ACTA ARITHMETICA
171.4 (2015)

Some q-supercongruences for truncated basic hypergeometric series

by

VICTOR J. W. GUO (Huaian) and JIANG ZENG (Lyon)

1. Introduction. We shall follow the standard q-notation from [4]. The q-shifted factorial is defined by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n = 1, 2, \ldots$, and $(a;q)_0 = 1$, while the *q-integer* is denoted by $[n] := \frac{1-q^n}{1-q}$. In a previous paper [6], we proposed several q-analogues of Rodriguez-Villegas and Mortenson type congruences for truncated hypergeometric series conjectured by Rodriguez-Villegas [11, 9], and proved the following q-analogue of one of their supercongruences:

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv \left(\frac{-1}{p}\right) q^{(1-p^2)/4} \pmod{[p]^2}. \tag{1.1}$$

Here and in what follows, p always denotes an odd prime, and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p.

Recently, by using the properties of generalized Legendre polynomials, Z.-H. Sun [13, Theorem 2.5] proved the following remarkable congruence:

$$\sum_{k=0}^{p-1} \binom{2k}{k}\binom{a}{k}\binom{-1-a}{k}\frac{1}{4^k} \equiv 0 \pmod{p^2}, \tag{1.2}$$

where a is a p-adic integer such that the least nonnegative residue of a modulo p is odd. It is interesting to note that (1.2) is a common generalization of several congruences due to van Hamme and Rodriguez-Villegas. On the other hand, van Hamme [17] proved the following congruence:

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 \frac{1}{64^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \text{ with } x \text{ odd}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

2010 *Mathematics Subject Classification*: Primary 11B65; Secondary 05A10, 05A30.
Key words and phrases: p-adic integer, congruences, supercongruences, basic hypergeometric series, q-binomial theorem, q-Chu–Vandermonde formula.

DOI: 10.4064/aa171-4-2 © Instytut Matematyczny PAN, 2015

a generalization of which was recently conjectured by H. Swisher [16, (H.3)]. The aim of this paper is to prove some q-supercongruences for certain truncated basic hypergeometric series generalizing the above results.

Recall that the *q-binomial coefficients* $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q^{n-k+1};q)_k}{(q;q)_k} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

The first aim of this paper is to give a unified q-analogue of (1.3) and Z.-W. Sun's generalization [14, Theorem 1.1(i)] of (1.3).

Theorem 1.1. *Let $0 \le s \le (p-1)/2$. Then modulo $[p]^2$,*

$$\sum_{k=0}^{(p-1)/2} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}^2 \begin{bmatrix} 2k \\ k+s \end{bmatrix}_{q^2} \frac{q^{2k}}{(-q^2;q^2)_k^2(-q;q)_{2k}} \tag{1.4}$$

$$\equiv \begin{cases} (-1)^s q^{(p-1)/2-s^2} \begin{bmatrix} (p-1)/2 \\ (p-2s-1)/4 \end{bmatrix}_{q^4}^2 \dfrac{(q^2;q^2)_{(p-2s-1)/2}(q^2;q^2)_{(p+2s-1)/2}}{(q^4;q^4)_{(p-1)/2}^2} & \\ \hfill \text{if } s \equiv \frac{p-1}{2} \pmod 2, \\ 0 \hfill \text{otherwise.} \end{cases}$$

When $s = 0$, the congruence (1.4) reduces to the following result.

Corollary 1.2. *Modulo $[p]^2$,*

$$\sum_{k=0}^{(p-1)/2} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}^3 \frac{q^{2k}}{(-q^2;q^2)_k^2(-q;q)_{2k}} \tag{1.5}$$

$$\equiv \begin{cases} q^{(p-1)/2} \begin{bmatrix} (p-1)/2 \\ (p-1)/4 \end{bmatrix}_{q^4}^2 \dfrac{1}{(-q^2;q^2)_{(p-1)/2}^2} & \text{if } p \equiv 1 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

To see that (1.5) is a q-analogue of (1.3) we need to recall a known result. Let p be a prime such that $p \equiv 1 \pmod 4$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$. Then we have the so-called Beukers–Chowla–Dwork–Evans congruence [3, 10]

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2},$$

which easily implies Sun's congruence [12, Lemma 3.4]

$$\binom{(p-1)/2}{(p-1)/4}^2 \frac{1}{2^{p-1}} \equiv 4x^2 - 2p \pmod{p^2}. \tag{1.6}$$

Using (1.6), it is clear that (1.5) reduces to (1.3) when $q \to 1$.

Generalizing a result of Mortenson [7, 8], Z.-W. Sun [15, (1.4)] obtained the congruence

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{2k+2s}{k+s}}{4^{2k+s}} \equiv \left(\frac{-1}{p}\right) \pmod{p^2} \quad \text{for } 0 \le s \le \frac{p-1}{2}. \tag{1.7}$$

For any p-adic integer x, let $\langle x\rangle_p$ denote the least nonnegative residue of x modulo p. The second aim of this paper is to give a unified q-analogue of (1.2) and (1.7).

THEOREM 1.3. *Let m and r be two positive integers with $p\nmid m$. Let $s \le \min\{\langle -r/m\rangle_p, \langle -(m-r)/m\rangle_p\}$ be a nonnegative integer. If $\langle -\frac{r}{m}\rangle_p \equiv s+1 \pmod 2$, then*

$$\sum_{k=s}^{(p-1)/2} \frac{(q^m;q^m)_{2k}(q^r;q^m)_k(q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}(q^{2m};q^{2m})_k^2} \equiv 0 \pmod{[p]^2}, \tag{1.8}$$

$$\sum_{k=s}^{p-1} \frac{(q^m;q^m)_{2k}(q^r;q^m)_k(q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}(q^{2m};q^{2m})_k^2} \equiv 0 \pmod{[p]^2}. \tag{1.9}$$

If $\langle -r/m\rangle_p \equiv s \pmod 2$, then

$$\begin{aligned}\sum_{k=s}^{(p-1)/2} &\frac{(q^m;q^m)_{2k}(q^r;q^m)_k(q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}(q^{2m};q^{2m})_k^2} \\ &\equiv q^{(s+\frac{p-1}{2})m}(q^m;q^{2m})_{\frac{\langle -r/m\rangle_p - s}{2}}(q^m;q^{2m})_{\frac{\langle -(m-r)/m\rangle_p - s}{2}} \\ &\quad\times \frac{(q^{-m\langle -r/m\rangle_p};q^m)_s(q^{-m\langle -(m-r)/m\rangle_p};q^m)_s}{(q^{2m};q^{2m})_{\frac{\langle -r/m\rangle_p + s}{2}}(q^{2m};q^{2m})_{\frac{\langle -(m-r)/m\rangle_p + s}{2}}} \pmod{[p]}.\end{aligned} \tag{1.10}$$

Letting $s=0$, $-r/m=a$ and $q\to 1$ in (1.9), we obtain (1.2). On the other hand, it is not difficult to see that (see [6]), for any prime $p\ge 5$,

$$(-1)^{\langle -1/3\rangle_p} = \left(\frac{-3}{p}\right), \quad (-1)^{\langle -1/4\rangle_p} = \left(\frac{-2}{p}\right), \quad (-1)^{\langle -1/6\rangle_p} = \left(\frac{-1}{p}\right).$$

Taking $r=1$ and $m=3,4,6$ in (1.8), we obtain

COROLLARY 1.4. *Let $p\ge 5$ be a prime and let s be a nonnegative integer. Then the following congruences hold modulo $[p]^2$:*

$$\sum_{k=s}^{(p-1)/2} {2k \brack k+s}_{q^3} \frac{(q;q^3)_k(q^2;q^3)_k q^{3k}}{(q^6;q^6)_k^2} \equiv 0 \text{ if } s \le \tfrac{p-1}{3} \text{ and } s \equiv \tfrac{1+(\frac{-3}{p})}{2} \pmod 2,$$

$$\sum_{k=s}^{(p-1)/2} {2k \brack k+s}_{q^4} \frac{(q;q^4)_k (q^3;q^4)_k q^{4k}}{(q^8;q^8)_k^2} \equiv 0 \text{ if } s \le \tfrac{p-1}{4} \text{ and } s \equiv \tfrac{1+(\frac{-2}{p})}{2} \pmod 2,$$

$$\sum_{k=s}^{(p-1)/2} {2k \brack k+s}_{q^6} \frac{(q;q^6)_k (q^5;q^6)_k q^{6k}}{(q^{12};q^{12})_k^2} \equiv 0 \text{ if } s \le \tfrac{p-1}{6} \text{ and } s \equiv \tfrac{1+(\frac{-1}{p})}{2} \pmod 2.$$

The proof of Theorem 1.3 is based on the following q-Clausen type summation formula, which seems to be new and interesting in its own right.

THEOREM 1.5. *Let n and s be nonnegative integers with $s \le n$. Then*

$$\left(\sum_{k=s}^{n} \frac{(q^{-2n};q^2)_k (x;q)_k q^k}{(q;q)_{k-s}(q;q)_{k+s}}\right)\left(\sum_{k=s}^{n} \frac{(q^{-2n};q^2)_k (q/x;q)_k q^k}{(q;q)_{k-s}(q;q)_{k+s}}\right) \tag{1.11}$$
$$= \frac{(-1)^n (q^2;q^2)_n^2 q^{-n^2}}{(q^2;q^2)_{n-s}(q^2;q^2)_{n+s}} \sum_{k=s}^{n} \frac{(-1)^k (q^2;q^2)_{n+k} (x;q)_k (q/x;q)_k q^{k^2-2nk}}{(q^2;q^2)_{n-k}(q;q)_{k-s}(q;q)_{k+s}(q;q)_{2k}}.$$

We also have the following q-analogue of (1.7), which reduces to (1.1) when $s=0$.

THEOREM 1.6. *Let p be an odd prime and let $0 \le s \le (p-1)/2$. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{(q;q^2)_k (q;q^2)_{k+s}}{(q^2;q^2)_k (q^2;q^2)_{k+s}} \equiv \left(\frac{-1}{p}\right) q^{(1-p^2)/4} \pmod{[p]^2}.$$

Finally, we shall prove the following result.

THEOREM 1.7. *Let p be an odd prime and let m, r be positive integers with $p \nmid m$ and $r < m$. Then for any integer s with $0 \le s \le \langle -(m-r)/m\rangle_p$,*

$$\sum_{k=0}^{p-s-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_{k+s}}{(q^m;q^m)_k (q^m;q^m)_{k+s}} \tag{1.12}$$
$$\equiv (-1)^{\langle -r/m\rangle_p} q^{-m\langle -r/m\rangle_p (\langle -r/m\rangle_p+1)/2} \pmod{[p]}.$$

In particular, if $p \equiv \pm 1 \pmod m$, then

$$\sum_{k=0}^{p-s-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_{k+s}}{(q^m;q^m)_k (q^m;q^m)_{k+s}} \equiv (-1)^{\langle -r/m\rangle_p} q^{\frac{r(m-r)(1-p^2)}{2m}} \pmod{[p]}. \tag{1.13}$$

Throughout the paper we will often use the fact that for any prime p, the q-integer $[p]$ is always an irreducible polynomial in $\mathbb{Q}[q]$. Hence, $\mathbb{Q}[q]/[p]$ is a field. Therefore, rational functions $a(q)/b(q)$ are well defined modulo $[p]$ or $[p]^r$ $(r \ge 1)$ on condition that $b(q)$ is relatively prime to $[p]$.

The rest of this paper is organized as follows. In Sections 2–5 we prove Theorem 1.1, Theorem 1.5, Theorem 1.3, and Theorems 1.6 and 1.7, respectively. We conclude the paper with some open problems.

2. Proof of Theorem 1.1. We first establish two lemmas.

LEMMA 2.1. *Let $0 \le k \le (p-1)/2$. Then*

$$\begin{bmatrix} (p-1)/2+k \\ 2k \end{bmatrix}_{q^2} \equiv (-1)^k \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \frac{q^{kp-k^2}}{(-q;q)_{2k}^2} \pmod{[p]^2}. \tag{2.1}$$

Proof. Since

$$(1-q^{p-2j+1})(1-q^{p+2j-1}) + (1-q^{2j-1})^2 q^{p-2j+1} = (1-q^p)^2,$$

we have

$$(1-q^{p-2j+1})(1-q^{p+2j-1}) \equiv -(1-q^{2j-1})^2 q^{p-2j+1} \pmod{[p]^2}.$$

It follows that

$$\begin{aligned} \begin{bmatrix} (p-1)/2+k \\ 2k \end{bmatrix}_{q^2} &= \frac{\prod_{j=1}^k (1-q^{p-2j+1})(1-q^{p+2j-1})}{(q^2;q^2)_{2k}} \\ &\equiv (-1)^k \frac{\prod_{j=1}^k (1-q^{2j-1})^2 q^{p-2j+1}}{(q^2;q^2)_{2k}} \\ &= (-1)^k \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \frac{q^{kp-k^2}}{(-q;q)_{2k}^2} \pmod{[p]^2}. \ \blacksquare \end{aligned}$$

LEMMA 2.2. *Let n and s be nonnegative integers with $s \le n$. Then*

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} 2k \\ k \end{bmatrix} \begin{bmatrix} 2k \\ k+s \end{bmatrix} \frac{(-1)^k q^{\binom{n-k}{2}}}{(-q;q)_k^2} \tag{2.2}$$
$$= \begin{cases} (-1)^s q^{(n^2-s^2)/2} \begin{bmatrix} n \\ (n-s)/2 \end{bmatrix}_{q^2}^2 \dfrac{(q;q)_{n-s}(q;q)_{n+s}}{(q^2;q^2)_n^2} & \text{if } n \equiv s \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We may rewrite the left-hand side of (2.2) as

$$\sum_{k=s}^{n} \frac{(q^{-n};q)_k (q^{n+1};q)_k (q^{1/2};q)_k (-q^{1/2};q)_k q^{k+\binom{n}{2}}}{(q;q)_k (q;q)_{k-s} (q;q)_{k+s} (-q;q)_k}$$
$$= \frac{(q^{-n};q)_s (q^{n+1};q)_s (q;q^2)_{2s} q^{s+\binom{n}{2}}}{(q^2;q^2)_s (q;q)_{2s}} {}_4\phi_3 \left[\begin{matrix} q^{s-n},\ q^{n+s+1},\ q^{s+1/2},\ -q^{s+1/2} \\ q^{s+1},\ -q^{s+1},\ q^{2s+1} \end{matrix} ; q, q \right].$$

The result then follows from Andrews' terminating q-analogue of Watson's formula [4, (II.17)]:

$$ {}_4\phi_3 \left[\begin{matrix} q^{-n},\ a^2 q^{n+1},\ b,\ -b \\ aq,\ -aq,\ b^2 \end{matrix} ; q, q \right] = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ \dfrac{b^n (q, a^2q^2/b^2; q^2)_{n/2}}{(a^2q^2, b^2q; q^2)_{n/2}} & \text{if } n \text{ is even}, \end{cases} \tag{2.3}$$

with the substitution of n, a and b by $n-s$, q^s and $q^{s+1/2}$, respectively. $\blacksquare$

Proof of Theorem 1.1. By the congruence (2.1), we have

$$\sum_{k=0}^{(p-1)/2} {2k \brack k}_{q^2}^2 {2k \brack k+s}_{q^2} \frac{q^{2k}}{(-q^2;q^2)_k^2(-q;q)_{2k}^2}$$
$$\equiv \sum_{k=0}^{(p-1)/2} {(p-1)/2+k \brack 2k}_{q^2} {2k \brack k}_{q^2} {2k \brack k+s}_{q^2} \frac{(-1)^k}{(-q^2;q^2)_k^2} q^{k^2+2k-pk} \pmod{[p]^2}.$$

The conclusion then follows from (2.2) with $n=(p-1)/2$ and $q\to q^2$. ∎

3. Proof of Theorem 1.5. We first establish four lemmas to make the proof easier.

Lemma 3.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n}(-1)^{n-k}{n \brack k}\frac{(aq^n;q)_k q^{\binom{n-k+1}{2}}}{(a;q)_k(1-xq^{-k})} = \frac{(ax;q)_n(q;q)_n}{(a;q)_n(xq^{-n};q)_{n+1}}, \tag{3.1}$$

$$\sum_{k=0}^{n}(-1)^{n-k}{n \brack k}q^{\binom{k}{2}}\frac{(q;q)_{n-k}}{(x;q)_{n-k+1}} = \frac{q^{\binom{n+1}{2}}}{1-xq^n}. \tag{3.2}$$

Proof. For (3.1), by partial fraction decomposition we have

$$\frac{(ax;q)_n(q;q)_n}{(a;q)_n(xq^{-n};q)_{n+1}} = \sum_{k=0}^{n}\frac{a_k}{1-xq^{-k}} \tag{3.3}$$

with

$$a_k = \lim_{x\to q^k}\frac{(1-xq^{-k})(ax;q)_n(q;q)_n}{(a;q)_n(xq^{-n};q)_{n+1}} = (-1)^{n-k}q^{\binom{n-k+1}{2}}{n \brack k}\frac{(aq^n;q)_k}{(a;q)_k}.$$

By the Gauss or q-binomial inversion (see, for example, [1, p. 77, Exercise 2.47]), the identity (3.2) is equivalent to

$$\frac{(q;q)_n}{(x;q)_{n+1}} = \sum_{k=0}^{n}{n \brack k}_q(-1)^k q^{\binom{k+1}{2}}\frac{1}{1-xq^k},$$

which corresponds to the $a=0$ case of (3.3) with $x\to xq^n$. ∎

Lemma 3.2. *Let n be a positive integer. Then*

$$(x;q)_n+(a/x;q)_n = (x;q)_n(a/x;q)_n + \sum_{k=0}^{n-1}\frac{(x;q)_k(a/x;q)_k(1-q^n)}{(q;q)_k(1-q^{n-k})}\sum_{j=0}^{k}(-1)^j{k \brack j}q^{\binom{j}{2}}(aq^{k+j};q)_{n-k}, \tag{3.4}$$

$$(x;q)_n+(a/x;q)_n = (x;q)_n(a/x;q)_n + (a;q)_n + \sum_{k=1}^{n-1}(x;q)_k(a/x;q)_k(1-q^n)\sum_{j=1}^{n-k}(-1)^j{n-k-1 \brack j-1}{k+j-1 \brack j-1}\frac{q^{\binom{j}{2}+kj}a^j}{1-q^j}. \tag{3.5}$$

Proof. We first prove (3.4). Taking $x=q^{-m}$ $(0\le m\le n-1)$, we have

$$
\begin{aligned}
(3.6)\qquad & \sum_{k=0}^{n-1}\frac{(x;q)_k(a/x;q)_k(1-q^n)}{(q;q)_k(1-q^{n-k})}\sum_{j=0}^{k}(-1)^j\begin{bmatrix}k\\ j\end{bmatrix}q^{\binom{j}{2}}(aq^{k+j};q)_{n-k}\\
&=\sum_{k=0}^{n-1}\frac{(q^{-m};q)_k(aq^m;q)_k(1-q^n)}{(q;q)_k(1-q^{n-k})}\sum_{j=0}^{k}(-1)^j\begin{bmatrix}k\\ j\end{bmatrix}q^{\binom{j}{2}}(aq^{k+j};q)_{n-k}\\
&=\sum_{k=0}^{m}(-1)^k\begin{bmatrix}m\\ k\end{bmatrix}q^{\binom{k}{2}-mk}\frac{(aq^m;q)_k(1-q^n)}{(1-q^{n-k})}\sum_{j=0}^{k}(-1)^j\begin{bmatrix}k\\ j\end{bmatrix}q^{\binom{j}{2}}\frac{(aq^j;q)_n}{(aq^j;q)_k}\\
&=\sum_{j=0}^{m}(-1)^j\begin{bmatrix}m\\ j\end{bmatrix}q^{\binom{j}{2}}(aq^j;q)_n\\
&\quad\times\sum_{k=j}^{m}(-1)^k\begin{bmatrix}m-j\\ k-j\end{bmatrix}q^{\binom{k}{2}-mk}\frac{(aq^m;q)_k(1-q^n)}{(aq^j;q)_k(1-q^{n-k})}.
\end{aligned}
$$

It follows from (3.1) that

$$
\begin{aligned}
&\sum_{k=j}^{m}(-1)^k\begin{bmatrix}m-j\\ k-j\end{bmatrix}q^{\binom{k}{2}-mk}\frac{(aq^m;q)_k(1-q^n)}{(aq^j;q)_k(1-q^{n-k})}\\
&\qquad=\frac{(aq^m;q)_j}{(aq^j;q)_j}\sum_{k=j}^{m}(-1)^k\begin{bmatrix}m-j\\ k-j\end{bmatrix}q^{\binom{m-k+1}{2}-\binom{m+1}{2}}\frac{(aq^{m+j};q)_{k-j}(1-q^n)}{(aq^{2j};q)_{k-j}(1-q^{n-k})}\\
&\qquad=\frac{(-1)^m(aq^m;q)_j(aq^{n+j};q)_{m-j}(q;q)_{m-j}(1-q^n)q^{-\binom{m+1}{2}}}{(aq^j;q)_j(aq^{2j};q)_{m-j}(q^{n-m};q)_{m-j+1}}\\
&\qquad=\frac{(-1)^m(a;q)_j(aq^{n+j};q)_{m-j}(q;q)_{m-j}(1-q^n)q^{-\binom{m+1}{2}}}{(a;q)_m(q^{n-m};q)_{m-j+1}}.
\end{aligned}
$$

Therefore, the right-hand side of (3.6) can be simplified as

$$
\begin{aligned}
(3.7)\qquad & \frac{(a;q)_{m+n}(1-q^n)q^{-\binom{m+1}{2}}}{(a;q)_m}\sum_{j=0}^{m}(-1)^{m-j}\begin{bmatrix}m\\ j\end{bmatrix}q^{\binom{j}{2}}\frac{(q;q)_{m-j}}{(q^{n-m};q)_{m-j+1}}\\
&=(aq^m;q)_n,
\end{aligned}
$$

where the equality follows from (3.2). Noticing that $(q^{-m};q)_n=0$ for $0\le m\le n-1$, we have proved that both sides of (3.4) are equal for $x=q^{-m}$ $(0\le m\le n-1)$, and by symmetry, for $x=aq^m$ $(0\le m\le n-1)$ too. Furthermore, both sides of (3.4) are of the form $x^{-n}P(x)$ with $P(x)$ being a polynomial in x of degree $2n$ with leading coefficient $(-1)^nq^{\binom{n}{2}}$. Hence, they must be identical. This proves (3.4).

By the q-binomial theorem (see, for example, [2, Theorem 3.3]), for $k \geq 1$,

$$(3.8)\qquad \sum_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}}(aq^{k+j};q)_{n-k}$$
$$= \sum_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} \sum_{i=0}^{n-k}(-1)^i \begin{bmatrix} n-k \\ i \end{bmatrix} q^{\binom{i}{2}+(k+j)i} a^i$$
$$= \sum_{i=0}^{n-k}(-1)^i \begin{bmatrix} n-k \\ i \end{bmatrix} q^{\binom{i}{2}+ik} a^i \sum_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}+ij}$$
$$= \sum_{i=1}^{n-k}(-1)^i \begin{bmatrix} n-k \\ i \end{bmatrix} (q^i;q)_k q^{\binom{i}{2}+ik} a^i.$$

Moreover, for $k = 0$, the left-hand side of (3.8) is clearly equal to $(a;q)_n$. Noticing that

$$\begin{bmatrix} n-k \\ i \end{bmatrix} \frac{(q^i;q)_k}{(q;q)_k(1-q^{n-k})} = \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k+i-1 \\ i-1 \end{bmatrix} \frac{1}{1-q^i},$$

we complete the proof of (3.5). ∎

Let n and h be positive integers and let m and s be nonnegative integers such that $s \leq m$ and $h \leq n-m$ (so $n > m$). Let

$$f(x;j,k) := (x;q)_j(x;q)_k(q^{-n};q)_j(q^{-n};q)_k$$
$$\times \frac{(q^{j-m-h+1};q)_{h-1}(q^{k-m-h+1};q)_{h-1}(q^{2j+k}-q^{2k+j})}{(-1)^{m-s-1}(q;q)_n^2(q;q)_{h-1}(q;q)_{j-s}(q;q)_{j+s}(q;q)_{k-s}(q;q)_{k+s}},$$

and, for integers $a, b \geq s$, let

$$L_{a,b}(x) := \sum_{j=s}^{a}\sum_{k=s}^{b} f(x;j,k).$$

Lemma 3.3. *Let* $A = (m^2+3m-s^2+s)/2 - m(n+h) - h^2 + h$. *Then*

$$(3.9)\qquad L_{m,n}(x) = \frac{(x;q)_s(x;q)_{m+h}(q^{s+1}/x;q)_{n-s-h}x^{n-s-h}q^A}{(q;q)_{m-s}(q;q)_{m+s}(q;q)_{n-s}(q;q)_{n+s}(q;q)_{n-m-h}}.$$

Proof. Without loss of generality, we assume that q is a complex number with $|q| < 1$. We first note that $f(x;j,k) = -f(x;k,j)$, and so $L_{m,m}(x) = 0$. Since both sides of (3.9) are polynomials in x of degree $m+n$ with the same leading coefficient, it suffices to show that these two polynomials have $m+n$ common roots, counted with multiplicity.

We proceed by dividing the roots of the right-hand side of (3.9) into four cases:

• If $s \ge 1$, then it is easily seen that $(x;q)_s^2$ divides $L_{m,n}(x)$, which means that the numbers q^{-r} $(0 \le r \le s-1)$ are roots of $L_{m,n}(x)$ with multiplicity 2.

• For r with $s \le r \le m$, we have $(q^{-r};q)_k = 0$ if $k > m$, and so $L_{m,n}(q^{-r}) = L_{m,m}(q^{-r}) = 0$.

• For r with $m+1 \le r \le m+h-1$, we have $r-m-h+1 \le 0$, and so $(q^{k-m-h+1};q)_{h-1} = 0$ for $m+1 \le k \le r$, while for $r < k \le n$ we have $(q^{-r};q)_k = 0$. Hence, we again get $L_{m,n}(q^{-r}) = L_{m,m}(q^{-r}) = 0$.

• For r with $s+1 \le r \le n-h$, it is clear that $L_{m,n}(q^r) = 0$ follows from the identity

$$\sum_{k=s}^{n} \frac{(q^{-n};q)_k (q^r;q)_k (q^{k-m-h+1};q)_{h-1}(1-q^{k-j})q^{2j+k}}{(q;q)_{k-s}(q;q)_{k+s}} = 0, \tag{3.10}$$

which is proved as follows: The left-hand side of (3.10) can be written as

$$\begin{aligned}(q^{-n};q)_s(q^r;q)_s \\ \times \sum_{k=s}^{n} \frac{(q^{-n+s};q)_{k-s}(q^{r+s};q)_{k-s}(q^{k-m-h+1};q)_{h-1}(1-q^{k-j})q^{2j+k}}{(q;q)_{k-s}(q;q)_{k+s}} \\ = (q^{-n};q)_s(q^r;q)_s \sum_{k=0}^{n-s}(-1)^k \begin{bmatrix} n-s \\ k \end{bmatrix} q^{-(n-s)k+\binom{k}{2}} R_k,\end{aligned} \tag{3.11}$$

where

$$R_k = \frac{(q^{r+s};q)_k (q^{k+s-m-h+1};q)_{h-1}(1-q^{k+s-j})q^{2j+k+s}}{(q;q)_{k+2s}}.$$

Since

$$\frac{(q^{r+s};q)_k}{(q;q)_{k+2s}} = \frac{(q^{k+2s+1};q)_{r-s-1}}{(q;q)_{r+s-1}},$$

we see that R_k is a polynomial in q^k of degree $r-s-1+h-1+2 \le n-s$ with constant term 0. By the q-binomial theorem (see [2, Theorem 3.3]),

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} x^k = (xq;q)_n,$$

and we have

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}-ik} = \begin{cases} 0 & \text{for } 1 \le i \le n, \\ (q;q)_n & \text{for } i = 0. \end{cases} \tag{3.12}$$

It follows that the right-hand side of (3.11) is equal to 0. Hence, the identity (3.10) holds.

Thus, we have found all the $m+n$ roots of $L_{m,n}(x)$, which are clearly the same as those of the right-hand side of (3.9). ∎

Remark. We can use the identity (3.12) to give a short proof of Jackson's terminating q-analogue of Dixon's identity (see [5]).

Lemma 3.4. *Let n and h be positive integers and let m and s be non-negative integers with $h \le n-m$ and $s \le m$. Then*

$$\begin{aligned}(3.13)\qquad &\sum_{j=s}^{m}\sum_{k=m+h}^{n}\frac{(q^{-2n};q^2)_j(q^{-2n};q^2)_k(1-q^{k-j})q^{j+k+jh}}{(q;q)_{j-s}(q;q)_{j+s}(q;q)_{k-s}(q;q)_{k+s}}\\ &\qquad\times\begin{bmatrix}k-m-1\\ h-1\end{bmatrix}\begin{bmatrix}m+h-j-1\\ h-1\end{bmatrix}\\ &=\frac{(-1)^{n-m-h}(q^2;q^2)_n^2(-q;q)_{2n-h}q^{m^2-n^2-2mn+mh}}{(q;q)_{m-s}(q;q)_{m+s}(q^2;q^2)_{n-s}(q^2;q^2)_{n+s}(q;q)_{h-1}(q^2,q^2)_{n-m-h}}.\end{aligned}$$

Proof. By the definition of q-binomial coefficients, we have $\begin{bmatrix}k-m-1\\ h-1\end{bmatrix}=0$ for $m+1\le k<m+h$. Hence, the left-hand side of (3.13) remains unchanged if we replace $\sum_{k=m+h}^{n}$ by $\sum_{k=m+1}^{n}$. Furthermore,

$$\begin{aligned}&\begin{bmatrix}k-m-1\\ h-1\end{bmatrix}\begin{bmatrix}m+h-j-1\\ h-1\end{bmatrix}\\ &\qquad=\frac{(q^{j-m-h+1};q)_{h-1}(q^{k-m-h+1};q)_{h-1}q^{(m-j)(h-1)-\binom{h}{2}}}{(-1)^{h-1}(q;q)_{h-1}^2}.\end{aligned}$$

The conclusion then follows from the identity (3.9) with $x=-q^{-n}$. ∎

Proof of Theorem 1.5. The left-hand side of (1.11) may be expanded as

$$\begin{aligned}(3.14)\qquad &\sum_{k=s}^{n}\frac{(q^{-2n};q^2)_k^2q^{2k}}{(q;q)_{k-s}^2(q;q)_{k+s}^2}(x;q)_k(q/x;q)_k\\ &+\sum_{s\le j<k\le n}\frac{(q^{-2n};q^2)_j(q^{-2n};q^2)_kq^{j+k}\big((x;q)_j(q/x;q)_k+(x;q)_k(q/x;q)_j\big)}{(q;q)_{j-s}(q;q)_{j+s}(q;q)_{k-s}(q;q)_{k+s}}.\end{aligned}$$

For $0\le j<k$, from (3.5) we deduce that

$$\begin{aligned}(3.15)\qquad &(x;q)_j(q/x;q)_k+(x;q)_k(q/x;q)_j\\ &=(x;q)_j(q/x;q)_j\big((xq^j;q)_{k-j}+(q^{j+1}/x;q)_{k-j}\big)\\ &=(x;q)_k(q/x;q)_k+(x;q)_j(q/x;q)_j(q^{2j+1};q)_{k-j}\\ &\quad+\sum_{i=1}^{k-j-1}(x;q)_{j+i}(q/x;q)_{j+i}(1-q^{k-j})\\ &\quad\times\sum_{h=1}^{k-j-i}(-1)^h\begin{bmatrix}k-j-i-1\\ h-1\end{bmatrix}\begin{bmatrix}i+h-1\\ h-1\end{bmatrix}\frac{q^{\binom{h+1}{2}+(i+2j)h}}{1-q^h}\end{aligned}$$

$$
\begin{aligned}
&= (x;q)_k(q/x;q)_k + (x;q)_j(q/x;q)_j \\
&\quad + \sum_{i=0}^{k-j-1} (x;q)_{j+i}(q/x;q)_{j+i}(1-q^{k-j}) \\
&\qquad \times \sum_{h=1}^{k-j-i} (-1)^h \begin{bmatrix} k-j-i-1 \\ h-1 \end{bmatrix} \begin{bmatrix} i+h-1 \\ h-1 \end{bmatrix} \frac{q^{\binom{h+1}{2}+(i+2j)h}}{1-q^h},
\end{aligned}
$$

where in the last step we have used the q-binomial theorem:

$$
(q^{2j+1};q)_{k-j} = 1 + \sum_{h=1}^{k-j} (-1)^h \begin{bmatrix} k-j \\ h \end{bmatrix} q^{\binom{h+1}{2}+2jh}.
$$

By (3.15), we may write (3.14) as $\sum_{m=s}^{n} a_m (x;q)_m (q/x;q)_m$, where

$$
\begin{aligned}
a_m &= \sum_{j=s}^{n} \frac{(q^{-2n};q^2)_j (q^{-2n};q^2)_m q^{j+m}}{(q;q)_{j-s}(q;q)_{j+s}(q;q)_{m-s}(q;q)_{m+s}} \\
&\quad + \sum_{j=s}^{m} \sum_{k=m+1}^{n} \frac{(q^{-2n};q^2)_j (q^{-2n};q^2)_k (1-q^{k-j}) q^{j+k}}{(q;q)_{j-s}(q;q)_{j+s}(q;q)_{k-s}(q;q)_{k+s}} \\
&\qquad \times \sum_{h=1}^{k-m} (-1)^h \begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix} \frac{q^{\binom{h+1}{2}+(m+j)h}}{1-q^h}.
\end{aligned} \tag{3.16}
$$

It is easy to see that

$$
\begin{aligned}
\sum_{j=s}^{n} \frac{(q^{-2n};q^2)_j q^j}{(q;q)_{j-s}(q;q)_{j+s}} &= \frac{(q^{-2n};q^2)_s q^s}{(q;q)_{2s}} \sum_{j=s}^{n} \frac{(q^{-2n+2s};q^2)_j q^{j-s}}{(q;q)_{j-s}(q^{2s+1};q)_{j-s}} \\
&= \frac{(q^{-2n};q^2)_s q^s}{(q;q)_{2s}} {}_2\phi_1 \left[\begin{matrix} q^{-n+s},\ -q^{-n+s}, \\ q^{2s+1} \end{matrix} ; q, q \right] \\
&= (-1)^{n-s} \frac{(q^{-2n};q^2)_s (-q^{n+s+1};q)_{n-s} q^{s-(n-s)^2}}{(q;q)_{2s}(q^{2s+1};q)_{n-s}}
\end{aligned}
$$

by the q-Chu–Vandermonde summation formula [4, Appendix (II.6)]. Hence,

$$
\begin{aligned}
&\sum_{j=s}^{n} \frac{(q^{-2n};q^2)_j (q^{-2n};q^2)_m q^{j+m}}{(q;q)_{j-s}(q;q)_{j+s}(q;q)_{m-s}(q;q)_{m+s}} \\
&\quad = (-1)^{n-s} \frac{(q^{-2n};q^2)_s (q^{-2n};q^2)_m (-q^{n+s+1};q)_{n-s} q^{m+s-(n-s)^2}}{(q;q)_{m-s}(q;q)_{m+s}(q;q)_{n+s}} \\
&\quad = \frac{(-1)^{n-m}(q^2;q^2)_n^2 (-q;q)_{2n} q^{m^2-n^2-2mn}}{(q;q)_{m-s}(q;q)_{m+s}(q^2;q^2)_{n-s}(q^2;q^2)_{n+s}(q^2,q^2)_{n-m}}.
\end{aligned} \tag{3.17}
$$

Substituting (3.17) and (3.13) into (3.16), we obtain

$$a_m = \frac{(-1)^{n-m}(q^2;q^2)_n^2 q^{m^2-n^2-2mn}}{(q;q)_{m-s}(q;q)_{m+s}(q^2;q^2)_{n-s}(q^2;q^2)_{n+s}} \times \sum_{h=0}^{n-m} \frac{(-q;q)_{2n-h} q^{\binom{h+1}{2}+2mh}}{(-1)^h (q;q)_h (q^2,q^2)_{n-m-h}}. \tag{3.18}$$

Replacing h by $n-m-h$, we have

$$\begin{aligned}
&\sum_{h=0}^{n-m} \frac{(-1)^h(-q;q)_{2n-h} q^{\binom{h+1}{2}+2mh}}{(q;q)_h (q^2,q^2)_{n-m-h}} \\
&= \frac{(-q;q)_{n+m}}{(q;q)_{n-m}} (-1)^{n-m} q^{\binom{n-m+1}{2}+2m(n-m)} \\
&\quad \times \sum_{h=0}^{n-m} \frac{(q^{m-n};q)_h(-q^{n+m+1};q)_h}{(-q;q)_h(q;q)_h} q^{-2hm} \\
&= \frac{(-q;q)_{n+m}}{(q;q)_{n-m}} (-1)^{n-m} q^{\binom{n-m+1}{2}+2m(n-m)} {}_2\phi_1\left[\begin{matrix} q^{-(n-m)}, & -q^{m+n+1} \\ & -q \end{matrix}; q, q^{-2m}\right] \\
&= \frac{(-q;q)_{n+m}(q^{-n-m};q)_{n-m}}{(q;q)_{n-m}(-q;q)_{n-m}} (-1)^{n-m} q^{\binom{n-m+1}{2}+2m(n-m)} \\
&= \frac{(q^2;q^2)_{m+n}}{(q^2;q^2)_{n-m}(q;q)_{2m}},
\end{aligned} \tag{3.19}$$

where we have used the q-Chu–Vandermonde summation formula. It follows from (3.18) and (3.19) that a_m is just the coefficient of $(x;q)_m(q/x;q)_m$ on the right-hand side of (1.11). ∎

4. Proof of Theorem 1.3. We first give a congruence modulo $[p]$.

LEMMA 4.1. *Let m and r be two positive integers with $p \nmid m$. Let $s \le \min\{\langle -r/m\rangle_p, \langle -(m-r)/m\rangle_p\}$ be a nonnegative integer. Then the following congruence holds modulo $[p]$:*

$$\sum_{k=s}^{(p-1)/2} \frac{(q^m;q^{2m})_k (q^r;q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} \tag{4.1}$$

$$\equiv \begin{cases} \dfrac{q^{\frac{(\langle -r/m\rangle_p+s)m}{2}} (q^m;q^{2m})_{\frac{\langle -r/m\rangle_p-s}{2}} (q^{-m\langle -r/m\rangle_p};q^m)_s}{(q^{2m};q^{2m})_{\frac{\langle -r/m\rangle_p+s}{2}}} & \text{if } \langle -r/m\rangle_p \equiv s \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that $\langle -r/m\rangle_p + \langle -(m-r)/m\rangle_p = p-1$, and so $s \le (p-1)/2$. Since p is an odd prime, we see that $(q^m;q^{2m})_k \equiv 0 \pmod{[p]}$ for $(p+1)/2 \le k \le p-s-1$, which means that

$$\sum_{k=s}^{(p-1)/2} \frac{(q^m;q^{2m})_k (q^r;q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} \equiv \sum_{k=s}^{p-s-1} \frac{(q^m;q^{2m})_k (q^r;q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} \pmod{[p]}.$$

Let $a = \frac{m\langle -r/m\rangle_p + r}{p}$. Then $m \mid r-ap$ and $r-ap = -m\langle -r/m\rangle_p \le 0$. It is clear that $(q^r;q^m)_k \equiv (q^{r-ap};q^m)_k \pmod{[p]}$ and $(q^{r-ap};q^m)_k = 0$ for $k > \langle -r/m\rangle_p$. Moreover, we have $p-s-1 \ge p - \langle -(m-r)/m\rangle_p - 1 = \langle -r/m\rangle_p \ge s$, and therefore

$$\begin{aligned}
&\sum_{k=s}^{p-s-1} \frac{(q^m;q^{2m})_k (q^{r-ap};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q;q)_{k+s}} \\
&\quad \equiv \sum_{k=s}^{\langle -r/m\rangle_p} \frac{(q^m;q^{2m})_k (q^{r-ap};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} \\
&\quad = \frac{(q^m;q^{2m})_s (q^{-m\langle -r/m\rangle_p};q^m)_s q^{ms}}{(q^m;q^m)_{2s}} \\
&\qquad \times {}_3\phi_2\left[\begin{matrix} q^{-m(\langle -r/m\rangle_p - s)}, q^{(s+1/2)m}, -q^{(s+1/2)m} \\ 0, q^{(2s+1)m} \end{matrix}; q^m, q^m\right] \pmod{[p]}.
\end{aligned}$$

The conclusion follows from Andrews' identity (2.3). ∎

Proof of Theorem 1.3. By Lemma 2.1, for $0 \le k \le (p-1)/2$, we have

$$\begin{aligned}
\frac{(q^m;q^m)_{2k}}{(q^{2m};q^{2m})_k^2} &= \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^{2m}} \frac{1}{(-q^m;q^m)_{2k}} \\
&\equiv (-1)^k q^{mk^2 - mkp} \begin{bmatrix} (p-1)/2+k \\ 2k \end{bmatrix}_{q^{2m}} (-q^m;q^m)_{2k} \\
&= \frac{(-1)^k q^{mk^2-mkp} (q^{2m};q^{2m})_{(p-1)/2+k}}{(q^{2m};q^{2m})_{(p-1)/2-k} (q^m;q^m)_{2k}} \pmod{[p]^2},
\end{aligned}$$

and so

$$\begin{aligned}
(4.2)\qquad &\sum_{k=s}^{(p-1)/2} \frac{(q^m;q^m)_{2k}(q^r;q^m)_k (q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}(q^{2m};q^{2m})_k^2} \\
&\equiv \sum_{k=s}^{(p-1)/2} \frac{(-1)^k (q^m;q^m)_{\frac{p-1}{2}+k} (q^r;q^m)_k (q^{m-r};q^m)_k q^{mk^2 - mk(p-1)}}{(q^{2m};q^{2m})_{\frac{p-1}{2}-k} (q^m;q^m)_{k-s}(q^m;q^m)_{k+s}(q^m;q^m)_{2k}} \pmod{[p]^2}.
\end{aligned}$$

Letting $q \to q^m$, $x = q^r$ and $n = (p-1)/2$ in Theorem 1.5, we see that the right-hand side of (4.2) can be written as

$$\begin{aligned}\tag{4.3} &\frac{(-1)^{(p-1)/2}(q^{2m};q^{2m})_{(p-1)/2-s}(q^{2m};q^{2m})_{(p-1)/2+s}q^{(p-1)^2/4}}{(q^{2m};q^{2m})_{(p-1)/2}^2} \\ &\qquad\times\bigg(\sum_{k=s}^{(p-1)/2}\frac{(q^{m(1-p)};q^{2m})_k(q^r;q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}}\bigg) \\ &\qquad\times\bigg(\sum_{k=s}^{(p-1)/2}\frac{(q^{m(1-p)};q^{2m})_k(q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}}\bigg).\end{aligned}$$

If $\langle -r/m\rangle_p \equiv s+1 \pmod 2$, then by the congruence (4.1), we have

$$\begin{aligned}\sum_{k=s}^{(p-1)/2}\frac{(q^{m(1-p)};q^{2m})_k(q^r;q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} &\equiv \sum_{k=s}^{(p-1)/2}\frac{(q^m;q^{2m})_k(q^r;q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} \\ &\equiv 0 \pmod{[p]},\end{aligned}$$

and also $\langle -(m-r)/m\rangle_p \equiv s+1 \pmod 2$, which means that

$$\sum_{k=s}^{(p-1)/2}\frac{(q^{m(1-p)};q^{2m})_k(q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} \equiv 0 \pmod{[p]}.$$

Noticing that $(q^{2m};q^{2m})_{(p-1)/2} \not\equiv 0 \pmod{[p]}$, we conclude that the right-hand side of (4.2) is congruent to 0 modulo $[p]^2$. This proves (1.8).

To prove (1.9), just observe that (see the proof of Lemma 4.1)

$$(q^r;q^m)_k \equiv (q^{m-r};q^m)_k \equiv 0 \pmod{[p]}$$

for $\max\{\langle -r/m\rangle_p, \langle -(m-r)/m\rangle_p\} < k \le p-1$, and

$$\begin{aligned}(q^r;q^m)_k(q^{m-r};q^m)_k &\equiv \frac{(q^m;q^m)_{2k}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}} \\ &\equiv 0 \pmod{[p]}\end{aligned}$$

for $(p-1)/2 < k \le \max\{\langle -r/m\rangle_p, \langle -(m-r)/m\rangle_p\}$.

Finally, (1.10) follows from factorizing (4.2) into (4.3), applying the first case of the congruence (4.1), and then using the aforementioned relation $\langle -r/m\rangle_p + \langle -(m-r)/m\rangle_p = p-1$. ■

5. Proof of Theorems 1.6 and 1.7. The following lemma is probably known. For the reader's convenience, we include a proof.

LEMMA 5.1. *Let m, n and s be nonnegative integers with $s \le n$. Then*

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m+k \\ n \end{bmatrix} q^{\binom{k}{2}-nk} = (-1)^n q^{-\binom{n+1}{2}}, \tag{5.1}$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} 2k+2s \\ k+s \end{bmatrix}_{q^2} \frac{q^{k^2-k-2nk}}{(-q^{2k+1};q)_{2s}} = (-1)^n q^{-n(n+1)}. \tag{5.2}$$

Proof. It is not difficult to see that (5.1) and (5.2) are equivalent, respectively, to

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m+n-k \\ n \end{bmatrix} q^{\binom{k+1}{2}} = 1, \tag{5.3}$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n-k \\ n \end{bmatrix}_{q^2} \frac{(q^{2n-2k+1};q^2)_s}{(q^{2n-2k+2};q^2)_s} q^{2\binom{k+1}{2}} = 1. \tag{5.4}$$

Since $\begin{bmatrix} m+n-k \\ n \end{bmatrix}$ can be written as a polynomial in q^{-k} of degree n with constant term $1/(q;q)_n$, the identity (5.3) follows from (3.12). On the other hand, since $0 \le s \le n$, we see that

$$\begin{bmatrix} 2n-k \\ n \end{bmatrix}_{q^2} \frac{(q^{2n-2k+1};q^2)_s}{(q^{2n-2k+2};q^2)_s} = \frac{(q^{2n-2k+2s+2};q^2)_{n-s}(q^{2n-2k+1};q^2)_s}{(q^2;q^2)_n}$$

is a polynomial in q^{-2k} of degree n with constant term $1/(q^2;q^2)_n$. Therefore, (5.4) follows from (3.12) with $q \to q^2$. ∎

Proof of Theorem 1.6. It is easy to see that

$$\frac{(q;q^2)_k}{(q^2;q^2)_k} = \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \frac{1}{(-q;q)_{2k}}.$$

Hence, by Lemmas 2.1 and 5.1, we have

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{(q;q^2)_k(q;q^2)_{k+s}}{(q^2;q^2)_k(q^2;q^2)_{k+s}} \qquad\qquad & \qquad (5.5)\\ = \sum_{k=0}^{(p-1)/2} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2k+2s \\ k+s \end{bmatrix}_{q^2} \frac{1}{(-q;q)_{2k}(-q;q)_{2k+2s}} & \\ \equiv \sum_{k=0}^{(p-1)/2} (-1)^k \begin{bmatrix} \frac{p-1}{2}+k \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} 2k+2s \\ k+s \end{bmatrix}_{q^2} \frac{(-q;q)_{2k} q^{k^2-kp}}{(-q;q)_{2k+2s}} & \\ = (-1)^{(p-1)/2} q^{(1-p^2)/4} \pmod{[p]^2}. \ ∎ & \end{aligned}$$

Proof of Theorem 1.7. Again, let $a = (m\langle -r/m\rangle_p + r)/p$. Then a is a positive integer, $m\,|\,ap - r$, and so

$$
\begin{aligned}
\frac{(q^r;q^m)_k}{(q^m;q^m)_k} &= \prod_{j=1}^{k} \frac{1-q^{mj+r-m}}{1-q^{mj}} \\
&\equiv (-1)^k \prod_{j=1}^{k} \frac{(1-q^{ap-mj-r+m})q^{mj+r-m}}{1-q^{mj}} \\
&= (-1)^k \begin{bmatrix} (ap-r)/m \\ k \end{bmatrix}_{q^m} q^{mk(k-1)/2+kr} \\
&\equiv (-1)^k \begin{bmatrix} \langle -r/m\rangle_p \\ k \end{bmatrix}_{q^m} q^{mk(k-1)/2-mk\langle -r/m\rangle_p} \pmod{[p]},
\end{aligned} \tag{5.6}
$$

$$
\begin{aligned}
\frac{(q^{m-r};q^m)_{k+s}}{(q^m;q^m)_{k+s}} &\equiv \prod_{j=1}^{k+s} \frac{1-q^{ap+mj-r}}{1-q^{mj}} = \begin{bmatrix} \frac{ap-r}{m}+k+s \\ k+s \end{bmatrix}_{q^m} \\
&= \begin{bmatrix} \langle -r/m\rangle_p + k + s \\ k+s \end{bmatrix}_{q^m} \pmod{[p]}.
\end{aligned} \tag{5.7}
$$

By the congruences (5.6) and (5.7), we have

$$
\begin{aligned}
&\sum_{k=0}^{p-s-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_{k+s}}{(q^m;q^m)_k (q^m;q^m)_{k+s}} \\
&\quad \equiv \sum_{k=0}^{p-s-1} (-1)^k \begin{bmatrix} \langle -r/m\rangle_p \\ k \end{bmatrix}_{q^m} \begin{bmatrix} \langle -r/m\rangle_p + k + s \\ k+s \end{bmatrix}_{q^m} q^{mk(k-1)/2-mk\langle -r/m\rangle_p} \\
&\quad = (-1)^{\langle -r/m\rangle_p} q^{-m\langle -r/m\rangle_p(\langle -r/m\rangle_p+1)/2} \pmod{[p]},
\end{aligned}
$$

where in the last step we have used $p-s-1 \ge p - \langle -(m-r)/m\rangle_p - 1 = \langle -r/m\rangle_p$ and the identity (5.1). This proves (1.12).

To prove (1.13), just notice that if $p \equiv \pm 1 \pmod{m}$, then $\frac{r(m-r)(1-p^2)}{2m}$ is an integer and

$$
\frac{-m\langle -r/m\rangle_p(\langle -r/m\rangle_p+1)}{2} \equiv \frac{r(m-r)(1-p^2)}{2m} \pmod{p}. \ \blacksquare
$$

6. Concluding remarks and open problems. It seems that the congruence (1.9) can be further generalized as follows.

Conjecture 6.1. *Let m and r be two positive integers with $p \nmid m$. Let $s \le p-1$ be a nonnegative integer. If $\langle -r/m\rangle_p \equiv s+1 \pmod 2$, then*

$$
\sum_{k=s}^{p-1} \frac{(q^m;q^m)_{2k}(q^r;q^m)_k(q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_{k-s}(q^m;q^m)_{k+s}(q^{2m};q^{2m})_k^2} \equiv 0 \pmod{[p]^2}. \tag{6.1}
$$

Note that if $s > \max\{\langle -r/m\rangle_p, \langle -(m-r)/m\rangle_p\}$, then (6.1) is obviously true, since in this case each summand on the left-hand side is congruent to 0 modulo $[p]^2$. It is easy to see that when $r=1$, $m=3,4,6$ and $q\to 1$, Conjecture 6.1 reduces to a result of Z.-W. Sun [14, Theorem 1.3(i)].

We conjecture that Theorem 1.7 can be further strengthened:

Conjecture 6.2. *Let m and r be positive integers with $p \equiv \pm 1 \pmod{m}$ and $r < m$. Then for any integer s with $0 \le s \le \langle -(m-r)/m\rangle_p$,*

$$\sum_{k=0}^{p-s-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_{k+s}}{(q^m;q^m)_k (q^m;q^m)_{k+s}} \equiv (-1)^{\langle -r/m\rangle_p} q^{r(m-r)(1-p^2)/(2m)} \pmod{[p]^2}.$$

Like [6, Conjecture 7.1], Conjecture 6.2 seems to have a further generalization

Conjecture 6.3. *Let m and $|r|$ be positive integers with $p \nmid m$ and $m \nmid r$. Then there exists a unique integer $f_{p,m,r}$ such that, for any s with $0 \le s \le \langle -(m-r)/m\rangle_p$,*

$$\sum_{k=0}^{p-s-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_{k+s}}{(q^m;q^m)_k (q^m;q^m)_{k+s}} \equiv (-1)^{\langle -r/m\rangle_p} q^{f_{p,m,r}} \pmod{[p]^2}.$$

Furthermore, the numbers $f_{p,m,r}$ satisfy the symmetry $f_{p,m,r} = f_{p,m,m-r}$ and the recurrence relation:

$$f_{p,m,m+r} = \begin{cases} -f_{p,m,r} & \text{if } r \equiv 0 \pmod{p}, \\ f_{p,m,r} - r & \text{otherwise.} \end{cases}$$

Here are some values of $f_{p,m,r}$:

$f_{3,2,1} = -2,\ f_{3,2,3} = -3,\ f_{3,2,5} = 3,\ f_{3,2,7} = -2,\ f_{3,2,9} = -9,$
$f_{3,2,11} = 9,\ f_{3,2,13} = -2,\ f_{5,3,1} = -8,\ f_{5,3,2} = -8,\ f_{5,3,4} = -9,$
$f_{5,3,5} = -10,\ f_{5,3,7} = -13,\ f_{5,3,8} = 10,\ f_{5,3,10} = -20,\ f_{5,3,11} = 2,$
$f_{5,3,13} = 20,\ f_{5,3,14} = -9,\ f_{5,3,16} = 7,\ f_{5,3,17} = -23,\ f_{5,3,19} = -9,$
$f_{5,8,1} = -23,\ f_{7,9,1} = -54,\ f_{7,9,2} = -21,\ f_{7,9,4} = -37,\ f_{7,9,5} = -37,$
$f_{7,9,7} = -21,\ f_{7,9,8} = -54,\ f_{7,9,10} = -55,\ f_{7,9,11} = -23\ f_{7,9,13} = -41,$
$f_{7,9,14} = -42,\ f_{7,9,16} = -22,\ f_{7,9,17} = -33.$

Finally, supercongruences (or q-supercongruences) have now been around for quite a long time, and it would be desirable to have some more conceptual proofs of these phenomena, such as combinatorial interpretations, connections to elliptic curves or to representations of p-adic groups.

Acknowledgments. This work was partially supported by the National Natural Science Foundation of China (grant no. 11371144).

References

[1] M. Aigner, *A Course in Enumeration*, Grad. Texts in Math. 238, Springer, 2007.

[2] G. E. Andrews, *The Theory of Partitions*, Cambridge Univ. Press, Cambridge, 1998.

[3] S. Chowla, B. Dwork, and R. Evans, *On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$*, J. Number Theory 24 (1986), 188–196.

[4] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia Math. Appl. 96, Cambridge Univ. Press, Cambridge, 2004.

[5] V. J. W. Guo and J. Zeng, *A short proof of the q-Dixon identity*, Discrete Math. 296 (2005), 259–261.

[6] V. J. W. Guo and J. Zeng, *Some q-analogues of supercongruences of Rodriguez-Villegas*, J. Number Theory 145 (2014), 301–316.

[7] E. Mortenson, *A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function*, J. Number Theory 99 (2003), 139–147.

[8] E. Mortenson, *Supercongruences between truncated ${}_2F_1$ hypergeometric functions and their Gaussian analogs*, Trans. Amer. Math. Soc. 355 (2003), 987–1007.

[9] E. Mortenson, *Supercongruences for truncated ${}_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. 133 (2005), 321–330.

[10] H. Pan, *An elementary approach to $\binom{(p-1)/2}{(p-1)/4}$ modulo p^2*, Taiwanese J. Math. 16 (2012), 2197–2202.

[11] F. Rodriguez-Villegas, *Hypergeometric families of Calabi–Yau manifolds*, in: Calabi–Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), Fields Inst. Comm. 38, Amer. Math. Soc., Providence, RI, 2003, 223–231.

[12] Z.-H. Sun, *Congruences concerning Legendre polynomials II*, J. Number Theory 133 (2013), 1950–1976.

[13] Z.-H. Sun, *Generalized Legendre polynomials and related supercongruences*, J. Number Theory 143 (2014), 293–319.

[14] Z.-W. Sun, *On sums involving products of three binomial coefficients*, Acta Arith. 156 (2012) 123–141.

[15] Z.-W. Sun, *Supercongruences involving products of two binomial coefficients*, Finite Fields Appl. 22 (2013), 24–44.

[16] H. Swisher, *On the supercongruence conjectures of van Hamme*, arXiv:1504.01028v1 (2015).

[17] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, in: p-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure Appl. Math. 192, Dekker, 1997, 223–236.

Victor J. W. Guo
School of Mathematical Sciences
Huaiyin Normal University
Huaian, Jiangsu 223300
People's Republic of China
E-mail: jwguo1977@aliyun.com

Jiang Zeng
Université de Lyon
Université Lyon 1
Institut Camille Jordan, UMR 5208 du CNRS
43, boulevard du 11 novembre 1918
F-69622 Villeurbanne Cedex, France
E-mail: zeng@math.univ-lyon1.fr

Received on 26.1.2015
and in revised form on 15.8.2015 (8064)