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Abstract. We investigate convergence and divergence of specific subsequences of partial sums with respect to the Walsh system on martingale Hardy spaces. By using these results we obtain a relationship of the ratio of convergence of the partial sums of the Walsh series and the modulus of continuity of the martingale. These conditions are in a sense necessary and sufficient.

1. Introduction. It is well-known (see e.g. [2] and [24]) that the Walsh system does not form a basis in the space L_1. Moreover, there exists a function f in the dyadic Hardy space H_1 such that the partial sums of f are not bounded in L_1-norm, but the partial sums S_n of the Walsh–Fourier series of every function $f \in L_1$ converge in measure (see also [7] and [12]).

Onneweer [16] showed that if the modulus of continuity of $f \in L_1[0,1)$ satisfies the condition

$$\omega_1(\delta, f) = o(1/\log(1/\delta)) \quad \text{as } \delta \to 0, \tag{1}$$

then the Walsh–Fourier series of f converges in L_1-norm. He also proved that condition (1) cannot be improved.

It is also known that a subsequence S_{m_k} of partial sums is bounded from L_1 to L_1 if and only if $\{m_k : k \geq 0\}$ has uniformly bounded variation. In [24, Ch. 1] it was proved that if $f \in L_1(G)$ and $\{m_n : n \geq 1\}$ is a subsequence of $\mathbb{N}$ such that

$$\omega_1(1/m_n, f) = o(1/L_S(m_n)) \quad \text{as } n \to \infty, \tag{2}$$

where $L_S(n)$ is the nth Lebesgue constant, then $S_{m_n} f$ converges in L_1-norm. Goginava and Tkebuchava [11] proved that condition (2) cannot be improved. Since (see [14] and e.g. [18])

$$V(n)/8 \leq L_S(n) \leq V(n), \tag{3}$$

condition (2) can be rewritten in the form

$$\omega_1(1/m_n, f) = o(1/V(m_n)) \quad \text{as } n \to \infty.$$

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In [20] it was proved that if $F \in H_p$ and

$$\omega_{H_p}(1/2^n, F) = o(1/(n^{[p]} 2^{(1/p-1)n})) \quad \text{as } n \to \infty, \tag{4}$$

where $0 < p \leq 1$ and $[p]$ denotes the integer part of p, then $S_n F \to F$ as $n \to \infty$ in $L_{p,\infty}$-norm. Moreover, it was shown there that condition (4) cannot be improved.

Uniform and pointwise convergence and some approximation properties of partial sums in L_1-norm were investigated by Goginava [8] (see also [11], [9]), Nagy [15] and Avdispahić and Memić [1]. Fine [4] obtained sufficient conditions for the uniform convergence which are in complete analogy with the Dini–Lipschitz conditions. Guličev [13] estimated the rate of uniform convergence of a Walsh–Fourier series by using Lebesgue constants and the modulus of continuity. These problems for Vilenkin groups were considered by Blahota [3], Fridli [5] and Gát [6].

The main aim of this paper is to find characterizations of boundedness of a subsequence of partial sums of the Walsh series of H_p martingales in terms of measure properties of a Dirichlet kernel corresponding to partial summation. As a consequence we get corollaries about the convergence and divergence of some specific subsequences of partial sums. For $p = 1$ a simple numerical criterion for the index of a partial sum in terms of its dyadic expansion is given which governs the convergence (or the ratio of divergence). Another type of result is a relationship of the ratio of convergence of the partial sums of the Walsh series and the modulus of continuity of the martingale. The conditions given below are in a sense necessary and sufficient.

2. Preliminaries. Let $\mathbb{N}_+$ denote the set of positive integers, and $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is, $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 gives measure $1/2$ to each singleton.

Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2's. The elements of G are represented by sequences $x := (x_0, x_1, \dots)$, where $x_k = 0$ or 1.

It is easy to give a base of neighborhoods of $x \in G$:

$$I_0(x) := G, \qquad I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \setminus I_n$ and $e_n := (0, \dots, 0, 1, 0, \dots) \in G$, $n \in \mathbb{N}$, with 1 in the nth place. Then it is easy to show that

$$\overline{I_M} = \bigcup_{s=0}^{M-1} I_s \backslash I_{s+1}. \tag{5}$$

Every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j 2^j$, where $n_j \in Z_2$ ($j \in \mathbb{N}$) and only a finite number of n_j's are not zero.

Let

$$\langle n\rangle := \min\{j \in \mathbb{N} : n_j \neq 0\}, \qquad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},$$

that is, $2^{|n|} \leq n \leq 2^{|n|+1}$. Set

$$d(n) = |n| - \langle n\rangle \quad \text{for all } n \in \mathbb{N}.$$

Define the *variation* of $n \in \mathbb{N}$ with binary coefficients $(n_k : k \in \mathbb{N})$ by

$$V(n) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|\,.$$

The norms (or quasi-norms) of the spaces $L_p(G)$ and $L_{p,\infty}(G)$ $(0<p<\infty)$ are respectively defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu, \qquad \|f\|_{L_{p,\infty}}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda).$$

The *kth Rademacher function* is defined by

$$r_k(x) := (-1)^{x_k} \qquad (x \in G,\, k \in \mathbb{N}).$$

Now, define the *Walsh system* $w := (w_n : n \in \mathbb{N})$ on G by

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see e.g. [18]).

If $f \in L_1(G)$ we define the Fourier coefficients, the partial sums of the Fourier series, and the Dirichlet kernels with respect to the Walsh system in the usual manner:

$$\widehat{f}(k) := \int_G f w_k \, d\mu \quad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \qquad D_n := \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N}_+).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n, \end{cases} \tag{6}$$

and

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}) \quad \text{for } n = \sum_{i=0}^{\infty} n_i 2^i. \tag{7}$$

Define the nth *Lebesgue constant* by

$$L_S(n) := \|D_n\|_1.$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n $(n \in \mathbb{N})$. Denote by $F = (F_n : n \in \mathbb{N})$ a martingale with respect to $\digamma_n$ $(n \in \mathbb{N})$ (for details see e.g. [22]).

The maximal function of the martingale F is defined by

$$F^* = \sup_{n\in\mathbb{N}} |F_n|.$$

In case $f \in L_1(G)$, the maximal function is also given by

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \Big| \int_{I_n(x)} f(u)\, d\mu(u) \Big|.$$

For $0 < p < \infty$ the *Hardy martingale space* $H_p(G)$ consists of all martingales F for which

$$\|F\|_{H_p} := \|F^*\|_p < \infty.$$

The best approximation of $f \in L_p(G)$ $(1 \le p < \infty)$ is defined as

$$E_n(f, L_p) = \inf_{\psi\in p_n} \|f - \psi\|_p,$$

where p_n is the set of all Walsh polynomials of order less than $n \in \mathbb{N}$.

The integrated modulus of continuity of $f \in L_p$ is defined by

$$\omega_p(1/2^n, f) = \sup_{h\in I_n} \|f(\cdot + h) - f(\cdot)\|_p.$$

The modulus of continuity in $H_p(G)$ $(0 < p \le 1)$ can be defined in the following way:

$$\omega_{H_p}(1/2^n, F) := \|F - S_{2^n} F\|_{H_p}.$$

Watari [21] showed that there are close connections between

$$\omega_p(1/2^n, f), \quad E_{2^n}(f, L_p), \quad \|f - S_{2^n} f\|_p, \quad p \ge 1,\ n \in \mathbb{N}.$$

In particular,

$$\frac{1}{2}\omega_p(1/2^n, f) \le \|f - S_{2^n} f\|_p \le \omega_p(1/2^n, f) \tag{8}$$

and

$$\frac{1}{2}\|f - S_{2^n} f\|_p \le E_{2^n}(f, L_p) \le \|f - S_{2^n} f\|_p.$$

A bounded measurable function a is called a *p-atom* if there exists a dyadic interval I such that

$$\int_I a\, d\mu = 0, \qquad \|a\|_\infty \le \mu(I)^{-1/p}, \qquad \operatorname{supp}(a) \subset I.$$

The dyadic Hardy martingale spaces H_p for $0 < p \le 1$ have an atomic characterization (see [19] and [23]):

THEOREM W. *A martingale $F = (F_n : n \in \mathbb{N})$ is in H_p $(0 < p \leq 1)$ if and only if there exists a sequence $(a_k : k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k : k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \qquad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{9}$$

Moreover,

$$\|F\|_{H_p} \backsim \inf\Big(\sum_{k=0}^{\infty} |\mu_k|^p\Big)^{1/p},$$

where the infimum is taken over all decompositions of F of the form (9).

It is easy to check that for every martingale $F = (F_n : n \in \mathbb{N})$ and every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n\to\infty} \int_G F_n(x) w_k(x)\, d\mu(x) \tag{10}$$

exists; it is called the *kth Walsh–Fourier coefficient* of F.

If $F := (E_n f : n \in \mathbb{N})$ is a regular martingale, generated by $f \in L_1(G)$, then $\widehat{F}(k) = \widehat{f}(k)$, $k \in \mathbb{N}$.

For the martingale

$$F = \sum_{n=0}^{\infty} (F_n - F_{n-1})$$

the conjugate transforms are defined as

$$\widetilde{F^{(t)}} = \sum_{n=0}^{\infty} r_n(t)(F_n - F_{n-1}),$$

where $t \in G$ is fixed. Note that $\widetilde{F^{(0)}} = F$. As is well known (see e.g. [22]),

$$\|\widetilde{F^{(t)}}\|_{H_p} = \|F\|_{H_p}, \qquad \|F\|_{H_p}^p \sim \int_G \|\widetilde{F^{(t)}}\|_p^p\, dt, \qquad \widetilde{S_n F^{(t)}} = S_n \widetilde{F^{(t)}}. \tag{11}$$

3. Formulation of main results

THEOREM 1. (a) *Let $0 < p < 1$ and $F \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_n F\|_{H_p} \leq c_p 2^{d(n)(1/p-1)} \|F\|_{H_p}.$$

(b) *Let $0 < p < 1$, $\{m_k : k \geq 0\}$ be any increasing sequence in $\mathbb{N}_+$ such that*

$$\sup_{k\in\mathbb{N}} d(m_k) = \infty, \tag{12}$$

and $\Phi:\mathbb{N}_+\to[1,\infty)$ *be any nondecreasing function satisfying*

$$\limsup_{k\to\infty}\frac{2^{d(m_k)(1/p-1)}}{\Phi(m_k)}=\infty. \tag{13}$$

Then there exists a martingale $F\in H_p$ *such that*

$$\sup_{k\in\mathbb{N}}\left\|\frac{S_{m_k}F}{\Phi(m_k)}\right\|_{L_{p,\infty}}=\infty.$$

Corollary 1. (a) *Let* $0<p<1$ *and* $F\in H_p$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that*

$$\|S_nF\|_{H_p}\le c_p(n\mu\{\operatorname{supp}D_n\})^{1/p-1}\|F\|_{H_p}.$$

(b) *Let* $0<p<1$, $\{m_k:k\ge0\}$ *be any increasing sequence in* $\mathbb{N}_+$ *such that*

$$\sup_{k\in\mathbb{N}}m_k\mu\{\operatorname{supp}D_{m_k}\}=\infty, \tag{14}$$

and $\Phi:\mathbb{N}_+\to[1,\infty)$ *be any nondecreasing function satisfying*

$$\limsup_{k\to\infty}\frac{(m_k\mu\{\operatorname{supp}D_{m_k}\})^{1/p-1}}{\Phi(m_k)}=\infty. \tag{15}$$

Then there exists a martingale $F\in H_p$ *such that*

$$\sup_{k\in\mathbb{N}}\left\|\frac{S_{m_k}F}{\Phi(m_k)}\right\|_{L_{p,\infty}}=\infty.$$

Corollary 2. *Let* $n\in\mathbb{N}$ *and* $0<p<1$. *Then there exists a martingale* $F\in H_p$ *such that*

$$\sup_{n\in\mathbb{N}}\|S_{2^n+1}F\|_{L_{p,\infty}}=\infty. \tag{16}$$

Corollary 3. *Let* $n\in\mathbb{N}$ *and* $0<p\le1$ *and* $F\in H_p$. *Then*

$$\|S_{2^n+2^{n-1}}F\|_{H_p}\le c_p\|F\|_{H_p}. \tag{17}$$

Theorem 2. (a) *Let* $n\in\mathbb{N}_+$ *and* $F\in H_1$. *Then there exists an absolute constant* c *such that*

$$\|S_nF\|_{H_1}\le cV(n)\|F\|_{H_1}.$$

(b) *Let* $\{m_k:k\ge0\}$ *be any increasing sequence in* $\mathbb{N}_+$ *such that*

$$\sup_{k\in\mathbb{N}}V(m_k)=\infty, \tag{18}$$

and $\Phi : \mathbb{N}_+ \to [1, \infty)$ *be any nondecreasing function satisfying*

$$\limsup_{k\to\infty} \frac{V(m_k)}{\Phi(m_k)} = \infty. \tag{19}$$

Then there exists a martingale $F \in H_1$ *such that*

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{m_k}F}{\Phi(m_k)} \right\|_1 = \infty.$$

THEOREM 3. *Let* $2^k < n \le 2^{k+1}$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that*

$$\|S_nF - F\|_{H_p} \le c_p 2^{d(n)(1/p-1)} \omega_{H_p}(1/2^k, F) \quad (0 < p < 1)$$

and

$$\|S_nF - F\|_{H_1} \le c_1 V(n) \omega_{H_1}(1/2^k, F). \tag{20}$$

THEOREM 4. (a) *Let* $0 < p < 1$, $F \in H_p$ *and* $\{m_k : k \ge 0\}$ *be a sequence of nonnegative integers such that*

$$\omega_{H_p}(1/2^{|m_k|}, F) = o(1/2^{d(m_k)(1/p-1)}) \quad \text{as } k \to \infty. \tag{21}$$

Then

$$\|S_{m_k}F - F\|_{H_p} \to 0 \quad \text{as } k \to \infty. \tag{22}$$

(b) *Let* $\{m_k : k \ge 0\}$ *be any increasing sequence in* $\mathbb{N}_+$ *satisfying* (12). *Then there exists a martingale* $F \in H_p$ *and a subsequence* $\{\alpha_k : k \ge 0\} \subset \{m_k : k \ge 0\}$ *for which*

$$\omega_{H_p}(1/2^{|\alpha_k|}, F) = O(1/2^{d(\alpha_k)(1/p-1)}) \quad \text{as } k \to \infty$$

and

$$\limsup_{k\to\infty} \|S_{\alpha_k}F - F\|_{L_{p,\infty}} > c_p > 0 \quad \text{as } k \to \infty, \tag{23}$$

where c_p *is an absolute constant depending only on* p.

COROLLARY 4. (a) *Let* $0 < p < 1$, $F \in H_p$ *and* $\{m_k : k \ge 0\}$ *be a sequence of nonnegative integers such that*

$$\omega_{H_p}(1/2^{|m_k|}, F) = o(1/(m_k\mu\{\operatorname{supp} D_{m_k}\})^{1/p-1}) \quad \text{as } k \to \infty. \tag{24}$$

Then (22) *is satisfied.*

(b) *Let* $\{m_k : k \ge 0\}$ *be any increasing sequence in* $\mathbb{N}_+$ *satisfying* (14). *Then there exists a martingale* $F \in H_p$ *and a subsequence* $\{\alpha_k : k \ge 0\} \subset \{m_k : k \ge 0\}$ *for which*

$$\omega_{H_p}(1/2^{|\alpha_k|}, F) = O(1/(\alpha_k\mu\{\operatorname{supp} D_{\alpha_k})^{1/p-1}\}) \quad \text{as } k \to \infty,$$

and (23) *is satisfied.*

THEOREM 5. (a) *Let* $F \in H_1$ *and* $\{m_k : k \geq 0\}$ *be a sequence of non-negative integers such that*

$$\omega_{H_1}(1/2^{|m_k|}, F) = o(1/V(m_k)) \quad \text{as } k \to \infty. \tag{25}$$

Then

$$\|S_{m_k}F - F\|_{H_1} \to 0 \quad \text{as } k \to \infty. \tag{26}$$

(b) *Let* $\{m_k : k \geq 0\}$ *be any increasing sequence in* $\mathbb{N}_+$ *satisfying* (18). *Then there exists a martingale* $F \in H_1$ *and a subsequence* $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ *for which*

$$\omega_{H_1}(1/2^{|\alpha_k|}, F) = O(1/V(\alpha_k)) \quad \text{as } k \to \infty$$

and

$$\limsup_{k\to\infty} \|S_{\alpha_k}F - F\|_1 > c > 0 \quad \text{as } k \to \infty, \tag{27}$$

where c is an absolute constant.

4. Proofs of the results

Proof of Theorem 1. Suppose that

$$\|2^{(1-1/p)d(n)}S_nF\|_p \leq c_p\|F\|_{H_p}. \tag{28}$$

By combining (11) and (28) we get

$$\begin{aligned} \|2^{(1-1/p)d(n)}S_nF\|_{H_p} &\leq c_p \int_G \|2^{(1-1/p)d(n)}\widetilde{S_nF^{(t)}}\|_p \, d\mu(t) \\ &= c_p \int_G \|2^{(1-1/p)d(n)}S_n\widetilde{F^{(t)}}\|_p \, d\mu(t) \leq c_p \int_G \|\widetilde{F^{(t)}}\|_{H_p} \, d\mu(t) \leq c_p\|F\|_{H_p}. \end{aligned} \tag{29}$$

By using Theorem W and (29), the proof of Theorem 1(a) will be complete if we show that

$$\int_G |2^{(1-1/p)d(n)}S_na|^p \, d\mu \leq c_p < \infty \tag{30}$$

for every p-atom a with support I and $\mu(I) = 2^{-N}$.

We may assume that $I = I_M$. It is easy to see that $S_na = 0$ when $2^M \geq n$. Therefore, we can suppose that $2^M < n$. Since $\|a\|_\infty \leq 2^{M/p}$ we can write

$$\begin{aligned} |2^{(1-1/p)d(n)}S_na(x)| &\leq 2^{(1-1/p)d(n)}\|a\|_\infty \int_{I_M} |D_n(x+t)| \, d\mu(t) \\ &\leq 2^{M/p}2^{(1-1/p)d(n)} \int_{I_M} |D_n(x+t)| \, d\mu(t). \end{aligned} \tag{31}$$

Let $x \in I_M$. Since $V(n) \leq d(n)$, by applying (3) we get

$$|2^{(1-1/p)d(n)}S_na| \leq 2^{M/p}2^{(1-1/p)d(n)}V(n) \leq 2^{M/p}d(n)2^{(1-1/p)d(n)}$$

and

$$\int\limits_{I_M} |2^{(1-1/p)d(n)} S_n a|^p \, d\mu \le d(n) 2^{(1-1/p)d(n)} < c_p < \infty. \tag{32}$$

Let $t \in I_M$ and $x \in I_s \backslash I_{s+1}$, $0 \le s \le M-1 < \langle n \rangle$ or $0 \le s < \langle n \rangle \le M-1$. Then $x+t \in I_s \backslash I_{s+1}$. By using (7) we get $D_n(x+t) = 0$ and

$$|2^{(1-1/p)d(n)} S_n a(x)| = 0.$$

Let $x \in I_s \backslash I_{s+1}$ and $\langle n \rangle \le s \le M-1$. Then $x+t \in I_s \backslash I_{s+1}$ for $t \in I_M$. By using (7) we can write

$$|D_n(x+t)| \le \sum_{j=0}^{s} n_j 2^j \le c 2^s.$$

If we apply (31) we get

$$|2^{(1-1/p)d(n)} S_n a(x)| \le 2^{(1-1/p)d(n)} 2^{M/p} \frac{2^s}{2^M} = 2^{\langle n \rangle (1/p-1)} 2^s. \tag{33}$$

By combining (5) and (33) we have

$$\begin{aligned}\int\limits_{\overline{I_M}} |2^{(1-1/p)d(n)} S_n a(x)|^p \, d\mu(x) &= \sum_{s=\langle n \rangle}^{M-1} \int\limits_{I_s \backslash I_{s+1}} |2^{\langle n \rangle (1/p-1)} 2^s|^p \, d\mu(x) \\ &\le c \sum_{s=\langle n \rangle}^{M-1} \frac{2^{\langle n \rangle (1-p)}}{2^{s(1-p)}} \le c_p < \infty.\end{aligned}$$

Let us prove Theorem 1(b). Under condition (13), there exists a subsequence $\{\alpha_k : k \ge 0\} \subset \{m_k : k \ge 0\}$ such that

$$\sum_{\eta=0}^{\infty} \frac{\Phi^{p/2}(\alpha_\eta)}{2^{d(\alpha_\eta)(1-p)/2}} < \infty. \tag{34}$$

Let

$$F_n = \sum_{\{k : |\alpha_k| < n\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{\Phi^{1/2}(\alpha_k)}{2^{d(\alpha_k)(1/p-1)/2}}, \qquad a_k = 2^{|\alpha_k|(1/p-1)} (D_{2^{|\alpha_k|+1}} - D_{2^{|\alpha_k|}}). \tag{35}$$

By combining Theorem W and (34) we conclude that $F = (F_n : n \in \mathbb{N}) \in H_p$. By a simple calculation we get

$$\text{(36)}\quad \widehat{F}(j) = \begin{cases} \Phi^{1/2}(\alpha_k)2^{(|\alpha_k|+\langle\alpha_k\rangle)(1/p-1)/2} & \text{if } j\in\{2^{|\alpha_k|},\ldots,2^{|\alpha_k|+1}-1\},\\ & \quad k=0,1,\ldots,\\ 0 & \text{if } j\notin\bigcup_{k=0}^{\infty}\{2^{|\alpha_k|},\ldots,2^{|\alpha_k|+1}-1\}. \end{cases}$$

Since

$$\text{(37)}\qquad D_{j+2^n}=D_{2^n}+w_{2^n}D_j \quad \text{when } j\le 2^n,$$

by applying (36) we have

$$\begin{aligned} \text{(38)}\quad \frac{S_{\alpha_k}F}{\Phi(\alpha_k)} &= \frac{1}{\Phi(\alpha_k)}\sum_{\eta=0}^{k-1}\sum_{v=2^{|\alpha_\eta|}}^{2^{|\alpha_\eta|+1}-1}\widehat{F}(v)w_v+\frac{1}{\Phi(\alpha_k)}\sum_{v=2^{|\alpha_k|}}^{\alpha_k-1}\widehat{F}(v)w_v \\ &= \frac{1}{\Phi(\alpha_k)}\sum_{\eta=0}^{k-1}\sum_{v=2^{|\alpha_\eta|}}^{2^{|\alpha_\eta|+1}-1}\Phi^{1/2}(\alpha_\eta)2^{(|\alpha_\eta|+\langle\alpha_\eta\rangle)(1/p-1)/2}w_v \\ &\quad+\frac{1}{\Phi(\alpha_k)}\sum_{v=2^{|\alpha_k|}}^{\alpha_k-1}\Phi^{1/2}(\alpha_k)2^{(|\alpha_k|+\langle\alpha_k\rangle)(1/p-1)/2}w_v \\ &= \frac{1}{\Phi(\alpha_k)}\sum_{\eta=0}^{k-1}\Phi^{1/2}(\alpha_\eta)2^{(|\alpha_\eta|+\langle\alpha_\eta\rangle)(1/p-1)/2}(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|+1}}) \\ &\quad+\frac{2^{(|\alpha_k|+\langle\alpha_k\rangle)(1/p-1)/2}w_{2^{|\alpha_k|}}D_{\alpha_k-2^{|\alpha_k|}}}{\Phi^{1/2}(\alpha_k)} =: \mathrm{I}+\mathrm{II}. \end{aligned}$$

By using (34) for I we can write

$$\begin{aligned} \text{(39)}\quad \|\mathrm{I}\|_{L_{p,\infty}}^p &\le \frac{1}{\Phi^p(\alpha_k)} \\ &\quad\times\sum_{\eta=0}^{k-1}\frac{\Phi^{p/2}(\alpha_\eta)}{2^{d(\alpha_\eta)(1-p)/2}}\|2^{|\alpha_\eta|(1/p-1)}(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|+1}})\|_{L_{p,\infty}}^p \\ &\le \frac{1}{\Phi^p(\alpha_k)}\sum_{\eta=0}^{\infty}\frac{\Phi^{p/2}(\alpha_\eta)}{2^{d(\alpha_\eta)(1-p)/2}}\le c<\infty. \end{aligned}$$

Let $x\in I_{\langle\alpha_k\rangle}\backslash I_{\langle\alpha_k\rangle+1}$. Under condition (7) we can show $|\alpha_k|\ne\langle\alpha_k\rangle$. It follows that $\langle\alpha_k-2^{|\alpha_k|}\rangle=\langle\alpha_k\rangle$. By combining (6) and (7) we have

$$\begin{aligned} \text{(40)}\quad |D_{\alpha_k-2^{|\alpha_k|}}| &= \left|(D_{2^{\langle\alpha_k\rangle+1}}-D_{2^{\langle\alpha_k\rangle}})+\sum_{j=\langle\alpha_k\rangle+1}^{|\alpha_k|-1}(\alpha_k)_j(D_{2^{i+1}}-D_{2^i})\right| \\ &= |-D_{2^{\langle\alpha_k\rangle}}| = 2^{\langle\alpha_k\rangle} \end{aligned}$$

and

$$(41)\qquad |\mathrm{II}| = \frac{2^{(|\alpha_k|+\langle\alpha_k\rangle)(1/p-1)/2}}{\Phi^{1/2}(\alpha_k)}|D_{\alpha_k-2^{|\alpha_k|}}(x)| = \frac{2^{|\alpha_k|(1/p-1)/2}2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)}.$$

By using (39) we see that

$$\begin{aligned}\left\|\frac{S_{\alpha_k}F}{\Phi(\alpha_k)}\right\|_{L_{p,\infty}}^p &\geq \|\mathrm{II}\|_{L_{p,\infty}}^p - \|\mathrm{I}\|_{L_{p,\infty}}^p\\ &\geq \frac{2^{(|\alpha_k|)(1/p-1)/2}2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)}\\ &\quad\times \mu\left\{x\in G : |\mathrm{II}| \geq \frac{2^{(|\alpha_k|)(1/p-1)/2}2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)}\right\}^{1/p}\\ &\geq \frac{2^{(|\alpha_k|)(1/p-1)/2}2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)}(\mu\{I_{\langle\alpha_k\rangle}\backslash I_{\langle\alpha_k\rangle+1}\})^{1/p}\\ &\geq c\frac{2^{d(\alpha_k)(1/p-1)/2}}{\Phi^{1/2}(\alpha_k)} \to \infty \quad \text{as } k\to\infty.\end{aligned}$$

Theorem 1 is proved. ■

Proof of Corollaries 1–3. By combining (6) and (7) we obtain

$$I_{\langle n\rangle}\backslash I_{\langle n\rangle+1} \subset \operatorname{supp} D_n \subset I_{\langle n\rangle}, \qquad 2^{-\langle n\rangle-1} \leq \mu\{\operatorname{supp} D_n\} \leq 2^{-\langle n\rangle}.$$

It follows that

$$\frac{2^{d(n)(1/p-1)}}{4} \leq (n\mu\{\operatorname{supp} D_n\})^{1/p-1} \leq 2^{d(n)(1/p-1)}.$$

Corollary 1 is proved.

To prove Corollary 2 we only have to calculate that

$$(42)\qquad |2^n+1| = n, \quad \langle 2^n+1\rangle = 0, \quad d(2^n+1) = n.$$

By using Theorem 1(b) we see that there exists a martingale $F=(F_n : n\in\mathbb{N}) \in H_p$ $(0<p<1)$ such that (16) is satisfied.

Let us prove Corollary 3. Analogously to (42) we can write

$$|2^n+2^{n-1}| = n, \quad \langle 2^n+2^{n-1}\rangle = n-1, \quad d(2^n+2^{n-1}) = 1.$$

By using Theorem 1(a) we immediately get (17) for all $0<p\leq 1$.

Corollaries 1–3 are proved. ■

Proof of Theorem 2. By using (3) we have

$$(43)\qquad \left\|\frac{S_nF}{V(n)}\right\|_1 \leq \|F\|_1 \leq \|F\|_{H_1}.$$

By combining (11) and (43), after similar steps to (29) we see that

$$\left\|\frac{S_nF}{V(n)}\right\|_{H_1} \sim \int_G \left\|\frac{\widetilde{S_nF^{(t)}}}{V(n)}\right\|_1 d\mu(t) \leq \|F\|_{H_1}. \tag{44}$$

Now, we prove part (b). Let $\{m_k : k \geq 0\}$ and $\Phi : \mathbb{N}_+ \to [1,\infty)$ be as in the hypothesis. By (19) there exists an increasing subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ of $\mathbb{N}_+$ such that

$$\sum_{k=1}^{\infty} \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} \leq \beta < \infty. \tag{45}$$

Let

$$F_n := \sum_{\{k:\,|\alpha_k|<n\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)}, \qquad a_k = D_{2^{|\alpha_k|+1}} - D_{2^{|\alpha_k|}}. \tag{46}$$

Analogously to Theorem 1, if we apply Theorem W and (45) we conclude that $F = (F_n : n \in \mathbb{N}) \in H_1$.

By a simple calculation we get

$$\widehat{F}(j) = \begin{cases} \dfrac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} & \text{if } j \in \{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1}-1\},\ k = 0,1,\dots \\ 0 & \text{if } j \notin \bigcup_{k=0}^{\infty}\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1}-1\}. \end{cases} \tag{47}$$

From (37) and (47) analogously to (38) we obtain

$$S_{\alpha_k}F = \sum_{\eta=0}^{k-1} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} (D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}) + \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} w_{2^{|\alpha_k|}} D_{\alpha_k - 2^{|\alpha_k|}}.$$

By combining (3) and (45) we have

$$\begin{aligned} \left\|\frac{S_{\alpha_k}F}{\Phi(\alpha_k)}\right\|_1 &\geq \frac{\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V^{1/2}(\alpha_k)} \|D_{\alpha_k - 2^{|\alpha_k|}}\|_1 - \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{k-1} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \\ &\geq \frac{V(\alpha_k - 2^{|\alpha_k|})\Phi^{1/2}(\alpha_k)}{8\Phi(\alpha_k)V^{1/2}(\alpha_k)} - \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{\infty} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \\ &\geq \frac{cV^{1/2}(\alpha_k)}{\Phi^{1/2}(\alpha_k)} \to \infty \quad \text{as } k \to \infty. \end{aligned}$$

Theorem 2 is proved. ∎

Proof of Theorem 3. Let $0 < p < 1$ and $2^k < n \le 2^{k+1}$. By using Theorem 1(a) we see that

$$\begin{aligned}\|S_nF-F\|_{H_p} &\le c_p\|S_nF-S_{2^k}F\|_{H_p}+c_p\|S_{2^k}F-F\|_{H_p}\\ &= c_p\|S_n(S_{2^k}F-F)\|_{H_p}+c_p\|S_{2^k}F-F\|_{H_p}\\ &\le c_p(1+2^{d(n)(1/p-1)})\omega_{H_p}(1/2^k,F)\\ &\le c_p2^{d(n)(1/p-1)}\omega_{H_p}(1/2^k,F).\end{aligned} \tag{48}$$

The proof of estimate (20) is analogous to that of (48).

Theorem 3 is proved. ∎

Proof of Theorem 4. Let $0 < p < 1$, $F \in H_p$ and $\{m_k : k \ge 0\}$ satisfy (21). By using Theorem 3 we see that (22) holds.

Let us prove part (b). Under condition (12), there exists a subsequence $\{\alpha_k : k \ge 0\} \subset \{m_k : k \ge 0\}$ such that

$$2^{d(\alpha_k)} \uparrow \infty \quad \text{as } k \to \infty, \qquad 2^{2(1/p-1)d(\alpha_k)} \le 2^{(1/p-1)d(\alpha_{k+1})}. \tag{49}$$

We set

$$F_n = \sum_{\{i:|\alpha_i|<n\}} \frac{a_i}{2^{(1/p-1)d(\alpha_i)}},$$

where a_i is defined by (35). Since a_i is a p-atom, if we apply Theorem W and (49) we conclude that $F \in H_p$. On the other hand,

(50)

$$\begin{aligned}F-S_{2^n}F &= (F^{(1)}-S_{2^n}F^{(1)},\dots,F^{(n)}-S_{2^n}F^{(n)},\dots,F^{(n+k)}-S_{2^n}F^{(n+k)})\\ &= (0,\dots,0,F^{(n+1)}-F^{(n)},\dots,F^{(n+k)}-F^{(n)},\dots)\\ &= \Big(0,\dots,0,\sum_{i=n}^{n+k}\frac{a_i}{2^{(1/p-1)d(\alpha_i)}},\dots\Big), \qquad k\in\mathbb{N}_+\end{aligned}$$

is a martingale. By combining (49) and Theorem W we get

$$\omega_{H_p}\Big(\frac{1}{2^{|\alpha_k|}},F\Big) \le \sum_{i=k}^{\infty}\frac{1}{2^{(1/p-1)d(\alpha_i)}} = O\Big(\frac{1}{2^{(1/p-1)d(\alpha_k)}}\Big) \quad \text{as } n\to\infty. \tag{51}$$

It is easy to show that

$$\widehat{F}(j) = \begin{cases} 2^{(1/p-1)\langle\alpha_k\rangle} & \text{if } j \in \{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\ k=0,1,\dots,\\ 0 & \text{if } j \notin \bigcup_{k=0}^{\infty}\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\}.\end{cases} \tag{52}$$

Analogously to (40) we can write

$$|D_{\alpha_k}| \ge 2^{\langle\alpha_k\rangle} \qquad \text{for } I_{\langle\alpha_k\rangle}\backslash I_{\langle\alpha_k\rangle+1}.$$

Since

$$\|D_{\alpha_k}\|_{L_{p,\infty}} \geq 2^{\langle\alpha_k\rangle}\mu\{x\in I_{\langle\alpha_k\rangle}\backslash I_{\langle\alpha_k\rangle+1} : |D_{\alpha_k}|\geq 2^{\langle\alpha_k\rangle}\}^{1/p}$$
$$\geq 2^{\langle\alpha_k\rangle}(\mu\{I_{\langle\alpha_k\rangle}\backslash I_{\langle\alpha_k\rangle+1}\})^{1/p} \geq 2^{\langle\alpha_k\rangle(1-1/p)}$$

by using (52) we have

$$\begin{aligned}\|S_{\alpha_k}F-F\|_{L_{p,\infty}} &\geq \|2^{(1/p-1)\langle\alpha_k\rangle}(D_{2^{|\alpha_k|+1}}-D_{\alpha_k})\|_{L_{p,\infty}}\\ &\quad -\Big\|\sum_{i=k+1}^{\infty}2^{(1/p-1)\langle\alpha_i\rangle}(D_{2^{|\alpha_i|+1}}-D_{2^{|\alpha_i|}})\Big\|_{L_{p,\infty}}\\ &=2^{(1/p-1)\langle\alpha_k\rangle}\|D_{\alpha_k}\|_{L_{p,\infty}}-2^{(1/p-1)\langle\alpha_k\rangle}\|D_{2^{|\alpha_k|+1}}\|_{L_{p,\infty}}\\ &\quad -\sum_{i\geq k+1}\frac{\|2^{(1/p-1)|\alpha_i|}(D_{2^{|\alpha_i|+1}}-D_{2^{|\alpha_i|}})\|_{L_{p,\infty}}}{2^{(1/p-1)d(\alpha_i)}}\\ &\geq c-\frac{1}{2^{(1/p-1)d(\alpha_k)}}-\sum_{i\geq k+1}\frac{1}{2^{(1/p-1)d(\alpha_i)}}\\ &\geq c-\frac{2}{2^{(1/p-1)d(\alpha_k)}}.\end{aligned}$$

Theorem 4 is proved. ■

Proof of Theorem 5. Let $F\in H_1$ and $\{m_k : k\geq 0\}$ satisfy (25). By using Theorem 3 we see that (26) holds.

Let us prove part (b). Under the conditions of this part, there exists a subsequence $\{\alpha_k : k\geq 0\}\subset\{m_k : k\geq 0\}$ such that

$$V(\alpha_k)\uparrow\infty \quad \text{as } k\to\infty, \qquad V^2(\alpha_k)\leq V(\alpha_{k+1}). \tag{53}$$

We set

$$F_n=\sum_{\{i:\,|\alpha_i|<n\}}\frac{a_i}{V(\alpha_i)},$$

where a_i is defined by (46). Since a_i is a 1-atom, if we apply Theorem W and (53) we conclude that $F=(F_n : n\in\mathbb{N})\in H_1$.

Analogously to (50), by (53) and Theorem W we can show that

$$F-S_{2^n}F=\left(0,\dots,0,\sum_{i=n}^{n+k}\frac{a_i}{V(\alpha_i)},\dots\right), \qquad k\in\mathbb{N}_+,$$

is a martingale and

$$\|F-S_{2^n}F\|_{H_1}\leq\sum_{i=n+1}^{\infty}\frac{1}{V(\alpha_i)}=O\left(\frac{1}{V(\alpha_n)}\right) \quad \text{as } n\to\infty.$$

It is easy to show that

$$\widehat{F}(j)=\begin{cases}1/V(\alpha_k) & \text{if } j\in\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\ k=0,1,\dots,\\ 0 & \text{if } j\notin\bigcup_{k=0}^{\infty}\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\end{cases}$$

which implies

$$\begin{aligned}\|F-S_{\alpha_k}F\|_1 &\geq \left\|\frac{D_{2^{|\alpha_k|+1}}-D_{\alpha_k}}{V(\alpha_k)}+\sum_{i=k+1}^{\infty}\frac{D_{2^{|\alpha_i|+1}}-D_{2^{|\alpha_i|}}}{V(\alpha_i)}\right\|_1\\ &\geq \frac{\|D_{\alpha_k}\|_1}{V(\alpha_k)}-\frac{\|D_{2^{|\alpha_k|+1}}\|_1}{V(\alpha_k)}-\left\|\sum_{i=k+1}^{\infty}\frac{D_{2^{|\alpha_i|+1}}-D_{2^{|\alpha_i|}}}{V(\alpha_i)}\right\|_1\\ &\geq \frac{1}{8}-\frac{1}{V(\alpha_k)}-\sum_{i=k+1}^{\infty}\frac{1}{V(\alpha_i)}\geq\frac{1}{8}-\frac{2}{V(\alpha_k)}.\end{aligned}$$

Theorem 5 is proved. ∎

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