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Nilakantha's accelerated series for π

by

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1. Introduction. Unbeknownst to its European discoverers—Gregory (1638–1675) and Leibniz (1646–1716)—the formula

(1) $$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

had been found in India already in the fourteenth or fifteenth century. It first appeared in Sanskrit verse in the book *Tantrasangraha* from about 1500 by the Indian mathematician, astronomer and universal genius Nilakantha (1445–1545). Unlike Gregory and Leibniz, Nilakantha also gave approximations of the tail sums and found a more rapidly converging series,

(2) $$\frac{\pi}{4} = \frac{3}{4} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)^3 - (2n+3)}.$$

The reader is referred to Roy [14] for more details on this fascinating story.

We show here that (2) is the first step of a certain series transformation that eventually leads to the accelerated series

(3) $$\pi = \sum_{n=0}^{\infty} \frac{(5n+3)n!(2n)!}{2^{n-1}(3n+2)!},$$

in much the same way as the difference operator leads to the *Euler transform*

(4) $$\pi = \sum_{n=0}^{\infty} \frac{2^{n+1}n!n!}{(2n+1)!}.$$

We call (3) the *Nilakantha transform* of (1) and note that it converges roughly as 13.5^{-n}, whereas the Euler transform converges as 2^{-n}. Applying

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the Nilakantha transformation to the Newton–Euler formula [8]

$$\frac{\pi}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}}{2n+1} \tag{5}$$

gives the accelerated series (A.5) (see the Appendix), also with convergence 13.5^{-n}. Similar transformations of these and other formulas lead to other accelerated series for π which are collected in the Appendix in a standardized form, with (3) corresponding to (A.1).

Several of the formulas in the Appendix are well known from the literature. A series equivalent to (A.1) is attributed to Gosper in [2, eq. (16.81)]. The series (A.2) is due to Adamchik and Wagon [1] and is one of the simplest of all "BBP-like" formulas [2, Chapter 10]. The even simpler BBP-formula (A.14) appears in [3, eq. (18)] with two terms at a time, attributed to Knuth.

I note that the formulas in the Appendix emerged quite naturally as accelerations of simple series, and all were derived by hand.

As we shall see, this acceleration method can also be used to transform divergent series into convergent ones, in a process more properly called series *deceleration*. For example, decelerating the divergent series

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-3)^n}{2n+1} \tag{6}$$

gives the convergent, fractional BBP-formula (A.18). This argument, of course, is no proof, but I also give an alternative, rigorous demonstration.

The general principle behind these formulas is an acceleration scheme that allows one to approximate an alternating, "sporadic" sum

$$S = a_0 - a_k + a_{2k} - a_{3k} + \cdots$$

from a finite number of terms $a_0, a_1, \ldots, a_n$ of the complete sequence. The general theory—in particular the idea of letting the a_i be the moments of a measure, writing S as an integral with respect to that measure, and approximating the integrand by means of Chebyshev polynomials—is strongly indebted to that in [7] which it generalizes from $k = 1$ to arbitrary k.

As another example of this method, we show how to compute numerically the constant

$$K = \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(n+1/2)}{2n+1} = -2.1577829966\ldots, \tag{7}$$

making the most of a small number of given integer and half-integer zeta-values, $\zeta(n)$ and $\zeta(n+1/2)$. The constant K is known as the *Schneckenkonstante* and arises in connection with the Spiral of Theodorus [6].

2. Alternating, sporadic series. Let μ be a finite, signed measure on $[0,1]$ with moments

$$a_i = \int_0^1 x^i \, d\mu, \qquad i \geq 0, \tag{8}$$

converging to zero for $i \to \infty$, and consider the alternating, sporadic series

$$S = \int_0^1 \frac{d\mu}{1+x^k} = \sum_{i=0}^{\infty} (-1)^i a_{ik} \tag{9}$$

for some integer $k \geq 1$. Let there be given a polynomial

$$P(x) = \sum_{i=0}^{m+k-1} b_i x^i \tag{10}$$

with $P(u) = 1$ for $u^k = -1$, and write

$$Q(x) = \frac{1 - P(x)}{1+x^k} = \sum_{i=0}^{m-1} c_i x^i. \tag{11}$$

Define a new measure μ' with density $P(x)$ with respect to μ, i.e., $d\mu' = P(x)d\mu$, and moments

$$a_i' = \int_0^1 x^i \, d\mu' = \sum_{j=0}^{m+k-1} b_j a_{i+j},$$

and consider the transformed series

$$S' = \int_0^1 \frac{d\mu'}{1+x^k} = \sum_{i=0}^{\infty} (-1)^i a_{ik}'.$$

Write the difference between the old and the new series as

$$\nabla S = S - S' = \int_0^1 Q(x) \, d\mu = \sum_{i=0}^{m-1} c_i a_i.$$

Repeating this process gives a sequence of measures $\mu^{(n)}$ with densities $d\mu^{(n)} = P(x)^n d\mu$ and moments

$$a_i^{(n)} = \int_0^1 x^i \, d\mu^{(n)} \tag{12}$$

as well as a sequence of transformed series

$$S^{(n)} = \int_0^1 \frac{d\mu^{(n)}}{1+x^k} = \sum_{i=0}^{\infty} (-1)^i a_i^{(n)} \tag{13}$$

with differences

$$\nabla S^{(n)} = S^{(n)} - S^{(n+1)} = \int_0^1 Q(x)\, d\mu^{(n)} = \sum_{i=0}^{m-1} c_i a_i^{(n)}. \tag{14}$$

After n steps one has

$$S = \sum_{i=0}^{n-1} \nabla S^{(i)} + S^{(n)}.$$

If M is the maximum of $|P(x)|$ on the interval $[0,1]$, then (13) gives the bound

$$|S^{(n)}| \le M^n \int_0^1 \frac{d|\mu|}{1+x^k}, \tag{15}$$

cf. Remark 1 below. So if $M < 1$, one gets the accelerated series

$$S = \sum_{n=0}^{\infty} \nabla S^{(n)} \tag{16}$$

with convergence M^n, i.e., $\nabla S^{(n)} = O(M^n)$.

REMARK 1. A *signed* measure may take negative as well as positive values. Even if μ were required to be positive, the transformed measure μ' would still take negative values if the density function $P(x)$ did. By Jordan's Decomposition Theorem [4], one can write $\mu = \mu^+ - \mu^-$ with unique positive measures μ^+ and μ^- with disjoint supports. The theory of integration with respect to a signed measure can thus be reduced to that of a usual, positive measure. Absolute integrals such as the one appearing in (15) are defined by means of the *total variation* $|\mu| = \mu^+ + \mu^-$.

The assumption that the moments (8) converge to zero is equivalent to $\mu(\{1\}) = 0$ and guarantees that

$$\frac{1}{1+x^k} = 1 - x^k + x^{2k} - \cdots$$

can be integrated termwise, say, by Lebesgue's Dominated Convergence Theorem.

It is a result of Hausdorff [12, Satz I] that a sequence of real numbers $a_0, a_1, a_2, \ldots$ is the sequence of moments of a finite, positive measure on $[0,1]$ if and only if it is *totally monotonic*, i.e.,

$$\nabla^n a_i \ge 0 \quad \text{for all } i, n \ge 0. \tag{17}$$

As above, $\nabla a_i = a_i - a_{i+1}$ denotes the negated forward difference operator. Similarly, also by Hausdorff [12, Satz II], the a_i are the moments of a finite, signed measure on $[0,1]$ if and only if

$$\sup_{n\geq 0}\sum_{i=0}^{n}\left|\binom{n}{i}\nabla^{n-i}a_i\right|<\infty. \tag{18}$$

The latter condition thus implies that (a_i) is the difference between two totally monotonic sequences. It is seen directly from the identity

$$\sum_{i=0}^{n}\binom{n}{i}\nabla^{n-i}a_i=a_0$$

that (17) implies (18).

Note that the moments (8) need not be of the same sign, not even eventually, so that in reality the series (9) is not necessarily alternating.

For later use we also note that $a_i=1/(i+1)$ and $a_i^*=1/(2i+1)$ are the moments of the usual Lebesgue measure μ and the measure μ^* with density $d\mu^*=d\mu/2\sqrt{x}$, respectively.

REMARK 2. In order to compute S numerically, we approximate it by the first difference ∇S. To minimize the error S', we have to choose $P(x)$ as a polynomial of high degree that approximates zero uniformly on $[0,1]$. In light of (11), this suggests taking

$$P(x)=1-(1+x^k)Q(x),$$

where the polynomial $Q(x)$ is a Chebyshev approximation to $1/(1+x^k)$. This will be carried out in more detail in Sections 4 and 6.

REMARK 3. On the other hand, if we wish to transform S into an exact, accelerated series (16), we have to choose $P(x)$ as a simple polynomial of low degree, so that the terms (14) can be computed explicitly. If, for example, $P(x)$ is of the form $x^j(1-x)^n$, then the terms of the transformed series S' are $a_i'=\nabla^n a_{i+j}$.

The binomial sum

$$\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{x+j}=\frac{n!}{x(x+1)\cdots(x+n)}$$

is well known. It holds as an identity in $\mathbb{Q}(x)$ and can be easily shown as a partial fraction decomposition. The substitution $x=i+1$ thus gives

$$\nabla^n a_i=\frac{n!i!}{(n+i+1)!} \tag{19}$$

for the sequence $a_i=1/(i+1)$. Similarly, letting $x=i+1/2$ gives

$$\nabla^n a_i^*=\frac{4^n n!(2i)!(n+i)!}{i!(2n+2i+1)!} \tag{20}$$

for $a_i^*=1/(2i+1)$. This will be of use later.

3. The Nilakantha transform. Let $k = 2$ throughout this section. Then $P(x)$ must satisfy $P(\pm i) = 1$. We can take for $P(x)$ any product of

$$-x^2, \qquad \frac{x(1-x)^2}{2}, \qquad -\frac{(1-x)^4}{4}.$$

Let

$$P(x) = \frac{x(1-x)^2}{2}, \tag{21}$$

so that

$$Q(x) = 1 - \frac{x}{2}.$$

The transformed series S' has terms

$$a_i' = \frac{\nabla^2 a_{i+1}}{2} = \frac{a_{i+1} - 2a_{i+2} + a_{i+3}}{2},$$

hence the first step of the transformation becomes

$$S = \left(a_0 - \frac{a_1}{2}\right) + \sum_{i=0}^{\infty} \frac{a_{i+1} - 2a_{i+2} + a_{i+3}}{2}. \tag{22}$$

The n times transformed series $S^{(n)}$ has terms

$$a_i^{(n)} = \frac{1}{2^n} \nabla^{2n} a_{i+n},$$

so that we get the Nilakantha transform

$$S = \sum_{n=0}^{\infty} \frac{1}{2^n} \nabla^{2n} \left(a_n - \frac{a_{n+1}}{2}\right) \tag{23}$$

with convergence 13.5^{-n}, since

$$M = P(1/3) = 2/27. \tag{24}$$

Example 4. To accelerate the Gregory–Leibniz series (1), we let $a_i = 1/(i+1)$. The first step (22) of the transformation is precisely Nilakantha's series (2). To compute the fully accelerated series (23), we use the identity (19) and get (A.1).

Example 5. Taking instead $P(x)$ as

$$-\frac{(1-x)^4}{4}, \qquad -\frac{x^3(1-x)^2}{2}, \qquad -\frac{x^4(1-x)^4}{4} \tag{25}$$

gives the three series (A.2), (A.3), (A.4) with $M = 1/4$, $54/3125$, $1/1024$, respectively.

Example 6. The Newton–Euler series (5) can be rewritten as

$$\frac{\pi}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3}$$

with two alternating, sporadic series corresponding to $a_i^* = 1/(2i+1)$ and $a_i^{**} = 1/(2i+3)$. Accelerating each series separately using

$$P(x) = \frac{x(1-x)^2}{2}, \ -\frac{(1-x)^4}{4}, \ -\frac{x^3(1-x)^2}{2}$$

and adding the results gives (A.5), (A.6), (A.7), respectively.

Remark 7. It follows in advance from (24) that (A.1) and (A.5) converge as 13.5^{-n}, but this is also evident from the expressions themselves, by Stirling's Formula. A similar remark applies to all other formulas in the Appendix.

Remark 8. The factors

$$\frac{(2n)!(2n)!(3n)!}{n!(6n)!}, \qquad \frac{(2n)!(5n)!(6n)!}{(3n)!(10n)!}$$

appearing in (A.5) and (A.7) happen to be reciprocal integers by a criterion of Landau (1900), anticipated by Chebyshev (1852) and Catalan (1874). Such expressions are not too common and have been completely classified (in a suitable sense) [5].

Remark 9. I stress that there is no evidence that Nilakantha derived (2) the way we have done here, much less that he knew (3). Of course, the transformation (22) is straightforward to verify directly, making (1) and (2) essentially equivalent.

4. Numerical approximations. Let $k \geq 1$ be given. The Chebyshev polynomials of the first kind, $T_m(x)$, are given recursively by $T_0(x) = 1$, $T_1(x) = x$ and

$$T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x).$$

The zeros of $T_m(x)$ are

$$\eta_i = \cos\frac{(2i-1)\pi}{2m}, \qquad i = 1, \ldots, m.$$

Let $Q(x)$ be the Chebyshev approximation of order m of $1/(1+x^k)$ on the interval $[0,1]$, i.e., $Q(x)$ is the polynomial of degree less than m agreeing with $1/(1+x^k)$ at the m points $(1+\eta_i)/2$. Since $(1+\eta_i)/2$ are the zeros of $T_m(1-2x)$, $Q(x)$ satisfies

$$Q(x) \equiv \frac{1}{1+x^k} \text{ modulo } T_m(1-2x)$$

and can be computed from this congruence by the Euclidean Algorithm. Thus, $P(x)$ will be the polynomial of degree less than $m+k$ with zeros $(1+\eta_i)/2$ and $P(u) = 1$ for $u^k = -1$. Lagrange interpolation gives the explicit expression

$$P(x) = T_m(1-2x) \sum_{u^k=-1} \frac{u}{\beta_m k} \cdot \frac{1+x^k}{u-x}$$

with $\beta_m = T_m(1-2u)$.

In order to evaluate the maximum M of $|P(x)|$ for $0 \le x \le 1$ as $m \to \infty$, first note $|T_m(1-2x)| \le 1$. For a fixed u, the numbers β_m satisfy $\beta_0 = 1$, $\beta_1 = 1-2u$ and the recursion $\beta_m = (2-4u)\beta_{m-1} - \beta_{m-2}$. Hence, $\beta_m = (\lambda_1^m + \lambda_2^m)/2$ with the roots λ_i of the characteristic polynomial $\lambda^2 - (2-4u)\lambda + 1$. We may suppose $|\lambda_1| > |\lambda_2|$, and conclude that $\beta_m \sim |\lambda_1|^m/2$. Finally, let λ be the minimum of $|\lambda_1|$ as u runs through the roots of unity with $u^k = -1$. Then $M = O(\lambda^{-m})$ as $m \to \infty$.

Some values of λ are given in Table 1. Note that the value $\lambda = 5.828$ for $k = 1$ was found in [7].

Table 1. Values of λ

k	1	2	3	4	5	6	7	8	9	10
λ	5.828	4.612	3.732	3.220	2.890	2.659	2.488	2.356	2.250	2.164

Example 10. Suppose we want to compute numerically the alternating, sporadic sum

$$S = \int_0^1 \frac{d\mu}{1+x^2} = \sum_{i=0}^{\infty} (-1)^i a_{2i},$$

and that we have at our disposal the terms $a_0, a_1, a_2, \ldots, a_{99}$. Letting $k = 1$ and $m = 50$ and using only every second term, $a_0, a_2, a_4, \ldots, a_{98}$, we expect a relative error of 5.828^{-50}, or 38 correct, significant digits of S. Letting $k = 2$ and $m = 100$, using all 100 available terms, we expect an error of 4.612^{-100}, or 66 correct digits.

Consider the constant (7), and rewrite it as

$$K = \zeta\left(\frac{1}{2}\right) - \sum_{i=0}^{\infty} (-1)^i \frac{\zeta(i+3/2)}{2i+3}$$

in order to bypass the singularity at $z = 1$. Suppose that the zeta-values $\zeta(i/2+3/2)$ are available for $i = 0, 1, \ldots, 99$. Using the first method gives 42 digits of S, while the second method gives 70 digits, in agreement with our expectations. The second method can be carried out in Pari as follows:

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T=subst(poltchebi(100),x,1-2*x)
Q=lift(1/Mod(1+x^2,T))
c(i)=polcoeff(Q,i)
a(i)=zeta(i/2+3/2)/(i+3)
K=zeta(1/2)-sum(i=0,99,c(i)*a(i))
```

We return to this example in Section 6.

REMARK 11. In the above example, it was assumed that the terms $a_0, a_1, \ldots, a_{99}$ were simply given in advance. In practice, one might obviously have to compute them first. Using $k = 2$ has the advantage that the integer zeta-values $\zeta(n)$ are much faster to compute than the half-integer values $\zeta(n+1/2)$. On the other hand, these values then need to be computed to a precision of up to 10 extra digits due to the numerically larger coefficients c_i. Also, the c_i can be computed particularly efficiently for $k = 1$ (cf. [7]).

REMARK 12. For $k = 2$, we can compare the (optimal) value $\lambda = 4.612$ obtained from Chebyshev polynomials with the values $\lambda = M^{-1/\deg P}$ from the polynomials $P(x)$ given in Section 3. Of these, the Nilakantha transformation (21) has the best convergence, i.e., $\lambda = 2.381$. The other three transformations (25) have $\lambda = 1.414, 2.252, 2.378$, respectively.

5. Geometrically converging series. Let μ be a finite, signed measure on $[0, 1]$ with *arbitrary* moments (8), and consider the alternating, geometrically converging series

$$S = \int_0^1 \frac{d\mu}{1 + \theta x^k} = \sum_{i=0}^{\infty} (-\theta)^i a_{ki} \tag{26}$$

for $k \geq 1$ and $0 < \theta < 1$. Let $P(x)$ be given as in (10) with $P(u/\theta^{1/k}) = 1$ for $u^k = -1$, and write

$$Q(x) = \frac{1 - P(x)}{1 + \theta x^k}.$$

As before, we define a sequence of measures $\mu^{(n)}$ with $d\mu^{(n)} = P(x)^n d\mu$ and moments (12), and we get a sequence of transformed series

$$S^{(n)} = \int_0^1 \frac{d\mu^{(n)}}{1 + \theta x^k}$$

with differences (14) as well as an accelerated series (16) with $\nabla S^{(n)} = O(M^n)$, where M is the maximum of $|P(x)|$ on $[0, 1]$.

EXAMPLE 13. The arcus tangent series

$$\frac{\arctan\sqrt{\theta}}{\sqrt{\theta}} = \sum_{i=0}^{\infty} \frac{(-\theta)^i}{2i+1} \tag{27}$$

has the form (26) with $k = 1$ and $a_i = 1/(2i + 1)$. To accelerate it, $P(x)$ must satisfy $P(-1/\theta) = 1$, and we can take any product of

$$-\theta x, \qquad \frac{\theta(1-x)}{\theta+1}.$$

Letting

$$P(x) = \frac{\theta(1-x)}{\theta+1} \tag{28}$$

and using (20) gives Euler's accelerated series

$$\frac{\arctan\sqrt{\theta}}{\sqrt{\theta}} = \frac{1}{\theta+1}\sum_{n=0}^{\infty}\left(\frac{4\theta}{\theta+1}\right)^n \frac{n!n!}{(2n+1)!} \tag{29}$$

with $M = \theta/(\theta+1)$.

Note that the original series (27) converges for $|\theta| < 1$, whereas the accelerated series (29) converges for $|\theta| < |\theta+1|$, or $\mathrm{Re}(\theta) > -1/2$. The preceding discussion shows that the two series agree for $0 < \theta < 1$. The Identity Theorem for holomorphic functions and the fact that uniform convergence preserves holomorphicity show that (29) holds for $\mathrm{Re}(\theta) > -1/2$.

Inserting $\theta = 1,\ 1/3,\ 3$ gives three classical formulas such as (4).

Also note that (4) is the Euler transform of the Gregory–Leibniz series, i.e., the acceleration corresponding to the negated difference operator ∇, or

$$P(x) = \frac{1-x}{2}.$$

Historical note 14. Euler develops the Euler transformation and derives (4) and (29) from (1) and (27) in his *Institutiones Calculi Differentialis* [9, Part II, Chapter 1] from 1755. Much earlier, he had given a series for $\arcsin^2 x$ essentially equivalent to (29) in a letter to Johann Bernoulli dated 10 December 1737 [15]. Euler proves (29) again (twice) as well as the Machin-like formula $\pi = 20\arctan 1/7 + 8\arctan 3/79$, and computes the two terms with 13 and 17 correct decimals, respectively, but without adding them, in 1779 [10]. He extends this calculation and computes 21 correct decimals of π in [11]. Several sources on the chronology of π state that Euler did this calculation in 1755 and/or in less than an hour. It seems from the above, though, that the calculation could not have been carried out before 1779. Regarding the duration, the relevant passage reads: "totusque hic calculus laborem unius circiter horae consum[p]sit" (and the entire calculation took about an hour's work).

Example 15. Taking

$$P(x) = -\frac{\theta^2 x(1-x)}{\theta+1}$$

rather than (28) gives the accelerated series

$$\frac{\arctan\sqrt{\theta}}{\sqrt{\theta}} = \frac{1}{4(\theta+1)}\sum_{n=0}^{\infty}\left(\frac{-4\theta^2}{\theta+1}\right)^n \binom{4n}{2n}^{-1}\left(\frac{3\theta+4}{4n+1} - \frac{\theta}{4n+3}\right)$$

with $M = \theta^2/4(\theta+1)$.

By the same argument as before, this formula holds on a complex domain bounded by the curve $|\theta|^2 = 4|\theta+1|$, or

$$(x^2+y^2)^2 = 16((x+1)^2+y^2)$$

in real variables. This quartic, algebraic curve is a *limaçon of Pascal*, named after Étienne Pascal, the father of Blaise Pascal, and first studied in 1525 by Albrecht Dürer [13] ([1]).

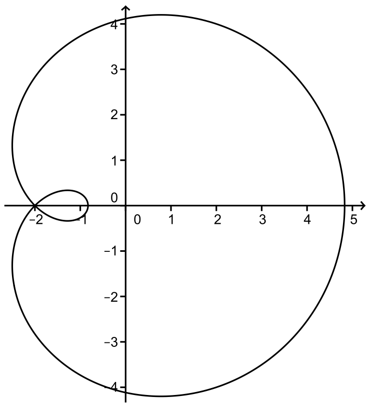

Fig. 1. Limaçon of Pascal

Inserting $\theta = 1,\ 1/3,\ 3$ gives (A.8), (A.9), (A.10).

Note that the small loop around –1 is not included in the domain of convergence, corresponding nicely to the fact that arctan has a singularity at $\pm i$.

Example 16. Letting

$$P(x) = -\frac{\theta^3 x(1-x)^2}{(\theta+1)^2}$$

gives the formidable expression

$$\frac{\arctan\sqrt{\theta}}{\sqrt{\theta}} = \frac{1}{9(\theta+1)^2}\sum_{n=0}^{\infty}\left(\frac{-16\theta^3}{(\theta+1)^2}\right)^n \times \frac{(2n)!(2n)!(3n)!}{n!(6n)!}\left(\frac{5\theta^2+15\theta+9}{6n+1} - \frac{\theta^2}{6n+5}\right)$$

with $M = 4\theta^3/27(\theta+1)^2$.

The domain of convergence is bounded by the sextic, limaçon-like curve

$$16(x^2+y^2)^3 = 729((x+1)^2+y^2)^2.$$

Inserting $\theta = 1,\ 1/3,\ 3$ gives (A.11), (A.12), (A.13).

([1]) I am grateful to my friend Kasper K. S. Andersen for identifying this curve.

Formulas (A.8) and (A.11) are examples of *van Wijngaarden's transformation* [7], i.e., they are the accelerations of the Gregory–Leibniz series corresponding to the polynomials

$$P(x) = -\frac{x(1-x)}{2}, \quad -\frac{x(1-x)^2}{4}.$$

Example 17. For $k = 2$ and $a_i = 1/(i+1)$, the general arctan series (27) cannot be accelerated as in the previous examples. It may, however, for specific choices of θ. Let $\theta = 1/3$. Then we must have $P(\pm i\sqrt{3})$ and can take any product of

$$-\frac{x^2}{3}, \qquad -\frac{(1-x)^3}{8}.$$

Letting

$$P(x) = -\frac{(1-x)^3}{8}, \quad \frac{x^2(1-x)^3}{24}$$

gives the series (A.14), (A.15) with $M = 1/8$, $9/6250$, respectively.

Example 18. The convergence of the accelerated series (16), and its identity with (26), was proved under Hausdorff's condition (18) and $0 < \theta < 1$. It is a common phenomenon, however, that acceleration techniques work in more general settings and even for divergent series [7, Remark 6]. Consider the divergent series (6), obtained by inserting $\theta = 3$ into (27). Let $k = 2$ and $\theta = 3$. Then $P(x)$ must satisfy

$$P\left(\frac{\pm i}{\sqrt{3}}\right) = 1.$$

We can take for $P(x)$ any product of

$$-3x^2, \qquad \frac{9x(1-x)^3}{8}, \qquad -\frac{27(1-x)^6}{64}.$$

Letting $P(x)$ be

$$\frac{9x(1-x)^3}{8}, \qquad -\frac{27x^3(1-x)^3}{8}, \qquad -\frac{27(1-x)^6}{64}$$

gives the three series (A.16), (A.17), (A.18) with $M = 243/2048$, $27/512$, $27/64$, respectively.

These formulas can be checked numerically to many digits, but of course the above argument is no proof (although I like to think that Euler would have appreciated it).

Remark 19. A quick, rigorous proof of (A.18) could go as follows. Write Mercator's Formula with six terms at a time,

$$-\log(1-z) = \sum_{n=0}^{\infty} z^{6n}\left(\frac{z}{6n+1} + \cdots + \frac{z^6}{6n+6}\right).$$

Insert $z = e^{i\pi/6}\sqrt{3}/2$ and take imaginary parts to get (A.18), q.e.d.

Similar proofs of (A.2) and (A.14) are possible: Insert $z = (1+i)/2$ and $z = e^{i\pi/3}/2$ into Mercator's Formula with four and three terms at a time, respectively.

6. Numerical approximations again. To approximate the geometrically converging series (26) numerically, let $k \geq 1$ be given, but now let $Q(x)$ agree with $1/(1+\theta x^k)$ at the points $(1+\eta_i)/2$, i.e.,

$$Q(x) \equiv \frac{1}{1+\theta x^k} \text{ modulo } T_m(1-2x).$$

Then

$$P(x) = T_m(1-2x) \sum_{u^k=-1} \frac{u}{\beta_m k} \cdot \frac{1+\theta x^k}{u-\theta^{1/k} x}$$

with $\beta_m = T_m(1-2u\theta^{-1/k})$. Again, $\beta_m \sim |\lambda_1|^m/2$ with λ_1 the numerically greater root of the characteristic polynomial $\lambda^2 - (2-4u\theta^{-1/k})\lambda + 1$. We conclude that $M = O(\lambda^{-m})$ as $m \to \infty$ with

$$\lambda = \min\{|\lambda_1| : u^k = -1\}.$$

Table 2 gives $\lambda = \lambda_\theta$ for various values of k and θ.

Table 2. Values of λ_θ

k	$\lambda_{1/2}$	$\lambda_{1/3}$	$\lambda_{1/4}$	$\lambda_{1/5}$	$\lambda_{1/6}$
1	9.899	13.93	17.94	21.95	25.96
2	6.129	7.328	8.352	9.263	10.09
3	4.607	5.254	5.782	6.236	6.636
4	3.829	4.264	4.612	4.905	5.160
5	3.357	3.685	3.942	4.157	4.343
6	3.040	3.303	3.508	3.678	3.823
7	2.811	3.031	3.202	3.342	3.462
8	2.636	2.827	2.973	3.094	3.196
9	2.499	2.667	2.796	2.902	2.991
10	2.388	2.539	2.654	2.748	2.828

EXAMPLE 20. We return to the computation of the constant K from Example 10. Write

$$K = \frac{\pi}{4} - 1 + \zeta\left(\frac{1}{2}\right) - \sum_{i=0}^{\infty} (-1)^i \frac{\zeta(i+3/2)-1}{2i+3}$$

to get a geometrically converging series, with $\theta = 1/2$. Suppose again the zeta-values $\zeta(i/2+3/2)$ are given for $i=0,1,\ldots,99$. Using only the terms $a_0, a_2, a_4, \ldots, a_{98}$, we expect an error of 9.899^{-50}, or 50 correct digits (cf. Table 2). Using all 100 available terms, we expect an error of 6.129^{-100}, or 79 correct digits. In practice, the two methods give 53 and 82 digits, respectively, confirming the theory. The second method can be carried out in Pari as follows:

```
T=subst(poltchebi(100),x,1-2*x)
Q=lift(1/Mod(1+x^2/2,T))
c(i)=polcoeff(Q,i)
a(i)=2^(i/2)*(zeta(i/2+3/2)-1)/(i+3)
K=Pi/4-1+zeta(1/2)-sum(i=0,99,c(i)*a(i))
```

Appendix. Series for π

(A.1) $$\frac{3\pi}{2} = \sum_{n=0}^{\infty} \frac{1}{2^n} \binom{3n}{n}^{-1} \left(\frac{4}{3n+1} + \frac{1}{3n+2} \right),$$

(A.2) $$\pi = \sum_{n=0}^{\infty} \left(\frac{-1}{4} \right)^n \left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3} \right),$$

(A.3) $$\frac{125\pi}{2} = \sum_{n=0}^{\infty} \left(\frac{-1}{2} \right)^n \binom{5n}{2n}^{-1} \times \left(\frac{208}{5n+1} - \frac{22}{5n+2} + \frac{8}{5n+3} - \frac{7}{5n+4} \right),$$

(A.4) $$1024\pi = \sum_{n=0}^{\infty} \left(\frac{-1}{4} \right)^n \binom{8n}{4n}^{-1} \times \left(\frac{3183}{8n+1} + \frac{117}{8n+3} - \frac{15}{8n+5} - \frac{5}{8n+7} \right),$$

(A.5) $$\frac{9\pi}{\sqrt{2}} = \sum_{n=0}^{\infty} 8^n \frac{(2n)!(2n)!(3n)!}{n!(6n)!} \left(\frac{19}{6n+1} + \frac{1}{6n+5} \right),$$

(A.6) $$16\sqrt{2}\pi = \sum_{n=0}^{\infty} (-64)^n \binom{8n}{4n}^{-1} \times \left(\frac{75}{8n+1} + \frac{13}{8n+3} - \frac{3}{8n+5} - \frac{5}{8n+7} \right),$$

(A.7) $$\frac{625\pi}{\sqrt{2}} = \sum_{n=0}^{\infty} (-8)^n \frac{(2n)!(5n)!(6n)!}{(3n)!(10n)!} \times \left(\frac{1339}{10n+1} + \frac{184}{10n+3} - \frac{16}{10n+7} - \frac{11}{10n+9} \right),$$

$$\text{(A.8)}\qquad 2\pi = \sum_{n=0}^{\infty}(-2)^n\binom{4n}{2n}^{-1}\left(\frac{7}{4n+1}-\frac{1}{4n+3}\right),$$

$$\text{(A.9)}\qquad \frac{8\pi}{\sqrt{3}} = \sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n\binom{4n}{2n}^{-1}\left(\frac{15}{4n+1}-\frac{1}{4n+3}\right),$$

$$\text{(A.10)}\qquad \frac{16\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}(-9)^n\binom{4n}{2n}^{-1}\left(\frac{13}{4n+1}-\frac{3}{4n+3}\right),$$

$$\text{(A.11)}\qquad 9\pi = \sum_{n=0}^{\infty}(-4)^n\frac{(2n)!(2n)!(3n)!}{n!(6n)!}\left(\frac{29}{6n+1}-\frac{1}{6n+5}\right),$$

$$\text{(A.12)}\qquad 24\sqrt{3}\pi = \sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n\frac{(2n)!(2n)!(3n)!}{n!(6n)!}\left(\frac{131}{6n+1}-\frac{1}{6n+5}\right),$$

$$\text{(A.13)}\qquad \frac{16\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}(-27)^n\frac{(2n)!(2n)!(3n)!}{n!(6n)!}\left(\frac{11}{6n+1}-\frac{1}{6n+5}\right),$$

$$\text{(A.14)}\qquad \frac{4\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}\left(\frac{-1}{8}\right)^n\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right),$$

$$\text{(A.15)}\qquad \frac{500\pi}{\sqrt{3}} = \sum_{n=0}^{\infty}\frac{1}{24^n}\binom{5n}{2n}^{-1}\left(\frac{872}{5n+1}+\frac{57}{5n+2}+\frac{12}{5n+3}+\frac{7}{5n+4}\right),$$

$$\text{(A.16)}\qquad \frac{256\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}\left(\frac{9}{8}\right)^n\binom{4n}{n}^{-1}\left(\frac{103}{4n+1}+\frac{72}{4n+2}+\frac{15}{4n+3}\right),$$

$$\text{(A.17)}\qquad \frac{1024\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}\left(\frac{-27}{8}\right)^n\binom{6n}{3n}^{-1}\left(\frac{637}{6n+1}+\frac{6}{6n+3}-\frac{27}{6n+5}\right),$$

$$\text{(A.18)}\qquad \frac{64\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}\left(\frac{-27}{64}\right)^n \times\left(\frac{16}{6n+1}+\frac{24}{6n+2}+\frac{24}{6n+3}+\frac{18}{6n+4}+\frac{9}{6n+5}\right).$$

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