denoted by $\mathbb{M}_n(\mathbb{Z})$, where $n \in \mathbb{N}$. We consider only *simple* graphs (i.e. graphs without multiple edges and loops) $\Gamma = (\Gamma_0, \Gamma_1)$ with $\Gamma_0 = \{v_1, \ldots, v_n\}$ the set of vertices and Γ_1 the set of edges, where $(v_i, v_j) \in \Gamma_1$ if there is an edge connecting v_i and v_j.

Assume that $\Gamma = (\Gamma_0, \Gamma_1)$ is a simple graph with the set of enumerated vertices $\Gamma_0 = \{v_1, \ldots, v_n\}$. We recall that the *adjacency matrix* of Γ is the $n \times n$ symmetric matrix

$$\mathrm{Ad}_\Gamma = [a_{ij}] \in \mathbb{M}_n(\mathbb{Z}) \tag{1.1}$$

with $a_{ij} = 1$ if $(v_i, v_j) \in \Gamma_1$, and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of Γ is defined to be

$$\chi_\Gamma(t) := \det(t \cdot I_n - \mathrm{Ad}_\Gamma) \in \mathbb{Z}[t] \tag{1.2}$$

where $I_n = [\delta_{ij}]$ is the identity matrix in $\mathbb{M}_n(\mathbb{Z})$. It is clear that $\chi_\Gamma(t)$ does not depend on the enumeration $v_1, \ldots, v_n$ of the vertices in Γ_0 (see [4] and [6]).

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Let $\mathbb{R}^n$ be the standard n-dimensional real vector space with the standard basis $e_1, \ldots, e_n$. Given $i \in \{1, \ldots, n\}$, the ith *reflection* of $\varGamma$ is defined to be the $\mathbb{R}$-linear automorphism $\sigma_i : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$\sigma_i(e_j) = e_j - (2\delta_{ij} - a_{ij})e_i. \tag{1.3}$$

The subgroup $W_\varGamma$ of the general linear group $\mathrm{GL}(\mathbb{R}^n) \cong \mathrm{GL}(n, \mathbb{R})$ generated by the reflections $\sigma_1, \ldots, \sigma_n$ of $\varGamma$ is called the *Weyl group* of $\varGamma$ and has the presentation

$$W_\varGamma = \langle \sigma_1, \ldots, \sigma_n : (\sigma_i \sigma_j)^{m_{ij}} = 1 \rangle \tag{1.4}$$

where $M = [m_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ is the matrix defined by $m_{ii} = 1$ for all $i = 1, \ldots, n$, and $m_{ij} = a_{ij} + 2$ for all $i \neq j$ (see [3, 11, 30]). The product $\varPhi_\varGamma = \sigma_1 \cdot \ldots \cdot \sigma_n \in W_\varGamma$ is defined to be the *Coxeter transformation* of $\varGamma$ (see [17]). Obviously, it depends on the enumeration of the vertices (see Remark 1.1 for details). We recall that the Coxeter transformations were first studied by Coxeter [5] who showed that their eigenvalues have remarkable properties (see also Bourbaki [3] and Humphreys [11]).

Throughout this paper, we assume that $\varGamma$ is a tree $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ with enumerated vertices $\mathcal{T}_0 = \{v_1, \ldots, v_n\}$, $\mathrm{Ad}_\mathcal{T} = [a_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ is its adjacency matrix, and

$$\varPhi_\mathcal{T} = \sigma_1 \cdot \ldots \cdot \sigma_n \in W_\mathcal{T} \tag{1.5}$$

is its Coxeter transformation with respect to the enumeration $v_1, \ldots, v_n$. The *Coxeter polynomial* of the tree $\mathcal{T}$ is defined to be the characteristic polynomial of $\varPhi_\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$, that is, the polynomial (see [11, 17, 25])

$$\mathrm{cox}_\mathcal{T}(t) := \det(t \cdot \mathrm{id}_{\mathbb{R}^n} - \varPhi_\mathcal{T}) \in \mathbb{Z}[t]. \tag{1.6}$$

Since $\mathcal{T}$ is a tree, the characteristic polynomial of $\varPhi_\mathcal{T}$ does not depend on the enumeration of the vertices. Indeed, if $v_{\epsilon(1)}, \ldots, v_{\epsilon(n)}$ is obtained from $v_1, \ldots, v_n$ by a permutation $\epsilon \in S_n$ then the Coxeter transformation $\varPhi^\epsilon_\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ corresponding to $v_{\epsilon(1)}, \ldots, v_{\epsilon(n)}$ is conjugate to $\varPhi_\mathcal{T}$ (see [25, Proposition 2.2], [11, Proposition 3.16], [3, 17] and the following remark for details).

Remark 1.1. (a) The Coxeter polynomial $\mathrm{cox}_\varDelta(t)$ is also defined and studied in [24, 25, 26] in a more general setting of loop-free edge-bipartite multigraphs $\varDelta = (\varDelta_0, \varDelta_1 = \varDelta_1^- \cup \varDelta_1^+)$, with $\varDelta_0 = \{v_1, \ldots, v_n\}$ and a separated bipartition $\varDelta_1 = \varDelta_1^- \cup \varDelta_1^+$ of the set of edges. The class of loop-free edge-bipartite multigraphs contains all simple graphs, loop-free multigraphs, and simple signed graphs (see [32]).

The definition of $\mathrm{cox}_\varDelta(t) \in \mathbb{Z}[t]$ for an edge-bipartite multigraph $\varDelta$ differs from (1.6) for simple graphs, and depends on the upper triangular Gram matrix $\check{G}_\varDelta = [d^\varDelta_{ij}] \in \mathrm{GL}(n, \mathbb{Z})$ where $d^\varDelta_{ij} = 1$ for $i = j$, $d^\varDelta_{ij}$ is the number of edges between v_i and v_j with $i < j$ lying in $\varDelta_1^+$, and $-d_{ij}^{\varDelta}$ is the number of edges between v_i and v_j with $i < j$ lying in $\varDelta_1^-$.

In [24, 25, 26], with any loop-free edge-bipartite multigraph $\varDelta = (\varDelta_0, \varDelta_1 = \varDelta_1^- \cup \varDelta_1^+)$ the Coxeter matrix $\mathrm{Cox}_\varDelta := -\check{G}_\varDelta \cdot \check{G}_\varDelta^{-\mathrm{tr}} \in \mathbb{M}_n(\mathbb{Z})$ is associated, and its characteristic polynomial

$$\text{(1.7)} \qquad \mathrm{cox}_\varDelta(t) := \det(t \cdot I_n - \mathrm{Cox}_\varDelta) \in \mathbb{Z}[t],$$

called the *Coxeter polynomial* of $\varDelta$, is self-reciprocal in the sense that $\mathrm{cox}_\varDelta(t) = t^n \,\mathrm{cox}_\varDelta(1/t)$ (see [23, Lemma 2.8(c3)–(c4)]). The *Coxeter transformation* of $\varDelta$ is defined to be the group automorphism

$$\text{(1.8)} \qquad \varPhi_\varDelta : \mathbb{Z}^n \to \mathbb{Z}^n, \qquad v \mapsto v \cdot \mathrm{Cox}_\varDelta .$$

It is proved in [25, Proposition 2.2] that when the underlying multigraph $\overline{\varDelta}$ of $\varDelta$ is a tree, the Coxeter polynomial does not depend on the enumeration of the vertices $v_1, \ldots, v_n$. Hence, in view of the sink-source reflection technique applied in [1, Proposition VII.4.7], the Coxeter polynomial $\mathrm{cox}_\varDelta(t)$ (1.7) of $\varDelta$ coincides with the Coxeter polynomial $\mathrm{cox}_{\overline{\varDelta}}(t)$ of the tree $\mathcal{T} = \overline{\varDelta}$ (in the sense of (1.6)).

The reader is also referred to the recent papers [12, 13], where the irreducible and reduced root systems in the sense of Bourbaki [3] are studied in connection with roots of positive connected edge-bipartite graphs.

(b) The Coxeter polynomial is also defined in [22, 27], for any finite poset $J \equiv (J, \preceq)$ with $J = \{1, \ldots, n\}$, as

$$\text{(1.9)} \qquad \mathrm{cox}_J(t) := \det(t \cdot I_n - \mathrm{Cox}_J) \in \mathbb{Z}[t]$$

where $\mathrm{Cox}_J = -C_J \cdot C_J^{-\mathrm{tr}} \in \mathbb{M}_n(\mathbb{Z})$ is the Coxeter matrix of J and $C_J := [c_{ij}] \in \mathbb{M}(\mathbb{Z})$ is its incidence matrix, with $c_{ij} = 1$ if $i \preceq j$, and $c_{ij} = 0$ if $i \not\preceq j$. It is shown that if the Hasse diagram $H := \mathcal{H}_J$ of J is a tree, then the Coxeter polynomial $\mathrm{cox}_J(t)$ (1.9) of J coincides with the Coxeter polynomial $\mathrm{cox}_H(t)$ of the tree $\mathcal{T} = H$ (in the sense of (1.6)).

By applying Remark 1.1(a) we get the following useful fact.

Corollary 1.2. *Assume that $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ is a tree with enumerated vertices $v_1, \ldots, v_n$ and let $\check{G}_\mathcal{T} = [d_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ be the upper triangular Gram matrix of $\mathcal{T}$, with $d_{11} = \cdots = d_{nn} = 1$, $d_{ij} = -1$ if $i < j$ and there is an edge (v_i, v_j) in $\mathcal{T}_1$, and $[d_{ij}] = 0$ otherwise.*

(a) *The Coxeter transformation $\varPhi_\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ (1.5) of the tree $\mathcal{T}$ restricts to the group automorphism $\varPhi_\mathcal{T} : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by*

$$\varPhi_\mathcal{T}(u) = u \cdot \mathrm{Cox}_\mathcal{T}$$

where $\mathrm{Cox}_\mathcal{T} := -\check{G}_\mathcal{T} \cdot \check{G}_\mathcal{T}^{-\mathrm{tr}} \in \mathbb{M}_n(\mathbb{Z})$ is the Coxeter matrix of $\mathcal{T}$ viewed as an edge-bipartite graph, with $\mathcal{T}_1^+$ empty.

(b) *The Coxeter polynomial* $\operatorname{cox}_{\mathcal{T}}(t)$ (1.6) *of the tree* $\mathcal{T}$ *coincides with the Coxeter polynomial* $\operatorname{cox}_{\mathcal{T}}(t) = \det(t \cdot I_n - \operatorname{Cox}_{\Delta})$ (1.7) *of* $\mathcal{T}$ *viewed as an edge-bipartite tree.*

(c) *The Coxeter polynomial* $\operatorname{cox}_{\mathcal{T}}(t)$ (1.6) *of* $\mathcal{T}$ *is self-reciprocal and does not depend on the enumeration of its vertices.*

Proof. We view $\mathcal{T}$ as an edge-bipartite graph, with $\mathcal{T}_1 = \mathcal{T}_1^- \cup \mathcal{T}_1^+$ where $\mathcal{T}_1^+$ is the empty set. Then the matrix $\check{G} = [d_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ coincides with the upper triangular Gram matrix $\check{G}_{\Delta} = [a_{ij}^{\Delta}]$ defined in Remark 1.1(a), and the corollary is a consequence of the remark. ∎

The most important families of trees are the trees of type ADE given in Figure 1. These are known as the *simply laced Dynkin diagrams*. There is a long list of objects which admit an ADE classification, meaning that there is an equivalence between equivalence classes of objects of the given type and the ADE graphs (see for example [9]). Examples of these objects include

- simply laced finite Coxeter groups,
- simply laced simple Lie algebras,
- platonic solids,
- quivers of finite representation types,
- Kleinian singularities,
- finite subgroups of SU(2).

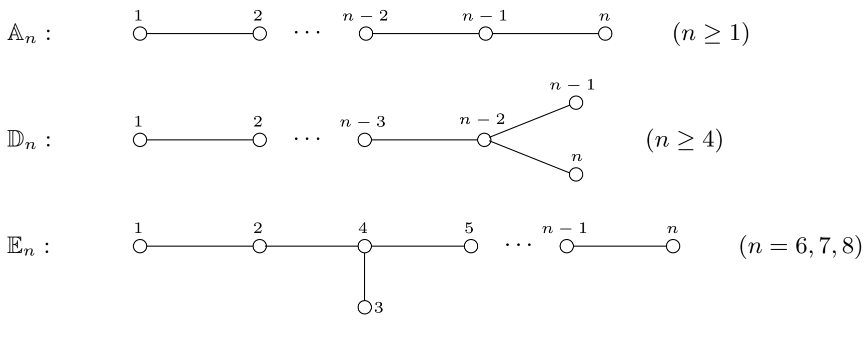

Fig. 1. Simply laced Dynkin diagrams

Note that the graphs $\mathbb{E}_n$ are defined in general for all $n \geq 3$, where $\mathbb{E}_3 = \mathbb{A}_2 \oplus \mathbb{A}_1$, and for $n \geq 4$ are defined as in Figure 1. The graphs $\mathbb{E}_n$ where studied extensively in [8] where their Coxeter polynomials were completely factored into cyclotomic and Salem polynomials. The Coxeter polynomials of the ADE graphs are well known and have been calculated many times (see for instance [2, 3, 7, 8, 25, 27, 30]). One of the main aims of this paper is to find a universal formula for the Coxeter polynomials of a family of trees which we denote by $S^{(i)}_{p_1,\ldots,p_k}$. For specific values of $i, k, p_1, \ldots, p_k \in \mathbb{N}$ we obtain the ADE graphs.

To define the trees $S^{(i)}_{p_1,\dots,p_k}$, we recall that the join of simple graphs $\Gamma_1,\dots,\Gamma_k$, with a fixed vertex v_i in each of the graphs Γ_i, is the graph obtained by adding a new vertex and joining it to v_i for all $i = 1,\dots,k$ (see [30]).

For $k, p_1,\dots,p_k \in \mathbb{N}$ and $i \in \{0, 1,\dots,k\}$, we define the tree $S^{(i)}_{p_1,\dots,p_k}$ to be the join of the Dynkin diagrams $\mathbb{D}_{p_1},\dots,\mathbb{D}_{p_i}$ and $\mathbb{A}_{p_{i+1}},\dots,\mathbb{A}_{p_k}$, in their vertices numbered 1, as shown in Figure 3.

The tree $S^{(0)}_{p_1,\dots,p_k}$ is the star $\mathbb{T}_{p_1-1,\dots,p_k-1}$ defined in [20], which is the join of the Dynkin diagrams $\mathbb{A}_{p_1-1},\dots,\mathbb{A}_{p_k-1}$. It is called a wild star in [14].

To the best of our knowledge the graphs $S^{(i)}_{p_1,\dots,p_k}$ for $i \ge 1$ are defined here for the first time. For particular values of i and p_j, we get some well-known trees. For example, for $k = 2$, $i = 0$, $p_1 = 1$, $p_2 = n - 2$ we obtain the Dynkin diagrams $\mathbb{A}_n$; for $k = 3$, $i = 0$, $p_1 = 1$, $p_2 = 1$, $p_3 = n - 3$ we obtain $\mathbb{D}_n$; for $k = 3$, $i = 0$, $p_1 = 1$, $p_2 = 2$, $p_3 = n - 4$ we obtain $\mathbb{E}_n$; and for $k = 3$, $i = 1$, $p_1 = n - 2$, $p_2 = p_3 = 1$ we obtain the Euclidean Dynkin diagrams $\widetilde{\mathbb{D}}_n$ (see Figure 2). Note that $S^{(0)}_{1,2,6} = \mathbb{E}_{10}$ and $\mathrm{cox}_{\mathbb{E}_{10}}(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$ is the well-known Lehmer polynomial which is conjectured to have the smallest Mahler measure among the monic integer non-cyclotomic polynomials (see [29]).

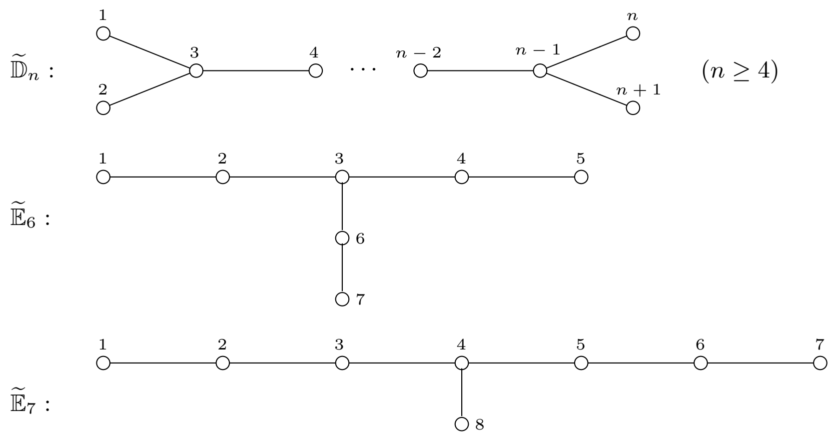

Fig. 2. The Euclidean diagrams $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$ and $\widetilde{\mathbb{E}}_7$

Let $p(t)$ be a monic polynomial with integer coefficients. We denote the set $\{z \in \mathbb{C} : p(z) = 0\}$ of its roots by $Z(p(t))$, and the maximum of $\{|z| : z \in Z(p)\}$ by $\rho(p(t))$. For example, $\rho(\mathrm{cox}_{\mathbb{A}_n}(t)) = \rho(\mathrm{cox}_{\mathbb{D}_n}(t)) = 1$, while $\rho(\mathrm{cox}_{\mathbb{E}_n}(t)) > 1$ for $n \ge 10$ (see [8] and [15]).

If the polynomial $p(t)$ is irreducible and all of its roots lie on the unit circle (or equivalently $\rho(p(t)) = 1$), then $p(t)$ is called a *cyclotomic polynomial*.

Assume now that the polynomial $p(t)$ is irreducible, non-cyclotomic with only one root outside the unit circle. If $p(t)$ has at least one root on the unit circle, it is called a *Salem polynomial*, while if it has no roots on the unit circle, it is called a *Pisot polynomial* (see [15]).

It is not difficult to see that cyclotomic and Salem polynomials are self-reciprocal. This follows from the following facts. A polynomial $p(t)$ of degree n is irreducible if and only if the polynomial $p^*(t) := t^n p(1/t)$, which we call the *reciprocal* of $p(t)$, is irreducible. If α lies on the unit circle then α is a root of $p(t)$ if and only if $1/\alpha$ is also a root of $p(t)$.

We recall from [15] the following definition.

DEFINITION 1.3.

(a) A tree $\mathcal{T}$ is said to be *cyclotomic* if all roots of the Coxeter polynomial $\operatorname{cox}_{\mathcal{T}}(t)$ are on the unit disk, or equivalently $\operatorname{cox}_{\mathcal{T}}(t)$ is a product of cyclotomic polynomials.

(b) A tree $\mathcal{T}$ is called a *Salem tree* if the Coxeter polynomial $\operatorname{cox}_{\mathcal{T}}(t)$ has only one root outside the unit circle, or equivalently $\operatorname{cox}_{\mathcal{T}}(t)$ is a product of a Salem polynomial and some cyclotomic polynomials.

2. Main results. In this paper we are mainly concerned with the case $k = 3$ (i.e. with the trees $S^{(i)}_{p,q,r}$) and prove four theorems about the Coxeter polynomials $\operatorname{cox}_{S^{(i)}_{p_1,\dots,p_k}}(t)$. In Theorem 2.1 we present a recursive relation for these polynomials and we use it in Theorem 2.2 to find the Coxeter polynomials of $S^{(i)}_{p,q,r}$ for all $i = 0, 1, 2, 3$. In Theorem 2.3 we show that the limits $\lim_{p\to\infty} \rho(\operatorname{cox}_{S^{(i)}_{p,q,r}}(t))$, $\lim_{q\to\infty} \rho(\operatorname{cox}_{S^{(i)}_{p,q,r}}(t))$ and $\lim_{r\to\infty} \rho(\operatorname{cox}_{S^{(i)}_{p,q,r}}(t))$ are Pisot numbers. We also show that

$$\lim_{p,q,r\to\infty} \rho(\operatorname{cox}_{S^{(i)}_{p,q,r}}(t)) = 2 \quad \text{for all } i = 0, 1, 2, 3.$$

It was shown by Lakatos [14] that

$$\lim_{p_1,\dots,p_k\to\infty} \rho(\operatorname{cox}_{S^{(0)}_{p_1,\dots,p_k}}(t)) = k - 1 \quad \text{for } k \in \mathbb{N}.$$

In Theorem 2.4 we generalize that result by showing that

$$\lim_{p_1,\dots,p_k\to\infty} \rho(\operatorname{cox}_{S^{(i)}_{p_1,\dots,p_k}}(t)) = k - 1 \quad \text{for all } i \in \{0, 1, \dots, k\}.$$

We mention here that the multiple limits $\lim_{p_1,\dots,p_i\to\infty} \alpha_n$ are the iterated limits $\lim_{p_1\to\infty}(\dots(\lim_{p_i\to\infty} \alpha_n))$.

THEOREM 2.1. *Let $k, p_1, \dots, p_k \in \mathbb{N}$ and $p_1 \geq 2$. Then*

$$\operatorname{cox}_{S^{(0)}_{p_1,\dots,p_k}}(t) = (t + 1) \operatorname{cox}_{S^{(0)}_{p_1-1,\dots,p_k}}(t) - t \operatorname{cox}_{S^{(0)}_{p_1-2,\dots,p_k}}(t).$$

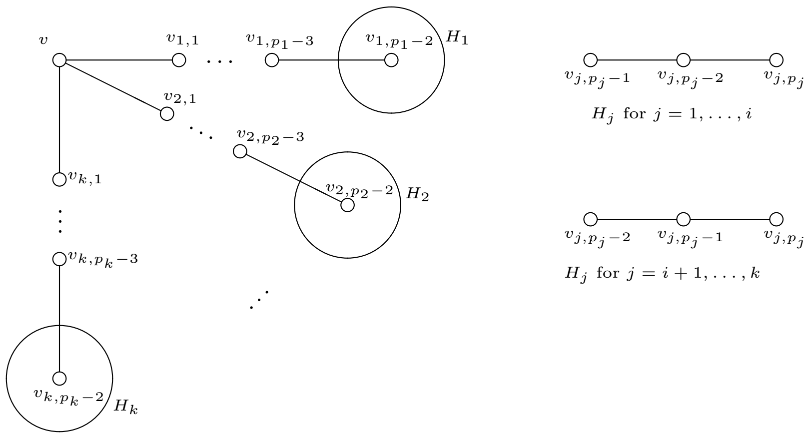

Fig. 3. The trees $S^{(i)}_{p_1,\dots,p_k}$

If $k \geq 2$ *and* $p_1 \geq 3$ *then*

$$\operatorname{cox}_{S^{(i)}_{p_1,\dots,p_k}}(t) = (t+1)[\operatorname{cox}_{S^{(i-1)}_{p_2,\dots,p_k,p_1-1}}(t) - t\operatorname{cox}_{S^{(i-1)}_{p_2,\dots,p_k,p_1-3}}(t)]$$

for all $i \in \{1, \dots, k\}$.

THEOREM 2.2.

(a) *For* $i \leq 2$,

$$\operatorname{cox}_{S^{(i)}_{p,q,r}}(t) = \frac{(t+1)^i}{t-1}[t^{r+2}F^{(i)}_{p,q}(t) - (F^{(i)}_{p,q})^*(t)],$$

where

$$F^{(0)}_{p,q}(t) = t^{p+q} - \operatorname{cox}_{\mathbb{A}_{p-1}}(t)\operatorname{cox}_{\mathbb{A}_{q-1}}(t),$$
$$F^{(1)}_{p,q}(t) = t^{p+q-2}(t-1) - (t^{p-2}+1)\operatorname{cox}_{\mathbb{A}_{q-1}}(t),$$
$$F^{(2)}_{p,q}(t) = t^{p+q-4}(t-1)^2 - (t^{p-2}+1)(t^{q-2}+1).$$

(b) *For* $i = 3$,

$$\operatorname{cox}_{S^{(3)}_{p,q,r}}(t) = (t+1)^3[t^r F^{(3)}_{p,q}(t) + (F^{(3)}_{p,q})^*(t)],$$

where $F^{(3)}_{p,q}(t) = F^{(2)}_{p,q}(t)$.

THEOREM 2.3.

(1) $\lim_{r\to\infty} \rho(\operatorname{cox}_{S^{(i)}_{p,q,r}}(t)) = \rho(F^{(i)}_{p,q}(t))$ *for* $i = 0, 1, 2,$ *and* $\rho(F^{(i)}_{p,q}(t))$ *is a Pisot number,*

$$(2)\qquad \lim_{p\to\infty}\rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))=\rho(F^{(i-1)}_{q,r}(t))\qquad \textit{for } i=1,2,3,$$

$$(3)\qquad \lim_{p,q\to\infty}\rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))=\rho(t^{r+2}-2t^{r+1}+1)\qquad \textit{for } i=0,1,2,$$

$$(4)\qquad \lim_{q,r\to\infty}\rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))=\rho(t^{p}-2t^{p-1}-1)\qquad \textit{for } i=1,2,3,$$

$$(5)\qquad \lim_{p,q,r\to\infty}\rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))=2\qquad \textit{for } i=0,1,2,3.$$

THEOREM 2.4. *For $k,p_1,\dots,p_k\in\mathbb{N}$ and all $i\in\{0,1,\dots,k\}$ we have*

$$\lim_{p_1,\dots,p_k\to\infty}\rho(\mathrm{cox}_{S^{(i)}_{p_1,\dots,p_k}}(t))=k-1.$$

REMARK 2.5. (a) Note that for $i=0$ or $i=3$ the trees $S^{(i)}_{p,q,r}$ and $S^{(i)}_{r,q,p}$ are the same, and therefore the case $i=0$ in (2) is given in (1). Similarly the limit $\lim_{p\to\infty}\rho(\mathrm{cox}_{S^{(0)}_{p,q,r}}(t))$ can be found using the result of (1). The same holds for (3) and (4): the double limit $\lim_{p,q\to\infty}\rho(\mathrm{cox}_{S^{(3)}_{p,q,r}}(t))$ is obtained from (4), and $\lim_{q,r\to\infty}\rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))$ from (3).

(b) In [15] it was shown by James McKee and Chris Smyth that if a non-cyclotomic tree is the join of cyclotomic trees then it is a Salem tree. The cyclotomic trees were classified in [28]; they are the subgraphs of the Euclidean diagram $\widetilde{\mathbb{E}}_8=\mathbb{E}_9$ and of the Euclidean diagrams of Figure 2 (see also [15, 19]). In [15] the Salem trees were classified and they include the joins of cyclotomic trees which are not cyclotomic. It follows from this classification that the cyclotomic cases of the trees $S^{(i)}_{p_1,\dots,p_k}$ are those for $k=i=2$ or $k=3$, $i=0$, $p_1=p_2=p_3=2$ or $k=3$, $i=0$, $p_1=1$, $p_2=p_3=3$ or $k=3$, $i=0$, $p_1=1$, $p_2=2$, $p_3=5$ and subgraphs of these. For all the other cases, $S^{(i)}_{p_1,\dots,p_k}$ are Salem trees.

(c) We recall that the *Mahler measure* of a monic integer polynomial $f(t)$ is

$$M(f)=\prod\{|z|:z\in Z(f(t)),\ |z|\geq 1\}$$

(see [29]). We can easily see that if f is cyclotomic, Salem or Pisot then $M(f)=\rho(f(t))$. Lehmer's problem asks if we can find f with Mahler measure arbitrarily close to 1. Since $\mathrm{cox}_{S^{(i)}_{p_1,\dots,p_k}}(t)$ has at most one root outside the unit circle, its Mahler measure is $\rho(\mathrm{cox}_{S^{(i)}_{p_1,\dots,p_k}}(t))$. Theorem 2.2 in connection with Lemma 3.3 can be used to verify Lehmer's conjecture for the family of the polynomials $\mathrm{cox}_{S^{(i)}_{p,q,r}}(t)$, asserting that the smallest Mahler measure, larger than 1, is the Mahler measure of $\mathrm{cox}_{S^{(0)}_{1,2,6}}(t)=\mathrm{cox}_{\mathbb{E}_{10}}(t)$ (see also [15] and the recent papers [16, 18]).

EXAMPLE 2.6. For the Dynkin diagrams $\mathbb{D}_n$, Theorem 2.2 gives

$$\begin{aligned}\operatorname{cox}_{\mathbb{D}_n}(t) &= \operatorname{cox}_{S^{(0)}_{1,1,n-3}}(t)\\ &= \frac{1}{t-1}(t^{n-1}(t^2-1)+t^2-1) = t^n + t^{n-1} + t + 1.\end{aligned}$$

For the Euclidean diagrams $\widetilde{\mathbb{D}}_n$, Theorem 2.2 gives

$$\begin{aligned}\operatorname{cox}_{\widetilde{\mathbb{D}}_n}(t) &= \operatorname{cox}_{S^{(1)}_{n-2,1,1}}(t)\\ &= \frac{t+1}{t-1}[t^3(t^{n-2}-t^{n-3}-t^{n-4}-1)+t^{n-2}+t^2+t-1]\\ &= (t^{n-2}-1)(t-1)(t+1)^2,\end{aligned}$$

and for the diagrams $\mathbb{E}_n$ it gives

$$\operatorname{cox}_{\mathbb{E}_n}(t) = \operatorname{cox}_{S^{(0)}_{1,2,n-4}}(t) = \frac{1}{t-1}[t^{n-2}(t^3-t-1)+t^3+t^2-1].$$

All these agree with the known formulas (see [7, 8] and [25, Proposition 2.3]).

We also prove the following theorem concerning joins of trees.

THEOREM 2.7. *Let $\mathcal{T}$ be the join of trees $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(k)}$, $k \geq 2$. Suppose that z is a root of $\operatorname{cox}_{\mathcal{T}^{(i)}}(t)$ with multiplicity m_i. Then z is also a root of $\operatorname{cox}_{\mathcal{T}}(t)$ with multiplicity at least*

$$\min\{m - m_i : i = 1, \dots, k\}$$

where $m = m_1 + \dots + m_k$.

REMARK 2.8. (a) According to [31] if $z \neq \pm 1$ is a common root z of the polynomials $\operatorname{cox}_{\mathcal{T}_1}(t), \dots, \operatorname{cox}_{\mathcal{T}_k}(t)$ then its multiplicity m_i is 1. Therefore in that case Theorem 2.7 shows that z is a root of $\operatorname{cox}_{\mathcal{T}}(t)$ with multiplicity at least $k-1$. This result was proved in [8, Theorem 3.1]. For $z = \pm 1$ however, z can be a root of $\operatorname{cox}_{\mathcal{T}}(t)$ with multiplicity less than $k-1$. For example, consider the join $\mathcal{T}$ of the Euclidean diagrams $\widetilde{\mathbb{D}}_4$ as shown in Figure 4. Then $\operatorname{cox}_{\mathcal{T}}(t)$ and $\operatorname{cox}_{\widetilde{\mathbb{D}}_4}(t)$ both have 1 as a root with multiplicity 2.

(b) Now suppose that $\mathcal{T}$ is a join of trees $\mathcal{T}_1, \mathcal{T}_2$ and z is a common root of $\operatorname{cox}_{\mathcal{T}_1}(t)$ and $\operatorname{cox}_{\mathcal{T}_2}(t)$. Then Theorem 2.7 generalizes a theorem due to Kolmykov [30] (see also [8, Theorem 1.5]) asserting that z is a root of $\operatorname{cox}_{\mathcal{T}}(t)$.

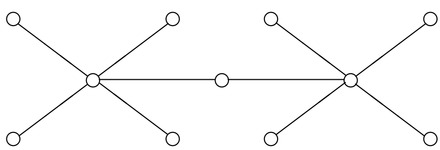

Fig. 4. The join of two $\widetilde{\mathbb{D}}_4$ diagrams

For the convenience of the reader we include all theorems that will be used, in several cases with proofs, thus making this paper self-contained. This is done in Section 3. In Section 4 we prove Theorems 2.1–2.4 and 2.7.

3. Generalities on Coxeter polynomials. The following proposition is due to Subbotin and Sumin; the proof below is taken from [30].

PROPOSITION 3.1. *Assume that $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ is a tree and let $e = (v_1, v_2) \in \mathcal{T}_1$ be a splitting edge of $\mathcal{T}$ that splits it into the trees $\mathcal{R} = (\mathcal{R}_0, \mathcal{R}_1)$ and $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1)$. Assume that $v_1 \in \mathcal{R}_0$ and $v_2 \in \mathcal{S}_0$. Then*

$$\operatorname{cox}_{\mathcal{T}}(t) = \operatorname{cox}_{\mathcal{R}}(t)\operatorname{cox}_{\mathcal{S}}(t) - t\operatorname{cox}_{\tilde{\mathcal{R}}}(t)\operatorname{cox}_{\tilde{\mathcal{S}}}(t)$$

where $\tilde{\mathcal{R}} = (\tilde{\mathcal{R}}_0, \tilde{\mathcal{R}}_1)$ and $\tilde{\mathcal{S}} = (\tilde{\mathcal{S}}_0, \tilde{\mathcal{S}}_1)$ are the subgraphs of $\mathcal{R}$ and $\mathcal{S}$ with vertex sets $\tilde{\mathcal{R}}_0 = \mathcal{R}_0 \setminus \{v_1\}$ and $\tilde{\mathcal{S}}_0 = \mathcal{S}_0 \setminus \{v_2\}$.

Proof. We enumerate the vertices of $\mathcal{R}$ and $\mathcal{S}$ as $\mathcal{R}_0 = \{u_1, \dots, u_k\}$ and $\mathcal{S}_0 = \{u_{k+1}, \dots, u_{k+m}\}$, where $v_1 = u_k$ and $v_2 = u_{k+1}$. Let $\widehat{e} = \{e_1, \dots, e_{k+m}\}$ be the standard basis of $\mathbb{R}^{k+m}$, and let V_1 be the vector subspace of $\mathbb{R}^{k+m}$ with basis $\widehat{e}_1 = \{e_1, \dots, e_k\}$ and V_2 the subspace of $\mathbb{R}^{k+m}$ with basis $\widehat{e}_2 = \{e_{k+1}, \dots, e_{k+m}\}$. Also let σ_i be the ith reflection of $\mathcal{T}$. Then $\Phi_{\mathcal{R}} = \sigma_1 \dots \sigma_k$ is the Coxeter transformation of $\mathcal{R}$, $\Phi_{\mathcal{S}} = \sigma_{k+1} \dots \sigma_{k+m}$ is the Coxeter transformation of $\mathcal{S}$, and $\Phi_{\mathcal{T}} = \Phi_{\mathcal{R}}\Phi_{\mathcal{S}}$ is the Coxeter transformation of $\mathcal{T}$. If R, S are the matrices corresponding to Φ_R, Φ_S with respect to the bases $\widehat{e}_1, \widehat{e}_2$, then with respect to the basis $\widehat{e}$ the Coxeter transformation $\Phi_{\mathcal{T}}$ corresponds to the matrix

$$\begin{pmatrix} R & E_{k1} \\ 0_{mk} & I_m \end{pmatrix} \cdot \begin{pmatrix} I_k & 0_{km} \\ E_{1k} & S \end{pmatrix},$$

where E_{ij} is the matrix with all entries zero except the i,j entry which is 1, and 0_{ij} is the $i \times j$ zero matrix. The Coxeter polynomial of $\mathcal{T}$ is then given by

$$\operatorname{cox}_{\mathcal{T}}(t) = \det(tI_{k+m} - \Phi_{\mathcal{T}}) = \det\begin{pmatrix} tI_k - R - E_{k,k} & -E_{k,1}S \\ -E_{1,k} & tI_m - S \end{pmatrix}.$$

Subtracting the $(k+1)$th row from the kth row we obtain

$$\operatorname{cox}_{\mathcal{T}}(t) = \det\begin{pmatrix} tI_k - R & -tE_{k,1} \\ -E_{1,k} & tI_m - S \end{pmatrix}.$$

Expanding the determinant with respect to the kth row we deduce that

$$\operatorname{cox}_{\mathcal{T}}(t) = \operatorname{cox}_{\mathcal{R}}(t)\operatorname{cox}_{\mathcal{S}}(t) - t\operatorname{cox}_{\tilde{\mathcal{R}}}(t)\operatorname{cox}_{\tilde{\mathcal{S}}}(t). \ \blacksquare$$

The following well-known lemma says that the eigenvalues of a bipartite graph are symmetric around 0 (see [4, 6]).

LEMMA 3.2. *Let Γ be a bipartite graph. If λ is an eigenvalue of the adjacency matrix* Ad_Γ, *then so is* $-\lambda$.

Proof. Enumerate the vertices of Γ in such a way that its adjacency matrix has the form

$$\mathrm{Ad}_\Gamma = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Suppose that $\binom{x}{y}$ is an eigenvector of Ad_Γ with eigenvalue λ. Then $\binom{-x}{y}$ is an eigenvector of Ad_Γ with eigenvalue $-\lambda$. ∎

The next lemma is due to Hoffman and Smith [10].

LEMMA 3.3. *If* $k, p_1, \ldots, p_k \in \mathbb{N}$, $0 \le i \le k$ *and* $p_j < p'_j$ *for some* $1 \le j \le k$, *then*

$$\rho(\mathrm{cox}_{S^{(i)}_{p_1,\ldots,p_j,\ldots,p_k}}(t)) \le \rho(\mathrm{cox}_{S^{(i)}_{p_1,\ldots,p'_j,\ldots,p_k}}(t)) \quad \text{if } j > i, \tag{1}$$

$$\rho(\mathrm{cox}_{S^{(i)}_{p_1,\ldots,p_j,\ldots,p_k}}(t)) \ge \rho(\mathrm{cox}_{S^{(i)}_{p_1,\ldots,p'_j,\ldots,p_k}}(t)) \quad \text{if } j \le i. \tag{2}$$

Moreover, equalities hold if and only if the tree $S^{(i)}_{p_1,\ldots,p'_j,\ldots,p_k}$ *is cyclotomic.*

We will also need the following lemma.

LEMMA 3.4. *Suppose that* $f_n(t) = t^n g(t) + h(t)$ *is a sequence of functions such that* g, h *are continuous,* $f_n(z_n) = 0$ *for all* $n \in \mathbb{N}$ *and that* $\lim_{n\to\infty} z_n = z_0$. *If* $|z_0| > 1$ *then* $g(z_0) = 0$, *while if* $|z_0| < 1$ *then* $h(z_0) = 0$.

Proof. Suppose that $|z_0| > 1$. The function h is continuous and $|g(z_n)| = |h(z_n)|/|z_n^n|$. Therefore $\lim_{n\to\infty} |g(z_n)| = 0$. Since $|g(z_0)| - |g(z_n)| \le |g(z_0) - g(z_n)| \to 0$ as $n \to \infty$, we conclude that $g(z_0) = 0$. The proof for $|z_0| < 1$ is similar. ∎

4. Proof of main theorems

Proof of Theorem 2.1. For $p_1 \ge 2$ we split the tree $S^{(0)}_{p_1,\ldots,p_k}$ by removing the edge (v_{1,p_1-1}, v_{1,p_1}) and we apply Proposition 3.1 to get

$$\begin{aligned}\mathrm{cox}_{S^{(0)}_{p_1,\ldots,p_k}}(t) &= \mathrm{cox}_{\mathbb{A}_1}(t)\,\mathrm{cox}_{S^{(0)}_{p_1-1,\ldots,p_k}}(t) - t\,\mathrm{cox}_{S^{(0)}_{p_1-2,\ldots,p_k}}(t) \\ &= (t+1)\,\mathrm{cox}_{S^{(0)}_{p_1-1,\ldots,p_k}}(t) - t\,\mathrm{cox}_{S^{(0)}_{p_1-2,\ldots,p_k}}(t).\end{aligned}$$

We have used the fact that $\mathrm{cox}_{\mathbb{A}_1}(t) = t + 1$, which can be easily verified from the definition of the Coxeter polynomial.

For $k \ge 2$, $p_1 \ge 3$ and $1 \le i \le k$, if we split the tree $S^{(0)}_{p_1,\ldots,p_k}$ by removing the edge (v_{1,p_1-2}, v_{1,p_1}) we end up with $\mathbb{A}_1$ and the join of $i-1$ Dynkin diagrams of types $\mathbb{D}_{p_2}, \ldots, \mathbb{D}_{p_i}$ and $k-i+1$ Dynkin diagrams of types

$\mathbb{A}_{p_{i+1}}, \ldots, \mathbb{A}_{p_k}, \mathbb{A}_{p_1-1}$. We apply Proposition 3.1 to the edge (v_{1,p_1-2}, v_{1,p_1}) to get

$$\mathrm{cox}_{S^{(i)}_{p_1,\ldots,p_k}}(t) = \mathrm{cox}_{\mathbb{A}_1}(t)[\mathrm{cox}_{S^{(i-1)}_{p_2,\ldots,p_k,p_1-1}}(t) - t\,\mathrm{cox}_{S^{(i-1)}_{p_2,\ldots,p_k,p_1-3}}(t)]. \ \blacksquare$$

Proof of Theorem 2.2. For simplicity of notation, we write u_j, v_j, w_j instead of $v_{1,j}, v_{2,j}, v_{3,j}$ respectively.

(a) Applying Proposition 3.1 to the splitting edge (v, u_1) of $S^{(0)}_{p,q,r}$ we get

$$\mathrm{cox}_{S^{(0)}_{p,q,r}}(t) = \mathrm{cox}_{\mathbb{A}_p}(t)\,\mathrm{cox}_{\mathbb{A}_{q+r+1}}(t) - t\,\mathrm{cox}_{\mathbb{A}_{p-1}}(t)\,\mathrm{cox}_{\mathbb{A}_q}(t)\,\mathrm{cox}_{\mathbb{A}_r}(t).$$

The polynomial $\mathrm{cox}_{\mathbb{A}_n}(t)$ can be easily calculated using Proposition 3.1. It satisfies the recurrence

$$\mathrm{cox}_{\mathbb{A}_n}(t) = \mathrm{cox}_{\mathbb{A}_{n-1}}(t) + t(\mathrm{cox}_{\mathbb{A}_{n-1}}(t) - \mathrm{cox}_{\mathbb{A}_{n-2}}(t))$$

and is given by the formula $\mathrm{cox}_{\mathbb{A}_n}(t) = t^n + t^{n-1} + \cdots + t + 1$. Therefore

$$\begin{aligned}(t-1)^3\,\mathrm{cox}_{S^{(0)}_{p,q,r}}(t) &= t^{p+q+r+4} - 2t^{p+q+r+3} + t^{p+r+2} + t^{q+r+2} - t^{r+2} \\ &\quad + t^{p+q+2} - t^{p+2} - t^{q+2} + 2t - 1 \\ &= t^{p+q+r+2}(t-1) - t^{r+2}(t^q - 1)\,\mathrm{cox}_{\mathbb{A}_{p-1}}(t) \\ &\quad + t^2(t^q - 1)\,\mathrm{cox}_{\mathbb{A}_{p-1}}(t) - t + 1,\end{aligned}$$

and hence

$$\begin{aligned}(t-1)\,\mathrm{cox}_{S^{(0)}_{p,q,r}}(t) &= t^{r+2}(t^{p+q} - \mathrm{cox}_{\mathbb{A}_{p-1}}(t)\,\mathrm{cox}_{\mathbb{A}_{q-1}}(t)) \\ &\quad + t^2\,\mathrm{cox}_{\mathbb{A}_{p-1}}(t)\,\mathrm{cox}_{\mathbb{A}_{q-1}}(t) - 1 \\ &= t^{r+2}F^{(0)}_{p,q}(t) - (F^{(0)}_{p,q})^*(t).\end{aligned}$$

For $i = 1, 2$ we use the recurrence relation of Theorem 2.1. For $i = 1$, we get

$$\begin{aligned}\mathrm{cox}_{S^{(1)}_{p,q,r}}(t) &= (t+1)[\mathrm{cox}_{S^{(0)}_{p-1,q,r}}(t) - t\,\mathrm{cox}_{S^{(0)}_{p-3,q,r}}(t)] \\ &= (t+1)t^{r+2}[F^{(0)}_{p-1,q}(t) - tF^{(0)}_{p-3,q}(t)] \\ &\quad - (t+1)[(F^{(0)}_{p-1,q})^*(t) - t(F^{(0)}_{p-3,q})^*(t)] \\ &= (t+1)t^{r+2}[F^{(0)}_{p-1,q}(t) - tF^{(0)}_{p-3,q}(t)] \\ &\quad - (t+1)[F^{(0)}_{p-1,q}(t) - tF^{(0)}_{p-3,q}(t)]^*.\end{aligned}$$

The last equality holds because of the following fact. For $m_1 \geq m_2 \in \mathbb{N}$ and two polynomials f, g with $\deg f = \deg(g) + m_1$ the reciprocal of $f(t) + t^{m_2}g(t)$ is $(f(t) + t^{m_2}g(t))^* = f^*(t) + t^{m_1-m_2}g^*(t)$. Therefore to finish the proof for $i = 1$ it is enough to show that

$$F^{(1)}_{p,q}(t) = F^{(0)}_{p-1,q}(t) - tF^{(0)}_{p-3,q}(t).$$

This is an easy verification:

$$\begin{aligned}
&F^{(0)}_{p-1,q}(t) - tF^{(0)}_{p-3,q}(t)\\
&\qquad = t^{p+q-2}(t-1) - \frac{t^{p-1}-1}{t-1}\operatorname{cox}_{\mathbb{A}_{q-1}}(t) + t\frac{t^{p-3}-1}{t-1}\operatorname{cox}_{\mathbb{A}_{q-1}}(t)\\
&\qquad = t^{p+q-2}(t-1) - (t^{p-2}+1)\operatorname{cox}_{\mathbb{A}_{q-1}}(t).
\end{aligned}$$

For $i = 2$, by Theorem 2.1 we get

$$\begin{aligned}
\operatorname{cox}_{S^{(2)}_{p,q,r}}(t) &= (t+1)[\operatorname{cox}_{S^{(1)}_{q,p-1,r}}(t) - t\operatorname{cox}_{S^{(1)}_{q,p-3,r}}(t)]\\
&= (t+1)t^{r+2}[F^{(1)}_{q,p-1}(t) - tF^{(1)}_{q,p-3}(t)]\\
&\quad - (t+1)[F^{(1)}_{q,p-1}(t) - tF^{(1)}_{q,p-3}(t)]^*,
\end{aligned}$$

and to finish the proof it is enough to verify that

$$F^{(2)}_{p,q}(t) = F^{(1)}_{q,p-1}(t) - tF^{(1)}_{q,p-3}(t).$$

(b) We apply Proposition 3.1 to the edge (w_{r-2}, w_r) of $S^{(3)}_{p,q,r}$ to obtain

$$\operatorname{cox}_{S^{(3)}_{p,q,r}}(t) = (t+1)\operatorname{cox}_{S^{(2)}_{p,q,r-1}}(t) - t(t+1)\operatorname{cox}_{S^{(2)}_{p,q,r-3}}(t).$$

Therefore

$$\begin{aligned}
\frac{t-1}{(t+1)^3}\operatorname{cox}_{S^{(3)}_{p,q,r}}(t) &= \frac{t-1}{(t+1)^2}\operatorname{cox}_{S^{(2)}_{p,q,r-1}}(t) - t\frac{t-1}{(t+1)^2}\operatorname{cox}_{S^{(2)}_{p,q,r-3}}(t)\\
&= t^{r+1}F^{(2)}_{p,q}(t) - (F^{(2)}_{p,q})^*(t) - t^rF^{(2)}_{p,q}(t) + t(F^{(2)}_{p,q})^*(t),
\end{aligned}$$

and hence

$$\operatorname{cox}_{S^{(3)}_{p,q,r}}(t) = (t+1)^3[t^rF^{(2)}_{p,q}(t) + (F^{(2)}_{p,q})^*(t)]. \blacksquare$$

REMARK 4.1. (a) For $i = 1$ we could have applied Proposition 3.1 to the splitting edge (u_{p-2}, u_p) and use $S^{(0)}_{p,q,r} = S^{(0)}_{q,r,p}$ to obtain

$$\operatorname{cox}_{S^{(1)}_{p,q,r}}(t) = (t+1)[t^pF^{(0)}_{q,r}(t) + (F^{(0)}_{q,r})^*(t)].$$

Similarly by noting that the graphs $S^{(1)}_{p,r,q}, S^{(1)}_{p,q,r}$ are the same, as also are $S^{(2)}_{p,q,r}, S^{(2)}_{q,p,r}$, Proposition 3.1 applied to the splitting edge (v_{q-2}, v_q) gives

$$\operatorname{cox}_{S^{(2)}_{p,q,r}}(t) = (t+1)^2[t^pF^{(1)}_{q,r}(t) + (F^{(1)}_{q,r})^*(t)].$$

(b) The polynomials $F^{(i)}_{p,q}(t)$ are explicitly given by

$$F^{(0)}_{p,q}(t) = \frac{t^p(t^{q+2} - 2t^{q+1} + 1) + t^q - 1}{(t-1)^2},$$

$$
\begin{aligned}
F^{(1)}_{p,q}(t) &= \frac{t^{p-2}(t^{q+2} - 2t^{q+1} + 1) - t^q + 1}{t-1} \\
&= \frac{t^q(t^p - 2t^{p-1} - 1) + t^{p-2} - 1}{t-1}, \\
F^{(2)}_{p,q}(t) &= t^{p-2}(t^q - 2t^{q-1} - 1) - t^{q-2} - 1.
\end{aligned}
$$

Proof of Theorem 2.3. (1) From Theorem 2.2 and Lemma 3.4 it is enough to show that the sequence $(\alpha_r)_{r\in\mathbb{N}}$ defined by $\alpha_r = \rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))$ is convergent. By Lemma 3.3, for $i = 0, 1, 2$ the sequence $(\alpha_r)_{r\in\mathbb{N}}$ is increasing. Since $\mathrm{cox}_{S^{(i)}_{p,q,r}}(t) = t^{r+2}F(t) + G(t)$ where $F(t)$, $G(t)$ are monic polynomials, $(\alpha_r)_{r\in\mathbb{N}}$ is also bounded, for if M is so large that $F(t), G(t) > 0$ for all $t \geq M$, then $z < M$ for all $z \in Z(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))$. Therefore the sequence $(\alpha_r)_{r\in\mathbb{N}}$ is indeed convergent.

We now prove that $\rho(F^{(i)}_{p,q})$ is a Pisot number (cf. [15, Lemma 4.3]). Let $\epsilon > 0$ be small enough and r be large enough such that $\rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t)) > 1+\epsilon$ and $|t^{r+2}F^{(i)}_{p,q}(t)| > |(F^{(i)}_{p,q})^*(t)|$ for every $|t| = 1 + \epsilon$. From Rouché's theorem (see [21]) it follows that $F^{(i)}_{q,r}(t)$ has only one root, say z_0, outside the unit circle. If z_0 were a Salem number then we would have $F^*(z_0) = 0$ and therefore $\mathrm{cox}_{S^{(i)}_{p,q,r}}(z_0) = 0$ for all large r, contrary to Lemma 3.3. Therefore $z_0 = \rho(F^{(i)}_{p,q}(t))$, and $\rho(F^{(i)}_{p,q}(t))$ is a Pisot number.

(2) As in (1) we define $\beta_p = \rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))$. From Lemma 3.3, for $i = 1, 2, 3$, the sequence $(\beta_p)_{p\in\mathbb{N}}$ is decreasing. Remark 4.1 implies that for $i = 1, 2$ we have

$$
\mathrm{cox}_{S^{(i)}_{p,q,r}}(t) = (t+1)^i[t^pF^{(i-1)}_{q,r}(t) + (F^{(i-1)}_{q,r})^*(t)]. \tag{4.1}
$$

From Theorem 2.2 and $\mathrm{cox}_{S^{(3)}_{p,q,r}}(t) = \mathrm{cox}_{S^{(3)}_{q,r,p}}(t)$ it follows that (4.1) also holds for $i = 3$. Therefore the sequence $(\beta_p)_{p\in\mathbb{N}}$ is bounded, and from Lemma 3.4 it converges to $\rho(F^{(i-1)}_{q,r}(t))$.

(3) For $q, r \in \mathbb{N}$ and $i \in \{0, 1, 2\}$ we define $\ell^{(i)}_{q,r} = \lim_{p\to\infty} \rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))$. By Lemma 3.3, $\ell_{q,r}$ is monotonic with respect to q. By (1) and (2) and the form of the polynomials $F^{(0)}_{q,r}(t), F^{(1)}_{q,r}(t)$, the sequence $(\ell^{(i)}_{q,r})_{q\in\mathbb{N}}$ is bounded, and hence convergent (note that $\ell^{(i)}_{q,r}$ equals $\rho(F^{(0)}_{q,r}(t))$ or $\rho(F^{(1)}_{q,r}(t))$). From Remark 4.1, Lemma 3.4 and the fact that $\ell_{q,r} > 1$ we deduce the formula of (3).

(4) The proof for this case is similar to (3). For $p, q \in \mathbb{N}$ and $i \in \{1, 2, 3\}$ we define $\ell^{(i)}_{p,q} = \lim_{r\to\infty} \rho(\mathrm{cox}_{S^{(i)}_{p,q,r}}(t))$. By Lemma 3.3, $\ell_{p,q}$ is monotonic in q. By (1), (2) and the form of $F^{(1)}_{p,q}(t), F^{(2)}_{p,q}(t)$ (see Remark 4.1), the sequence

$(\ell_{p,q}^{(i)})_{q\in\mathbb{N}}$ is bounded, and hence convergent ($\ell_{p,q}^{(i)}$ is equal to $\rho(F_{p,q}^{(1)}(t))$ or $\rho(F_{p,q}^{(2)}(t))$). From Lemma 3.4 and $\ell_{p,q} > 1$ we deduce the formula of (4).

(5) The case $i = 0$ was proved by Lakatos [14], and so we only consider the cases $i = 1, 2, 3$. Let $\ell_p^{(i)} = \lim_{q,r\to\infty} \rho(\mathrm{cox}_{S_{p,q,r}^{(i)}}(t))$. From (4), $\ell_p^{(i)} = \rho(H(t))$ where $H(t) := t^p - 2t^{p-1} - 1$. Hence $\lim_{p,q,r\to\infty} \rho(\mathrm{cox}_{S_{p,q,r}^{(i)}}(t)) = \lim_{p\to\infty} \rho(H(t)) = 2$. ■

Proof of Theorem 2.4. For $i \in \{0, 1, \dots, k-1\}$ we have

$$\mathrm{cox}_{S_{p_1,\dots,p_k}^{(i)}}(t) = \frac{t^{p_k+1}F(t) - F^*(t)}{t-1}$$

where

$$F(t) = \mathrm{cox}_{S_{p_1,\cdots,p_{k-1}}^{(i)}}(t) - \mathrm{cox}_{\mathbb{D}_{p_1}(t)}(t)\dots\mathrm{cox}_{\mathbb{D}_{p_i}}(t)\,\mathrm{cox}_{\mathbb{A}_{p_{i+1}}}(t)\dots\mathrm{cox}_{\mathbb{A}_{p_{k-1}}}(t).$$

Since the Coxeter polynomials of $S_{p_1,\dots,p_k}^{(i)}$ and $\mathbb{D}_{p_j}, \mathbb{A}_{p_j}$ are self-reciprocal (see Corollary 1.2(c)), we have

$$F^*(t) = \mathrm{cox}_{S_{p_1,\dots,p_{k-1}}^{(i)}}(t) - t\,\mathrm{cox}_{\mathbb{D}_{p_1}(t)}(t)\dots\mathrm{cox}_{\mathbb{D}_{p_i}}(t)\,\mathrm{cox}_{\mathbb{A}_{p_{i+1}}}(t)\dots\mathrm{cox}_{\mathbb{A}_{p_{k-1}}}(t).$$

Proposition 3.1 applied to the splitting edge $(v, v_{k,1})$ yields

$$\begin{aligned}\mathrm{cox}_{S_{p_1,\dots,p_k}^{(i)}}(t) &= \mathrm{cox}_{S_{p_1,\dots,p_{k-1}}^{(i)}}(t)\,\mathrm{cox}_{\mathbb{A}_{p_k}}(t)\\ &\quad - t\,\mathrm{cox}_{\mathbb{D}_{p_1}(t)}(t)\dots\mathrm{cox}_{\mathbb{D}_{p_i}}(t)\,\mathrm{cox}_{\mathbb{A}_{p_{i+1}}}(t)\dots\mathrm{cox}_{\mathbb{A}_{p_{k-1}}}(t)\,\mathrm{cox}_{\mathbb{A}_{p_k-1}}(t)\\ &= \mathrm{cox}_{S_{p_1,\dots,p_{k-1}}^{(i)}}(t)\frac{t^{p_k+1}-1}{t-1}\\ &\quad - t\,\mathrm{cox}_{\mathbb{D}_{p_1}(t)}(t)\dots\mathrm{cox}_{\mathbb{D}_{p_i}}(t)\,\mathrm{cox}_{\mathbb{A}_{p_{i+1}}}(t)\dots\mathrm{cox}_{\mathbb{A}_{p_{k-1}}}(t)\frac{t^{p_k}-1}{t-1},\end{aligned}$$

which is exactly the polynomial $\frac{t^{p_k+1}F(t)-F^*(t)}{t-1}$.

Therefore $\lim_{p_k\to\infty} \rho(\mathrm{cox}_{S_{p_1,\dots,p_k}^{(i)}}(t)) = \rho(F)$. Similar formulas hold for $i = k$ and inductively we show that

$$\lim_{p_2,\dots,p_k\to\infty} \rho(\mathrm{cox}_{S_{p_1,\dots,p_k}^{(i)}}(t)) = \rho(G)$$

where

$$G(t) = \begin{cases} t^{p_1} - (k-1)t^{p_1-1} - k + 2 & \text{if } i \neq 0,\\ t^{p_1+1} - (k-1)t^{p_1} + k - 2 & \text{if } i = 0.\end{cases}$$

Hence the assertion follows. ■

Proof of Theorem 2.7. Let $\mathcal{T}^{(i)} = (\mathcal{T}_0^{(i)}, \mathcal{T}_1^{(i)})$ where $\mathcal{T}_0^{(i)}$ is the set of vertices of $\mathcal{T}^{(i)}$. We denote by $\mathcal{T}^{[i]}$ the join of the graphs $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(i)}$ at the vertices $v_i \in \mathcal{T}_0^{(i)}$. The graph $\mathcal{T}^{(i)}$ looks like the one in Figure 5.

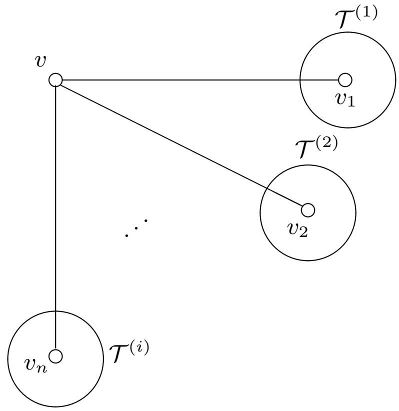

Fig. 5. The join of the graphs $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(i)}$

Let $i \in \{2, \dots, k\}$. Applying Proposition 3.1 to the edge (v, v_i) we get

$$\operatorname{cox}_{\mathcal{T}^{[i]}}(t) = \operatorname{cox}_{\mathcal{T}^{[i-1]}}(t) \operatorname{cox}_{\mathcal{T}^{(i)}}(t) - t \operatorname{cox}_{\mathcal{T}^{(1)}}(t) \dots \operatorname{cox}_{\mathcal{T}^{(i-1)}}(t) \operatorname{cox}_{\widetilde{\mathcal{T}^{(i)}}}(t),$$

where we denote by $\widetilde{\mathcal{T}^{(i)}}$ the induced subgraph of $\mathcal{T}^{(i)}$ with the set of vertices $\widetilde{\mathcal{T}^{(i)}}_0 = \mathcal{T}^{(i)}_0 \setminus \{v_i\}$.

Set $P_k(t) = \operatorname{cox}_{\mathcal{T}^{(1)}}(t) \dots \operatorname{cox}_{\widetilde{\mathcal{T}^{(i)}}}(t) \dots \operatorname{cox}_{\mathcal{T}^{(k)}}(t)$. Then

$$\begin{aligned}
\operatorname{cox}_{\mathcal{T}^{[k]}}(t) &= \operatorname{cox}_{\mathcal{T}^{[k-1]}}(t) \operatorname{cox}_{\mathcal{T}^{(k)}}(t) - tP_k(t) \\
&= \operatorname{cox}_{\mathcal{T}^{[k-2]}}(t) \operatorname{cox}_{\mathcal{T}^{(k-1)}}(t) \operatorname{cox}_{\mathcal{T}^{(k)}}(t) \\
&\quad - t \operatorname{cox}_{\mathcal{T}^{(1)}}(t) \dots \operatorname{cox}_{\mathcal{T}^{(k-2)}}(t) \operatorname{cox}_{\widetilde{\mathcal{T}^{(k-1)}}}(t) \operatorname{cox}_{\mathcal{T}^{(k)}}(t) - tP_k(t) \\
&= \operatorname{cox}_{\mathcal{T}^{[k-2]}}(t) \operatorname{cox}_{\mathcal{T}^{(k-1)}}(t) \operatorname{cox}_{\mathcal{T}^{(k)}}(t) - t(P_{k-1}(t) + P_k(t)) \\
&\quad \dots \\
&= \operatorname{cox}_{\mathcal{T}^{[0]}}(t) \operatorname{cox}_{\mathcal{T}^{(1)}}(t) \dots \operatorname{cox}_{\mathcal{T}^{(k)}}(t) - t(P_1(t) + \dots + P_k(t)) \\
&= (t+1) \operatorname{cox}_{\mathcal{T}^{(1)}}(t) \dots \operatorname{cox}_{\mathcal{T}^{(k)}}(t) - t(P_1(t) + \dots + P_k(t)).
\end{aligned}$$

Since z is a root of $P_i(t)$ of multiplicity $m - m_i$, the theorem follows. ■

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