COLLOQUIUM MATHEMATICUM

VOL. 141 2015 NO. 2

UPPER BOUNDS FOR CERTAIN TRIGONOMETRIC SUMS INVOLVING COSINE POWERS

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Abstract. We establish upper bounds for certain trigonometric sums involving cosine powers. Part of these results extend previous ones valid for the sum

$$\sum_{m=1}^{k-1} \frac{|\sin(\pi rm/k)|}{\sin(\pi m/k)}.$$

We apply our results to estimate character sums in an explicit and elementary way.

1. Introduction and statement of the results. Let k, r, b be integers with $k \geq 3$, $b \geq 1$. We define the sums

$$S_k(r,b) = \sum_{m=1}^{[k/2]} \frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)} \cos^b \frac{\pi m}{k},$$
$$S_k^{(1)}(r,b) = \sum_{m=1}^{[k/2]} \frac{|\sin(\pi rm/k)|}{\sin(\pi m/k)} \cos^b \frac{\pi m}{k}.$$

In [17] we established elementary character sum estimates by estimating sums of the type $S_k^{(1)}(r,b)$. More specifically, we estimated the sum

$$S_k^{(1)}(2r, 2t+1)$$

for $t = [\sqrt{k}]$, with the following two results:

$$\begin{aligned} S_k^{(1)}(2r,2t+1) &\leq \frac{3}{4\pi} k \log k + g_1(k), \\ S_k^{(1)}(2r,2t+1) &\leq \frac{\sqrt{3}}{2\sqrt{2}\,\pi} k \log k + g_2(k), \end{aligned} \tag{1.1}$$

2010 *Mathematics Subject Classification*: Primary 11L03, 11L40; Secondary 42A05.
Key words and phrases: trigonometric sum, cosine powers, primitive character, character sum, Pólya–Vinogradov inequality.

DOI: 10.4064/cm141-2-9

where $g_1(k), g_2(k)$ are explicit secondary terms. The sum $S_k^{(1)}(r,b)$ is a weighted analog of

$$\sum_{m=1}^{[k/2]} \frac{|\sin(\pi rm/k)|}{\sin(\pi m/k)}.$$

Several authors have investigated this sum in its analogous form

$$\sum_{m=1}^{k-1} \frac{|\sin(\pi rm/k)|}{\sin(\pi m/k)} = 2\sum_{m=1}^{[k/2]} \frac{|\sin(\pi rm/k)|}{\sin(\pi m/k)} - \lambda,$$

where $\lambda = 1$ if k is even and r is odd, and $\lambda = 0$ otherwise. Upper bounds for this sum were obtained first by Vinogradov [18] and later by Niederreiter [10]. The first result with the main term best possible is due to Cochrane [2], who proved that

$$\sum_{m=1}^{k-1} \frac{|\sin(\pi rm/k)|}{\sin(\pi m/k)} < \frac{4}{\pi^2} k \log k + g(k,r), \tag{1.2}$$

where $g(k,r)$ is an explicit secondary term. Subsequent results with sharper remainders $g(k,r)$ (not always explicit) were presented by Peral [11], Yu [19], Cochrane and Peral [3], and Alzer and Koumandos [1].

In the present work we estimate the sum $S_k(r,b)$ by proving the following theorem.

THEOREM 1.1. *Let $k \geq 3$, $b \geq 1$ be integers. Then*

$$S_k(r,b) \leq \sum_{\substack{m=1 \\ m \text{ odd}}}^{[k/2]} \frac{\cos^b(\pi m/k)}{\sin(\pi m/k)} + \kappa, \tag{1.3}$$

where $\kappa = 0$ if k is even, and $\kappa = 1/(2\sqrt{b})$ if k is odd.

The proof of Theorem 1.1 will be given in Section 2. Estimating from above the bound in (1.3) we obtain the following more explicit bounds.

COROLLARY 1.2. *Let $k \geq 3$ and $b \geq 2$ be integers and set $t = [b/2]$. Then*

$$S_k(r,b) < \frac{k}{2\pi}\left(\log k - \frac{1}{2}\log t\right) + \left(\frac{1}{2} - \frac{\log 2}{2\pi} - \frac{\gamma}{4\pi}\right)k + \frac{\pi t}{4k} + \kappa, \tag{1.4}$$

$$S_k(r,b) < \frac{1}{8\pi}\frac{k^3}{b+1}\cos^{b+1}\frac{\pi}{k} + \frac{k}{2}\cos^b\frac{\pi}{k} + \kappa, \tag{1.5}$$

where $\gamma = 0.577\ldots$ is Euler's constant, and κ is as in Theorem 1.1.

Corollary 1.2 will be proved in Section 3. For $t \geq k^2$ estimate (1.4) becomes rather weak; estimate (1.5) then begins to take over.

Denote by B any upper bound for $S_k(r,b)$ depending only on k and b. The use of the Fourier expansion

$$|\sin x| = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2(nx)}{4n^2-1} \tag{1.6}$$

yields immediately the bound

$$S_k^{(1)}(r,b) \le \frac{4}{\pi}B. \tag{1.7}$$

In particular, (1.4) gives

$$S_k^{(1)}(r,b) < \frac{2k}{\pi^2}\left(\log k - \frac{1}{2}\log t\right) + \frac{4}{\pi}\left(\frac{1}{2} - \frac{\log 2}{2\pi} - \frac{\gamma}{4\pi}\right)k + \frac{t}{k} + \frac{4}{\pi}\kappa, \tag{1.8}$$

which generalizes and sharpens our previous results (1.1). It also extends previous results of the type (1.2) to the case of the weighted sums $S_k^{(1)}(r,b)$ where $b \ge 2$. It is important to note that (1.1) is elementary, while (1.7) and (1.8) are not, because their derivation depends on the Fourier expansion (1.6). Their sources, that is, estimates (1.3), (1.4) and (1.5), are all elementary however.

We now apply some of our results to establish upper bounds for character sums. Let χ be a non-trivial primitive Dirichlet character of modulus k. This implies that $k \ge 3$. The character χ is called *even* or *odd* if $\chi(-1) = 1$ or $\chi(-1) = -1$ respectively. Set

$$s = s(\chi) = \max_r \left|\sum_{n=1}^{r}\chi(n)\right| \quad \text{and} \quad S = S(\chi) = \max_{a,r}\left|\sum_{n=a}^{r}\chi(n)\right|.$$

Obviously, $s \le S \le 2s$ and it is well known that $S = 2s$ whenever χ is even. Thus, any upper bound for s yields, in an obvious way, an upper bound for S and vice versa.

The majority of the results concerning the Pólya–Vinogradov inequality appear in the form

$$\frac{s}{\sqrt{k}\log k} \le c_1 + f_1(k) \quad \text{or} \quad \frac{S}{\sqrt{k}\log k} \le c_2 + f_2(k), \tag{1.9}$$

where c_1, c_2 are explicit constants and the remainders $f_1(k), f_2(k)$ are functions such that $f_1(k) = o(1)$ and $f_2(k) = o(1)$, as $k \to \infty$. Any such result is called *explicit* or *non-explicit* according as the function $f_i(k)$, $i = 1,2$, is explicit or not. The size of the constant c_i, $i = 1,2$, measures how sharp each individual result is. Explicit results are usually weaker than non-explicit ones. Also, *elementary results* (obtained by elementary methods) are usually weaker than *non-elementary* ones. Sharper versions of (1.9) where instead of $\sqrt{k}\log k$ appear functions of smaller order have been proved either in special cases of characters χ, or under unproved conditions: cf. Montgomery and Vaughan [9], Granville and Soundararajan [8], and Goldmakher [7]. Denote the constant c_i, $i = 1, 2$, by c_i^+ or c_i^- if χ is even or odd respectively, in case there is reason to distinguish between even and odd characters.

We apply (1.4) with $b = 2t+1$, $t = \left[\sqrt{k}\right]$ to establish an explicit estimate of the type (1.9) with $c_1^- = 3/(4\pi)$.

THEOREM 1.3. *Let χ be a non-trivial odd primitive Dirichlet character of modulus k. Then*

$$s < \frac{3}{4\pi}\sqrt{k}\log k + \left(2 - \frac{\log 2}{\pi} - \frac{\gamma}{2\pi}\right)\sqrt{k} + 1. \tag{1.10}$$

Theorem 1.3 will be proved in Section 4. Estimate (1.10) is partially sharper than any previous elementary estimate in the sense that each of these results yields for the constant c_1^- a value $> 3/(4\pi) = 0.238\ldots$. Also, although elementary, it is partially sharper than the non-elementary results by Pólya [12] and Qiu [14]; indeed Pólya's result yields $c_1^- = 1/\pi$, while Qiu's result yields $c_1^- = 4/\pi^2$. It may therefore be possible to get sharp values for the constant c_1^- in an elementary way. For a thorough comparison we mention the recent records for the constant c_1^-: The sharpest elementary result prior to our current work is due to Frolenkov [5] (2011); his result yields

$$c_1^- = 1/(4\log(1+\sqrt{2})) + \varepsilon = 0.283\ldots + \varepsilon \qquad \text{for every } \varepsilon > 0$$

(here the constant ε does not vitiate the explicit nature of the result). In the category of non-elementary and explicit results, sharpest ones appear in Pomerance [13] (2011), Frolenkov [4] (2011), and Frolenkov and Soundararajan [6] (2013); each of these yields $c_1^- = 1/(2\pi) = 0.159\ldots$. The sharpest non-elementary and non-explicit result is due to Granville and Soundararajan [8] (2007), with $c_1^- = 1/(4\pi) = 0.079\ldots$ if k is cubefree, and $c_1^- = 1/(3\pi) = 0.106\ldots$ otherwise.

2. Estimation of the sum $S_k(r, b)$. Set $k_0 = [k/2]$ for simplicity. As an arithmetic function of r, $S_k(r, b)$ is periodic with period k and satisfies $S_k(k - r, b) = S_k(r, b)$, and thus attains all its values for $r = 0, 1, \ldots, k_0$. Obviously, $S_k(0, b) = 0$. We shall prove that $S_k(r, b)$ attains its maximum at $r = k_0$. This is evident for $k = 3$. For $k \geq 4$ we shall prove the assertion by proving that $S_k(r, b)$ increases with r in the range $r = 0, 1, \ldots, k_0$. More precisely, we shall prove that $S_k(r + 1, b) \geq S_k(r, b)$ for $r = 0, 1, \ldots, k_0 - 1$. For this purpose, we set

$$D_{r,b} = S_k(r + 1, b) - S_k(r, b)$$

and prove the following preliminary lemma.

LEMMA 2.1. *Let $k \geq 4$ and $b \geq 1$ be integers.*

(i) *We have*

$$D_{r,b} = \sum_{m=1}^{k_0} \sin \frac{\pi(2r+1)m}{k} \cos^b \frac{\pi m}{k}. \tag{2.1}$$

(ii) *For $b \geq 3$ the difference $D_{r,b}$ satisfies the recurrence formula*

$$D_{r,b} = \tfrac{1}{4} D_{r-1,b-2} + \tfrac{1}{2} D_{r,b-2} + \tfrac{1}{4} D_{r+1,b-2}, \tag{2.2}$$

while for $b = 2$ we have

$$\begin{aligned} D_{r,2} = \frac{1}{4} \sum_{m=1}^{k_0} \sin \frac{\pi(2r-1)m}{k} + \frac{1}{2} \sum_{m=1}^{k_0} \sin \frac{\pi(2r+1)m}{k} \\ + \frac{1}{4} \sum_{m=1}^{k_0} \sin \frac{\pi(2r+3)m}{k}. \end{aligned} \tag{2.3}$$

(iii) *¡ $D_{r,1} \geq 0$ for $r = 0, 1, \ldots, k_0 - 1$.*
(iv) *$D_{r,2} > 0$ for $r = 0, 1, \ldots, k_0 - 1$.*

Proof. (i) We have

$$\begin{aligned} D_{r,b} &= \sum_{m=1}^{k_0} \left[\sin^2 \frac{\pi(r+1)m}{k} - \sin^2 \frac{\pi r m}{k} \right] \frac{\cos^b(\pi m/k)}{\sin(\pi m/k)} \\ &= \sum_{m=1}^{k_0} \sin \frac{\pi m}{k} \sin \frac{\pi(2r+1)m}{k} \frac{\cos^b(\pi m/k)}{\sin(\pi m/k)} \\ &= \sum_{m=1}^{k_0} \sin \frac{\pi(2r+1)m}{k} \cos^b \frac{\pi m}{k}. \end{aligned}$$

(ii) For $b \geq 3$ we have

$$\begin{aligned} D_{r,b} &= \sum_{m=1}^{k_0} \sin \frac{\pi(2r+1)m}{k} \cos^2 \frac{\pi m}{k} \cos^{b-2} \frac{\pi m}{k} \\ &= \sum_{m=1}^{k_0} \sin \frac{\pi(2r+1)m}{k} \left(\frac{1}{2} + \frac{1}{2} \cos \frac{2\pi m}{k} \right) \cos^{b-2} \frac{\pi m}{k} \\ &= \frac{1}{2} \sum_{m=1}^{k_0} \sin \frac{\pi(2r+1)m}{k} \cos^{b-2} \frac{\pi m}{k} \\ &\quad + \frac{1}{2} \sum_{m=1}^{k_0} \sin \frac{\pi(2r+1)m}{k} \cos \frac{2\pi m}{k} \cos^{b-2} \frac{\pi m}{k}. \end{aligned}$$

Since

$$\sin\frac{\pi(2r+1)m}{k}\cos\frac{2\pi m}{k}=\frac{1}{2}\sin\frac{\pi(2r-1)m}{k}+\frac{1}{2}\sin\frac{\pi(2r+3)m}{k},$$

we obtain (2.2), and (2.3) follows in the same way.

(iii) By (i) we have

$$D_{r,1}=\sum_{m=1}^{k_0}\sin\frac{\pi(2r+1)m}{k}\cos\frac{\pi m}{k}=\frac{1}{2}\sum_{m=1}^{k_0}\sin\frac{2\pi(r+1)m}{k}+\frac{1}{2}\sum_{m=1}^{k_0}\sin\frac{2\pi rm}{k}.$$

In case k is even the evaluation of the sums in the right member yields

$$D_{r,1}=\begin{cases}\frac{1}{2}\cot\frac{\pi(r+1)}{k}, & r \text{ even},\\ \frac{1}{2}\cot\frac{\pi r}{k}, & r \text{ odd}.\end{cases}$$

It follows that $D_{r,1}\ge 0$ for $r=0,1,\dots,k_0-1$. In fact $D_{r,1}=0$ iff $k\equiv 2 \pmod 4$ and $r=k_0-1$. In case k is odd we obtain in the same way

$$D_{r,1}=\begin{cases}\frac{1}{4}\cot\frac{\pi(r+1)}{2k}-\frac{1}{4}\tan\frac{\pi r}{2k}, & r \text{ even},\\ -\frac{1}{4}\tan\frac{\pi(r+1)}{2k}+\frac{1}{4}\cot\frac{\pi r}{2k}, & r \text{ odd}.\end{cases}$$

It follows that $D_{r,1}>0$ for $r=0,1,\dots,k_0-1$, since $\cot x>\tan y$ for all x,y such that $0<x,y<\pi/4$.

(iv) Evaluating the sums in (2.3) we obtain

$$D_{r,2}=\frac{1}{8}\cot(2r-1)\frac{\pi}{2k}+\frac{1}{4}\cot(2r+1)\frac{\pi}{2k}+\frac{1}{8}\cot(2r+3)\frac{\pi}{2k}.$$

For $r=0$ the sum is obviously positive, and so $D_{r,2}>0$ in this case. For $0<r<k_0-1$ we have $0<(2r-1)\pi/2k,(2r+1)\pi/2k,(2r+3)\pi/2k<\pi/2$, and so $D_{r,2}>0$ in this case as well. For $r=k_0-1$ a calculation shows that $D_{r,2}>0$. ∎

We now prove the monotonicity of the sum $S_k(r,b)$.

THEOREM 2.2. *Let $k\ge 4$, $b\ge 1$ be integers. Then*

$$S_k(r+1,b)\ge S_k(r,b)\quad \textit{for } r=0,1,\dots,k_0-1.$$

Proof. We use induction on b. The result is true for $b=1$ and $b=2$ by Lemma 2.1(iii) & (iv). Assume that it is true for all exponents $<b$; we shall prove it for b. For $r=0$ the result holds trivially because $S_k(0,b)=0$ and $S_k(1,b)$ is positive. For $1\le r\le k_0-2$ the result holds by the induction hypothesis and formula (2.2). It remains to examine the case $r=k_0-1$. If k is even we have $D_{r+1,b-2}=-D_{r,b-2}$, and so

$$D_{r,b}=\tfrac{1}{4}D_{r-1,b-2}+\tfrac{1}{2}D_{r,b-2}-\tfrac{1}{4}D_{r,b-2}=\tfrac{1}{4}D_{r-1,b-2}+\tfrac{1}{4}D_{r,b-2}\ge 0$$

because of the induction hypothesis. For k odd we have $D_{r+1,b-2} = 0$, and so

$$D_{r,b} = \tfrac{1}{4} D_{r-1,b-2} + \tfrac{1}{2} D_{r,b-2} \geq 0$$

by the induction hypothesis. ■

Proof of Theorem 1.1. We have proved that $\max_r S_k(r, b) = S_k(k_0, b)$. If k is even we have $k_0 = k/2$ and so $\sin(\pi k_0 m/k) = \sin(\pi m/2)$. It follows that

$$\max_r S_k(r, b) = \sum_{\substack{m=1 \\ m \text{ odd}}}^{k_0} \frac{\cos^b(\pi m/k)}{\sin(\pi m/k)}. \tag{2.4}$$

For k odd we have $k_0 = (k-1)/2$ and so

$$\begin{aligned} \sin^2 \frac{\pi k_0 m}{k} = \sin^2 \frac{\pi(k-1)m}{2k} &= \sin^2 \left(\frac{\pi m}{2} - \frac{\pi m}{2k}\right) \\ &= \begin{cases} \sin^2(\pi m/2k), & m \text{ even}, \\ \cos^2(\pi m/2k), & m \text{ odd}, \end{cases} \end{aligned}$$

and

$$\frac{\sin^2(\pi k_0 m/k)}{\sin(\pi m/k)} = \begin{cases} \dfrac{1}{2} \tan \dfrac{\pi m}{2k}, & m \text{ even}, \\ \dfrac{1}{\sin(\pi m/k)} - \dfrac{1}{2} \tan \dfrac{\pi m}{2k}, & m \text{ odd}. \end{cases}$$

It follows that

$$\begin{aligned} \max_r S_k(r, b) = &\sum_{\substack{m=1 \\ m \text{ odd}}}^{k_0} \frac{\cos^b(\pi m/k)}{\sin(\pi m/k)} + \frac{1}{2} \sum_{\substack{m=1 \\ m \text{ even}}}^{k_0} \tan \frac{\pi m}{2k} \cos^b \frac{\pi m}{k} \\ &- \frac{1}{2} \sum_{\substack{m=1 \\ m \text{ odd}}}^{k_0} \tan \frac{\pi m}{2k} \cos^b \frac{\pi m}{k} \end{aligned}$$

(in case $k = 3$ the second sum in the right member is zero since its range of summation is empty). Denote the sums in the right member by $\Sigma_1, \Sigma_2, \Sigma_3$ respectively.

Consider the function

$$f(x) = \tan \frac{x}{2} \cos^b x, \qquad 0 \leq x \leq \pi/2.$$

There is $\xi \in (0, \pi/2)$ such that $f(x)$ is strictly increasing in $[0, \xi]$, attains a maximum at $x = \xi$, and is strictly decreasing in $[\xi, \pi/2]$. Obviously $f(\xi) \leq 1$. The following calculation yields the more accurate bound $f(\xi) < 1/(2\sqrt{b})$. Indeed, ξ is the unique root of the derivative $f'(x)$ in $(0, \pi/2)$ and

$$f'(x) = \frac{1}{2} \frac{1}{\cos^2(x/2)} \cos^b x - b \tan \frac{x}{2} \cos^{b-1} x \sin x.$$

It follows that $b = \cos\xi/\sin^2\xi$. Set $\tan(\xi/2) = \omega$. Then $b = (1-\omega^4)/(4\omega^2)$. This implies that

$$\omega^2 = -2b + \sqrt{4b^2+1} = -2b + 2b\sqrt{1+\frac{1}{4b^2}} < -2b + 2b\left(1+\frac{1}{8b^2}\right) = \frac{1}{4b},$$

and so $f(\xi) = \omega\cos^b\xi < \omega = 1/(2\sqrt{b})$. We now have

$$\Sigma_2 < \frac{k}{2\pi}\int_0^{\pi/2} f(x)\,dx + f(\xi) \quad \text{and} \quad \Sigma_3 > \frac{k}{2\pi}\int_0^{\pi/2} f(x)\,dx - f(\xi),$$

and hence $\frac{1}{2}\Sigma_2 - \frac{1}{2}\Sigma_3 < f(\xi) \le 1/(2\sqrt{b})$. We have proved that

$$\max_r S_k(r,b) < \sum_{\substack{m=1\\ m\ \text{odd}}}^{k_0} \frac{\cos^b(\pi m/k)}{\sin(\pi m/k)} + \frac{1}{2\sqrt{b}} \tag{2.5}$$

when k is odd. Results (2.4) and (2.5) together prove Theorem 1.1. ∎

3. Proof of Corollary 1.2. The function $f(x) = (\cos^b x)/\sin x$ is strictly decreasing in the interval $0 \le x \le \pi/2$. Hence (1.3) yields the new estimate

$$S_k(r,t) \le \frac{k}{2\pi}\int_{\pi/k}^{\pi/2} \frac{\cos^b x}{\sin x}\,dx + \frac{\cos^b(\pi/k)}{\sin(\pi/k)} + \kappa. \tag{3.1}$$

Corollary 1.2 is then immediate in view of the following lemma.

LEMMA 3.1. *Let $k \ge 3$ and $b \ge 2$ be integers. Set $t = [b/2]$ and*

$$I_b = \int_{\pi/k}^{\pi/2} \frac{\cos^b x}{\sin x}\,dx.$$

Then

$$I_b < \log k - \frac{1}{2}\log t - \left(\log 2 + \frac{\gamma}{2}\right) + t\frac{\pi^2}{2k^2}, \tag{3.2}$$

$$I_b < \frac{1}{4}\,\frac{k^2}{b+1}\cos^{b+1}\frac{\pi}{k}. \tag{3.3}$$

Proof. In case $b = 2t$ we have

$$I_{2t} = -\log\tan\frac{\pi}{2k} - \sum_{j=1}^{t}\frac{\cos^{2j-1}(\pi/k)}{2j-1}. \tag{3.4}$$

It follows that

$$I_{2t} < \log k - \sum_{j=1}^{t}\frac{[1-\pi^2/(2k^2)]^{2j-1}}{2j-1} \le \log k - \sum_{j=1}^{t}\frac{1-(2j-1)\pi^2/(2k^2)}{2j-1}.$$

Furthermore,

$$I_{2t} < \log k - \sum_{j=1}^{t} \frac{1}{2j-1} + t\frac{\pi^2}{2k^2} < \log k - \left(\frac{1}{2}\log t + \log 2 + \frac{\gamma}{2}\right) + t\frac{\pi^2}{2k^2}.$$

This proves inequality (3.2) in case $b = 2t$. Since

$$-\log \tan \frac{x}{2} = \frac{1}{2}\log \cot^2 \frac{x}{2} = \frac{1}{2}\log \frac{1+\cos x}{1-\cos x},$$

the right member of (3.4) is equal to $R_t(\cos(\pi/k))$, where $R_t(x)$ is the remainder of the Maclaurin approximation

$$\frac{1}{2}\log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2t-1}}{2t-1} + R_t(x), \qquad 0 < x < 1.$$

Since $R_t(x) < x^{2t+1}/((2t+1)(1-x^2))$ we obtain

$$R_t(\cos(\pi/k)) < \frac{\cos^{2t+1}(\pi/k)}{(2t+1)\sin^2(\pi/k)} < \frac{k^2}{4}\,\frac{\cos^{2t+1}(\pi/k)}{2t+1}.$$

This proves inequality (3.3) in case $b = 2t$. If $b = 2t+1$ we have

$$I_{2t+1} = -\log \sin \frac{\pi}{k} - \sum_{j=1}^{t} \frac{\cos^{2j}(\pi/k)}{2j}. \tag{3.5}$$

It follows that

$$I_{2t+1} < \log \frac{k}{2} - \sum_{j=1}^{t} \frac{[1-\pi^2/(2k^2)]^{2j}}{2j} \le \log \frac{k}{2} - \sum_{j=1}^{t} \frac{1-2j\pi^2/(2k^2)}{2j}.$$

Furthermore,

$$I_{2t+1} < \log \frac{k}{2} - \frac{1}{2}\sum_{j=1}^{t} \frac{1}{j} + t\frac{\pi^2}{2k^2} < \log \frac{k}{2} - \frac{1}{2}(\log t + \gamma) + t\frac{\pi^2}{2k^2}.$$

This proves inequality (3.2) when $b = 2t+1$. Since

$$-\log \sin x = -\tfrac{1}{2}\log \sin^2 x = -\tfrac{1}{2}\log(1-\cos^2 x),$$

the right member of (3.5) is equal to $\frac{1}{2}R_t(\cos^2(\pi/k))$, where $R_t(x)$ is the remainder of the Maclaurin approximation

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^t}{t} + R_t(x), \qquad 0 < x < 1.$$

Since $R_t(x) < x^{t+1}/((t+1)(1-x))$ we obtain

$$\frac{1}{2}R_t(\cos^2(\pi/k)) < \frac{1}{2}\,\frac{\cos^{2t+2}(\pi/k)}{(t+1)\sin^2(\pi/k)} < \frac{k^2}{8}\,\frac{\cos^{2t+2}(\pi/k)}{t+1}.$$

This proves inequality (3.3) for $b = 2t+1$. ∎

Since

$$\frac{\cos^b(\pi/k)}{\sin(\pi/k)} < \frac{k}{2}\cos^b\frac{\pi}{k} < \frac{k}{2},$$

relations (3.1)–(3.3) together yield the proof of Corollary 1.2. ∎

4. Proof of Theorem 1.3. Denote by $G(r,\chi)$ the Gauss sum associated with the odd primitive character χ. To prove Theorem 1.3 we start out with the identity

$$-iG(1,\chi)\sum_{n=1}^{r}\overline{\chi}(n) = \sum_{m=1}^{k-1}\chi(m)\frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)}\cos\frac{\pi m}{k} + \frac{i}{2}G(r,\chi), \tag{4.1}$$

proved and used in [16]. This identity is a variant of the standard Schur representation [15] in the following trigonometric form:

$$-iG(1,\chi)\sum_{n=1}^{r}\overline{\chi}(n) = \sum_{m=1}^{k-1}\chi(m)\frac{\sin(\pi(r+1)m/k)\sin(\pi rm/k)}{\sin(\pi m)/k}, \qquad \chi \text{ odd}.$$

At this point we need a lemma which is an analogue of [17, Lemma 1].

LEMMA 4.1. *Let χ be a non-trivial odd primitive Dirichlet character of modulus k and let t be a non-negative integer. Then*

$$\sum_{m=1}^{k-1}\chi(m)\frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)}\cos\frac{\pi m}{k} = \sum_{m=1}^{k-1}\chi(m)\frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)}\cos^{2t+1}\frac{\pi m}{k} + \Delta_t,$$

where Δ_t is a complex number such that $|\Delta_t| \le t\sqrt{k}$.

Proof. We argue by induction on t. The lemma is trivially true for $t = 0$. Assume that it is true for $t > 0$. We have

$$\begin{aligned}\sum_{m=1}^{k-1}\chi(m)\frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)}\cos^{2(t+1)+1}\frac{\pi m}{k} &= \sum_{m=1}^{k-1}\chi(m)\frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)}\cos^{2t+1}\frac{\pi m}{k} \\ &\quad - \sum_{m=1}^{k-1}\chi(m)\sin^2\frac{\pi rm}{k}\sin\frac{\pi m}{k}\cos^{2t+1}\frac{\pi m}{k}.\end{aligned}$$

The induction hypothesis then implies

$$\begin{aligned}\sum_{m=1}^{k-1}\chi(m)\frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)}\cos\frac{\pi m}{k} &= \sum_{m=1}^{k-1}\chi(m)\frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)}\cos^{2(t+1)+1}\frac{\pi m}{k} \\ &\quad + \Big[\Delta_t + \sum_{m=1}^{k-1}\chi(m)\sin^2\frac{\pi rm}{k}\sin\frac{\pi m}{k}\cos^{2t+1}\frac{\pi m}{k}\Big].\end{aligned}$$

The sum

$$\sum_{m=1}^{k-1} \chi(m) \sin^2 \frac{\pi rm}{k} \sin \frac{\pi m}{k} \cos^{2t+1} \frac{\pi m}{k}$$

is a linear combination of Gauss sums and it is $\leq \sqrt{k}$ in absolute value. The induction hypothesis implies then that the quantity in the brackets is $\leq (t+1)\sqrt{k}$ in absolute value. This implies that the lemma is true for $t+1$. ■

Proof of Theorem 1.3. Identity (4.1) becomes, in view of Lemma 4.1,

$$-iG(1,\chi) \sum_{n=1}^{r} \overline{\chi}(n) = \sum_{m=1}^{k-1} \chi(m) \frac{\sin^2(\pi rm/k)}{\sin(\pi m/k)} \cos^{2t+1} \frac{\pi m}{k} + \varDelta_t + \frac{i}{2} G(r,\chi),$$

and immediately yields the estimate

$$s \leq \frac{2}{\sqrt{k}} S_k(r, 2t+1) + t + \frac{1}{2}.$$

Substituting (1.4) into the last result and choosing $t = [\sqrt{k}]$ we obtain

$$s < \frac{2}{\sqrt{k}} \left\{ \frac{k}{2\pi} \left(\log k - \frac{1}{2} \log d \right) + \left(\frac{1}{2} - \frac{\log 2}{2\pi} - \frac{\gamma}{4\pi} \right) k + \frac{\pi t}{4k} + \kappa \right\} + t + \frac{1}{2},$$

or equivalently,

$$s < \frac{3}{4\pi} \sqrt{k} \log k + \left(2 - \frac{\log 2}{\pi} - \frac{\gamma}{2\pi} \right) \sqrt{k} + r(k),$$

where

$$r(k) = \frac{\sqrt{k}}{\pi} \left(\frac{1}{4} \log k - \frac{1}{2} \log t \right) + \frac{\pi t}{2k\sqrt{k}} + \frac{2}{\sqrt{k}} \kappa - \{\sqrt{k}\} + \frac{1}{2}.$$

It is easily seen that

$$r(k) < -\frac{\sqrt{k}}{2\pi} \log\left(1 - \frac{1}{\sqrt{k}} \right) + \frac{\pi}{2k} + \frac{2}{\sqrt{2k\sqrt{k} - k}} + \frac{1}{2}. \tag{4.2}$$

A computation shows that $r(k) < 1$, for $3 \leq k \leq 9$, while estimate (4.2) implies that $r(k) < 1$ for $k \geq 10$. This finishes the proof of Theorem 1.3.

Acknowledgements. I would like to express my thanks to the referee for many valuable suggestions and comments.

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Received 4 August 2014;
revised 1 June 2015 (6334)