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Continued fraction expansions for complex numbers—a general approach

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1. Introduction. A. Hurwitz [4] introduced, in 1887, continued fraction expansions for complex numbers with Gaussian integers as partial quotients, via the nearest integer algorithm (also known subsequently as the Hurwitz algorithm), and established some basic properties concerning convergence of the sequence of convergents; he also proved an analogue of the classical Lagrange theorem characterizing quadratic surds as the numbers with eventually periodic continued fractions. Analogous results were also proved for the nearest integer algorithms with respect to Eisenstein integers as partial quotients, in place of Gaussian integers.

Application of complex continued fractions, typically involving the nearest integer algorithm, to questions in Diophantine approximation analogous to the theory for simple continued fractions for real numbers, was taken up by various authors during the last century (see [6], [7], [5], [3], and other references cited therein).

In [1], where we considered the question of values of binary quadratic forms with complex coefficients over pairs of Gaussian integers, we extended the study of continued fractions to other possible algorithms in place of the nearest integer algorithm, and also introduced certain nonalgorithmic constructions for continued fraction expansions, via what was called iteration sequences; the partial quotients for the continued fractions were however retained to be Gaussian integers. In this paper we set up a broader framework for studying continued fraction expansions for complex numbers, and prove certain general results on convergence, analogue of the Lagrange theorem, speed of convergence etc. Our results in particular generalize those of Hurwitz in the case of the nearest integer algorithms with respect to Gaussian integers and Eisenstein integers.

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2. Preliminaries on continued fraction expansions. We begin with a general formulation of the notion of continued fraction expansion, with flexible choices for the partial quotients. Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{C}^*$ the set of nonzero numbers in $\mathbb{C}$. When $z \in \mathbb{C}$ can be expressed as

$$z = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

with $a_j \in \mathbb{C}^*$ for all $j \in \mathbb{N}$ (natural numbers), where the right hand side is assigned the usual meaning as the limit of the truncated expressions (assuming that they represent genuine complex numbers and the limit exists—see below), we consider the expression as above to be a continued fraction expansion for z; though our main application will be with a_n's in specific rings, we shall first discuss some results in which a_n can be more general complex numbers. The above concept can be formulated more systematically as follows.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}^*$. We associate to it two sequences $\{p_n\}_{n=-1}^{\infty}$ and $\{q_n\}_{n=-1}^{\infty}$ defined recursively by the relations

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = a_0, \quad p_{n+1} = a_{n+1}p_n + p_{n-1} \quad \text{for all } n \geq 0, \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1} \quad \text{for all } n \geq 0. \end{aligned}$$

If $q_n \neq 0$ for all n then we can form p_n/q_n, and if they converge, as $n \to \infty$, to a complex number z, we say that $\{a_n\}_{n=0}^{\infty}$ *defines a continued fraction expansion of* z; in this case we express z as $[a_0, a_1, a_2, \ldots]$.

In conformity with the nomenclature adopted in [1] we call $\{p_n\}, \{q_n\}$ the *$\mathcal{Q}$-pair of sequences associated to* $\{a_n\}_{n=0}^{\infty}$ ($\mathcal{Q}$ signifies "quotient"). The ratios p_n/q_n with $q_n \neq 0$ are called the *convergents* corresponding to the $\mathcal{Q}$-pair, or to the sequence $\{a_n\}_{n=0}^{\infty}$. We note that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$ for all $n \geq 0$, as may be verified inductively.

Given a $z \in \mathbb{C}^*$, "candidates" for continued fraction expansions for z can be arrived at by setting $a_n = z_n - z_{n+1}^{-1}$ for all $n \geq 0$, where $\{z_n\}_{n=0}^{\infty}$ is a sequence in $\mathbb{C}^*$ such that $z_0 = z$ and for all $n \geq 1$, $|z_n| \geq 1$ and $z_{n+1} \neq z_n^{-1}$. We shall call such a sequence an *iteration sequence for* z, and $\{a_n\}_{n=0}^{\infty}$ the associated sequence of *partial quotients*. (In [1] "iteration sequences" were introduced, with slightly different conditions, and a_n's restricted to Gaussian integers.) Whether a sequence of partial quotients so constructed indeed defines a continued fraction expansion for z is an issue that needs to be considered however.

We begin by noting the following general properties.

Proposition 2.1. *Let $z \in \mathbb{C}^*$, and let $\{z_n\}$ be an iteration sequence for z. Let $\{a_n\}_{n=0}^{\infty}$ be the associated sequence of partial quotients, and let*

$\{p_n\}, \{q_n\}$ be the $\mathcal{Q}$-pair of sequences associated to $\{a_n\}$. Then for all $n \geq 0$:

(i) $q_n z - p_n = (-1)^n (z_1 \cdots z_{n+1})^{-1}$;

(ii) *if* $|p_n| > |z_1|^{-1}$ *then* $q_n \neq 0$;

(iii) $(z_{n+1} q_n + q_{n-1}) z = z_{n+1} p_n + p_{n-1}$;

(iv) *if* $|q_{n-1}| < |q_n|$ *then* $|z - p_n/q_n| \leq |q_n|^{-2}(|z_{n+1}| - |q_{n-1}/q_n|)^{-1}$;

(v) *if* q_n*'s are nonzero and* $|q_n| \to \infty$ *then* p_n/q_n *converges to* z *as* $n \to \infty$.

Proof. (i) We argue by induction. Note that as $p_0 = a_0$, $q_0 = 1$ and $z - a_0 = z_1^{-1}$, the statement holds for $n = 0$. Now let $n \geq 1$ and suppose that the assertion holds for $0, 1, \ldots, n-1$. Then

$$\begin{aligned} q_n z - p_n &= (a_n q_{n-1} + q_{n-2}) z - (a_n p_{n-1} + p_{n-2}) \\ &= a_n (q_{n-1} z - p_{n-1}) + (q_{n-2} z - p_{n-2}) \\ &= (-1)^{n-1} (z_1 \cdots z_n)^{-1} a_n + (-1)^{n-2} (z_1 \cdots z_{n-1})^{-1} \\ &= (-1)^n (z_1 \cdots z_n)^{-1} (-a_n + z_n) = (-1)^n (z_1 \cdots z_{n+1})^{-1}. \end{aligned}$$

(ii) For $n \geq 0$, if $|p_n| > |z_1|^{-1}$ then by (i) we have

$$|q_n z| \geq |p_n| - |z_1 \cdots z_{n+1}|^{-1} \geq |p_n| - |z_1|^{-1} > 0,$$

and hence $q_n \neq 0$.

(iii) For $n \geq 0$, by (i) we have

$$\begin{aligned} z_{n+1}(q_n z - p_n) &= (-1)^n (z_1 \cdots z_{n+1})^{-1} z_{n+1} = (-1)^n (z_1 \cdots z_n)^{-1} \\ &= -(q_{n-1} z - p_{n-1}), \end{aligned}$$

and hence (iii) follows.

(iv) By (iii) we get

$$\begin{aligned} |(z_{n+1} q_n + q_{n-1})(q_n z - p_n)| &= |(z_{n+1} p_n + p_{n-1}) q_n - (z_{n+1} q_n + q_{n-1}) p_n)| \\ &= |p_{n-1} q_n - q_{n-1} p_n| = 1. \end{aligned}$$

Also, $|z_{n+1} q_n + q_{n-1}| \geq |q_n|(|z_{n+1}| - |q_{n-1}/q_n|)$, and since $|z_{n+1}| \geq 1$ and $|q_{n-1}| < |q_n|$ we have $|z_{n+1}| - |q_{n-1}/q_n| > 0$. Thus

$$|z - p_n/q_n| = |q_n|^{-1} |z_{n+1} q_n + q_{n-1}|^{-1} \leq |q_n|^{-2} (|z_{n+1}| - |q_{n-1}/q_n|)^{-1}.$$

(v) If q_n are nonzero and $|q_n| \to \infty$ then

$$|z - p_n/q_n| = |q_n|^{-1} |z_1 \cdots z_{n+1}|^{-1} \leq |q_n|^{-1} \to 0 \quad \text{as } n \to \infty. \ \blacksquare$$

We next specialize to sequences $\{a_n\}_{n=0}^{\infty}$ contained in discrete subrings of $\mathbb{C}$; by a subring we shall always mean one containing 1, the multiplicative identity. When $\{a_n\}_{n=0}^{\infty}$ is contained in a discrete subring $\mathit{\Gamma}$, from the recurrence relations it follows that for the corresponding $\mathcal{Q}$-pair $\{p_n\}, \{q_n\}$, we have $p_n, q_n \in \mathit{\Gamma}$ for all n.

Proposition 2.2. *Let the notation be as in Proposition* 2.1 *and suppose further that:*

(i) $\{a_n\}_{n=0}^{\infty}$ *is contained in a discrete subring* Γ *of* $\mathbb{C}$;
(ii) *there exists* $\alpha > 0$ *such that* $|z_n| \geq 1 + \alpha$ *for all* $n \geq 1$.

Then $q_n \neq 0$ *for all* $n \geq 0$, *and* $p_n/q_n \to z$ *as* $n \to \infty$. *Also, for all* n *such that* $|q_{n-1}| < |q_n|$,

$$|z - p_n/q_n| \leq \alpha^{-1}|q_n|^{-2}.$$

Proof. Since Γ is a discrete subring of $\mathbb{C}$, for any $p \in \Gamma \setminus \{0\}$ we have $|p| \geq 1$. Now if $q_n = 0$ for some $n \geq 1$, then by Proposition 2.1(i) we should have $|p_n| = |z_1 \cdots z_{n+1}|^{-1} \in (0, 1)$, which is not possible since $p_n \in \Gamma$. Hence $q_n \neq 0$ for all $n \geq 0$. Since $q_n \in \Gamma$, this implies that $|q_n| \geq 1$ for all n. Therefore,

$$|z - p_n/q_n| = |q_n|^{-1}|q_n z - p_n| = |q_n|^{-1}|z_1 \cdots z_{n+1}|^{-1} \leq (1+\alpha)^{-n} \to 0,$$

and hence $p_n/q_n \to z$ as $n \to \infty$.

When $|q_{n-1}| < |q_n|$, by Proposition 2.1 we have

$$|z - p_n/q_n| \leq |q_n|^{-2}(|z_{n+1}| - |q_{n-1}/q_n|)^{-1} \leq \alpha^{-1}|q_n|^{-2},$$

since $|z_n| \geq 1 + \alpha$. ∎

A standard way to generate iteration sequences is via algorithms. Let Λ be a countable subset of $\mathbb{C}$ such that for every $z \in \mathbb{C}$ there exists $\lambda \in \Lambda$ such that $|z - \lambda| \leq 1$. By a Λ-valued *algorithm* we mean a map $f : \mathbb{C} \to \Lambda$ such that $|z - f(z)| \leq 1$ for all $z \in \mathbb{C}$. Let K denote the subfield of $\mathbb{C}$ generated by Λ; we note that K is also countable. For any $z \in \mathbb{C} \setminus K$ a Λ-valued algorithm f as above yields an iteration sequence defined by $z_0 = z$ and $z_{n+1} = (z_n - f(z_n))^{-1}$ for all $n \geq 0$; for $z \in \mathbb{C} \setminus K$, it may be observed successively that all z_n are in $\mathbb{C} \setminus K$ and hence $z_n \neq f(z_n)$, so $z_n - f(z_n) \neq 0$.

Definition 2.3. We call the set $\{z - f(z) \mid z \in \mathbb{C} \setminus K\}$ the *fundamental set* of the algorithm f.

When Λ is a discrete subring of $\mathbb{C}$ we have the following.

Theorem 2.4. *Let* Γ *be a discrete subring of* $\mathbb{C}$ *and let* $f : \mathbb{C} \to \Gamma$ *be a* Γ*-valued algorithm such that the fundamental set of* f *is contained in a ball of radius* r *centered at* 0, *where* $0 < r < 1$. *Let* K *be the subfield generated by* Γ. *Let* $z \in \mathbb{C} \setminus K$ *and let* $\{z_n\}_{n=0}^{\infty}$ *be the iteration sequence for* z *with respect to* f. *Let* $\{a_n\}_{n=0}^{\infty}$ *be the associated sequence of partial quotients, and* $\{p_n\}, \{q_n\}$ *the corresponding* $\mathcal{Q}$*-pair. Then:*

(i) $q_n \neq 0$ *for all* $n \geq 0$, *and* $p_n/q_n \to z$ *as* $n \to \infty$;
(ii) *for every* n *such that* $|q_{n-1}| < |q_n|$ *we have* $\left|z - \frac{p_n}{q_n}\right| \leq \frac{r}{1-r}|q_n|^{-2}$.

Proof. Under the given hypothesis, $|z_n - a_n| \leq r$ for all $n \geq 0$. Hence for all $n \geq 1$ we have $|z_n| = |z_{n-1} - a_{n-1}|^{-1} \geq r^{-1}$. Thus condition (ii) of Proposition 2.2 holds with $\alpha = r^{-1} - 1$, and hence the theorem follows from the proposition. ∎

When Λ is a discrete subset we have an algorithm f arising canonically, where we choose, for $z \in \mathbb{C}$, $f(z)$ to be the element of Λ nearest to z; this map is defined uniquely only for z in the complement of a countable set of lines (consisting of points which are equidistant from two distinct points of Λ), but we consider it extended to $\mathbb{C}$ through some convention—the specific choice of the extension will not play any role in our discussion. We call this the *nearest element algorithm* with respect to Λ; when Λ is a ring of "integers", such as the Gaussian or Eisenstein integers, the algorithm will be referred to as the *nearest integer algorithm* of the corresponding ring.

REMARK 2.5. It can be seen that any discrete subring Γ of $\mathbb{C}$ (containing 1), other than $\mathbb{Z}$, has the form $\mathbb{Z}[i\sqrt{k}]$ or $\mathbb{Z}\big[\frac{1}{2}+\frac{i}{2}\sqrt{4l-1}\big]$ with $k, l \in \mathbb{N}$. From among these, the requirement that there be an element of Γ within distance 1 from every z in $\mathbb{C}$ (enabling continued fraction expansions to be defined for all $z \in \mathbb{C}$) is met for $\mathbb{Z}[i\sqrt{k}]$, $1 \le k \le 3$, and $\mathbb{Z}\big[\frac{1}{2}+\frac{i}{2}\sqrt{4l-1}\big]$, $1 \le l \le 3$; for $k = 1$ and $l = 1$ these are the rings of Gaussian integers and Eisenstein integers respectively. With respect to the nearest integer algorithm the fundamental set is the square with vertices at $\pm\frac{1}{2} + \pm\frac{\sqrt{k}}{2}i$ for $\Gamma = \mathbb{Z}[i\sqrt{k}]$, $k = 1, 2, 3$, and for $\Gamma = \mathbb{Z}\big[\frac{1}{2}+\frac{i}{2}\sqrt{\tau}\big]$ with $\tau = 3, 7$ or 11 it is a hexagon (not regular in the last two cases) with vertices at

$$\pm\frac{1}{2} \pm \frac{\tau-1}{4\sqrt{\tau}}i \quad \text{and} \quad \pm\frac{\tau+1}{4\sqrt{\tau}}i$$

respectively; thus the vertices lie on the circle, centered at the origin, with radius $\frac{1}{2}\sqrt{(1+k)}$, $k = 1, 2, 3$, in the former case, and $\frac{\tau+1}{4\sqrt{\tau}}$ with $\tau = 3, 7, 11$ in the latter; consequently, the fundamental set is contained in the open unit ball, except for $\mathbb{Z}[i\sqrt{3}]$. Hence, except when $\Gamma = \mathbb{Z}[i\sqrt{3}]$ (a case not considered in the literature), by Theorem 2.4, we have $q_n \neq 0$ for all $n \geq 0$, and $p_n/q_n \to z$ as $n \to \infty$ for the continued fraction expansion with respect to the respective nearest integer algorithms.

REMARK 2.6. The second assertion in Theorem 2.4 highlights the usefulness of establishing the monotonicity of $\{|q_n|\}$, to complete the picture; the monotonicity condition will also be involved in proving the analogue of the Lagrange theorem (see Theorem 4.2). The latter was proved by Hurwitz for the nearest integer algorithms with respect to the rings of Gaussian integers and Eisenstein integers. It was proved by Lund for the nearest integer algorithm on $\mathbb{Z}[i\sqrt{2}]$, as noted in [5], where it is also stated without proof that monotonicity holds for $\mathbb{Z}\big[\frac{1}{2}+\frac{i}{2}\sqrt{\tau}\big]$, $\tau = 3, 7$ or 11, for the nearest integer algorithm as well as another variation of it (in each case; see [5] for details). These verifications involve elaborate arguments involving "succession rules", which are certain restrictions that hold for the succeeding partial quotient in the expansion.

In [1] we established monotonicity for a variety of algorithms with values in the ring of Gaussian integers, under a general condition. In the following section we extend the idea and introduce a condition on the partial quotients which ensures such monotonicity independent of the algorithm involved, and even of the domain for drawing the partial quotients.

3. Monotonicity of the denominators of the convergents. In this section we describe certain general conditions which ensure that the denominators of the convergents grow monotonically in size, viz. $|q_{n+1}| > |q_n|$ for all $n \geq 0$ in the notation as above.

For $z \in \mathbb{C}$ and $r > 0$ we denote by $B(z,r)$ and $\bar{B}(z,r)$ respectively the open and closed balls with center at z and radius r. We note that if $|z| > r$ then $\bar{B}(z,r) \subset \mathbb{C}^*$ and the sets $B(z,r)^{-1}$ and $\bar{B}(z,r)^{-1}$ (consisting of the inverses of elements from the respective sets) are respectively

$$B\left(\frac{\bar{z}}{|z|^2 - r^2}, \frac{r}{|z|^2 - r^2}\right) \quad \text{and} \quad \bar{B}\left(\frac{\bar{z}}{|z|^2 - r^2}, \frac{r}{|z|^2 - r^2}\right).$$

DEFINITION 3.1. A sequence $\{a_n\}_{n=0}^{\infty}$ in $\mathbb{C}$ is said to satisfy *Condition* $\mathcal{C}$ if $|a_n| > 1$ for all $n \geq 1$, and whenever $|a_{n+1}| < 2$ for some $n \geq 1$ then

$$|(|a_{n+1}|^2 - 1)a_n + \bar{a}_{n+1}| \geq |a_{n+1}|^2.$$

THEOREM 3.2. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$ satisfying Condition $\mathcal{C}$ and let $\{p_n\}, \{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Then $|q_{n+1}| > |q_n|$ for all $n \geq 1$.*

Proof. Suppose, if possible, that there exists $n \geq 1$ such that $|q_{n+1}| \leq |q_n|$, and let $m \geq 1$ be the smallest such number. Thus $|q_{m+1}| \leq |q_m|$ and $|q_{n+1}| > |q_n|$ for $n = 1, \dots, m-1$. In particular $q_n \neq 0$ for $n = 1, \dots, m$. For all $0 \leq n \leq m$ let $r_n = q_{n+1}/q_n$; then $|r_n| > 1$ for $n = 0, 1, \dots, m-1$, and $|r_m| \leq 1$. From the recurrence relations for $\{q_n\}$ we have $r_n = a_{n+1} + r_{n-1}^{-1}$ for all $1 \leq n \leq m$. In particular $r_{m-1}^{-1} \in \bar{B}(-a_{m+1}, |r_m|) \subset \bar{B}(-a_{m+1}, 1)$, and since $|a_{m+1}| > 1$, this implies

$$r_{m-1} \in \bar{B}\left(\frac{-\bar{a}_{m+1}}{|a_{m+1}|^2 - 1}, \frac{1}{|a_{m+1}|^2 - 1}\right).$$

We have $r_{m-1} = a_m + r_{m-2}^{-1}$, and together with the preceding conclusion we get

$$a_m \in \bar{B}\left(\frac{-\bar{a}_{m+1}}{|a_{m+1}|^2 - 1}, |r_{m-2}^{-1}| + \frac{1}{|a_{m+1}|^2 - 1}\right).$$

In turn, since $|r_{m-2}| > 1$, we have

$$a_m \in B\left(\frac{-\bar{a}_{m+1}}{|a_{m+1}|^2 - 1}, 1 + \frac{1}{|a_{m+1}|^2 - 1}\right).$$

Thus

$$|(|a_{m+1}|^2-1)a_m+\bar{a}_{m+1}|<(|a_{m+1}|^2-1)+1=|a_{m+1}|^2.$$

On the other hand, since $r_m=a_{m+1}+r_{m-1}^{-1}$ we have $|a_{m+1}|\le|r_m|+|r_{m-1}^{-1}|<2$. Together with the above conclusion this contradicts the hypothesis. Therefore $|r_n|>1$ for all $n\ge 0$, or equivalently $|q_{n+1}|>|q_n|$ for all $n\ge 0$. ∎

REMARK 3.3. Let Γ be a discrete subring of $\mathbb{C}$ and $f:\mathbb{C}\to\Gamma$ be a Γ-valued algorithm such that the fundamental set of f is contained in a ball of radius $0<r<1$. Let $z\in\mathbb{C}^*\setminus K$, where K is the subfield generated by Γ, and let $\{a_n\}_{n=0}^{\infty}$ be the sequence of partial quotients for z with respect to f, and $\{p_n\},\{q_n\}$ be the $\mathcal{Q}$-pair corresponding to $\{a_n\}_{n=0}^{\infty}$. If $\{a_n\}_{n=0}^{\infty}$ satisfies Condition $\mathcal{C}$, then by Theorem 3.2, $|q_{n+1}|>|q_n|$ for all $n\ge 0$, and by Theorem 2.4, $|z-p_n/q_n|\le c|q_n|^{-2}$ for all $n\ge 0$ with $c=r/(1-r)$. From a Diophantine point of view these are only weak estimates—but seem to be of significance on account of generality of their context. In [5] optimal values for such a constant c are described for continued fraction expansions with respect to the nearest integer algorithms, and also a variation in the case of $\mathbb{Z}\big[\frac{1}{2}+\frac{i}{2}\sqrt{\tau}\big]$, $\tau=3$, 7 or 11. It would be interesting to know similar optimal values for more general algorithms.

REMARK 3.4. Let $\mathfrak{G}=\mathbb{Z}[i]$ denote the ring of Gaussian integers. Let $z\in\mathbb{C}$ and $\{z_n\}_{n=0}^{\infty}$ be an iteration sequence for z such that $a_n=z_n-z_{n+1}^{-1}\in\mathfrak{G}$ for all $n\ge 0$. For $a\in\mathfrak{G}$, we have $1<|a|<2$ if and only if $a=\pm1\pm i$, or equivalently $|a|=\sqrt{2}$. Thus in this case Condition $\mathcal{C}$ reduces to the condition that for all $n\ge 1$, we have $|a_n|>1$ and either $|a_{n+1}|\ge 2$ or $|a_n+\bar{a}_{n+1}|\ge 2$. This corresponds to Condition (H′) in [1], used for obtaining a conclusion as in Theorem 3.2 above; a special case of Theorem 3.2 was obtained in [1, Theorem 6.11], only after proving other results about the asymptotic growth of $|q_n|$'s.

It may also be recalled here that the sequence $\{a_n\}$ obtained by application of the nearest (Gaussian) integer algorithm, starting with a $z\in\mathbb{C}\setminus\mathbb{Q}(i)$, may not satisfy Condition $\mathcal{C}$ (the second part) (see [1, §5] for details). The sequences corresponding to the nearest integer algorithm satisfy a weaker condition, named Condition (H) in [1], which also suffices to obtain the conclusion as in Theorem 3.2; the condition however is rather technical and not amenable to generalization.

In [1] another algorithm, named PPOI (acronym for partially preferring odd integers), was introduced, producing a continued fraction expansion in terms of Gaussian integers for which Condition (H′) is satisfied. We shall however show in the following sections that in the case of the Eisenstein integers the sequences corresponding to the nearest integer algorithm, as also certain other algorithms, satisfy Condition $\mathcal{C}$.

4. Lagrange theorem for continued fractions. In this section we prove an analogue of the classical Lagrange theorem, about the continued fraction expansion being eventually periodic if and only if the number is a quadratic surd. We follow the previous notation.

Let K be a subfield of $\mathbb{C}$. A number $z \in \mathbb{C}$ is called a *quadratic surd* over K if $z \notin K$ and it is a root of a quadratic polynomial over K.

PROPOSITION 4.1. *Let $z \in \mathbb{C} \setminus K$ and $\{z_n\}_{n=0}^{\infty}$ be an iteration sequence for z such that $|z_n| > 1$ for all $n \geq 1$. Let $a_n = z_n - z_{n+1}^{-1}$, $n \geq 0$, be the corresponding sequence of partial quotients and suppose that a_n, $n \geq 0$, are all contained in a discrete subring Γ of $\mathbb{C}$ contained in K. Let $\{p_n\}, \{q_n\}$ be the corresponding $\mathcal{Q}$-pair. If $z_m = z_n$ for some $0 \leq m < n$, then z is a quadratic surd over K.*

Proof. Clearly, for all $m \geq 0$, $\{z_{m+k}\}_{k=0}^{\infty}$ is an iteration sequence for z_m, and z is a quadratic surd if and only if z_m is. Hence we may assume that $z_m = z$, or equivalently that $m = 0$. Let $n \geq 1$ be such that $z_n = z$. By Proposition 2.1 we have

$$(q_{n-1}z - p_{n-1})z_n = (-1)^{n-1}(z_1 \cdots z_n)^{-1} z_n = (q_{n-2}z - p_{n-2}).$$

Since by hypothesis $z_n = z$, we get $q_{n-1}z^2 - (p_{n-1} + q_{n-2})z + p_{n-2} = 0$. Suppose, if possible, that $q_{n-1} = 0$. Then $|p_{n-1}| = |q_{n-1}z - p_{n-1}| = |z_1 \cdots z_n|^{-1} \in (0, 1)$, which is not possible since p_{n-1} is contained in a discrete subring Γ of $\mathbb{C}$. Thus $q_{n-1} \neq 0$, and we see that z satisfies a quadratic polynomial over K. Since $z \notin K$, it follows that z is a quadratic surd over K. ∎

We now prove the following converse of this. The proof follows what is now a standard strategy (cf. [2] for instance) for proving such a result, with variations in the hypothesis; the main purpose here is to bring out a general formulation which at the same time is focused enough and amenable to a brief treatment.

THEOREM 4.2. *Let Γ be a discrete subring of $\mathbb{C}$ and K be the quotient field of Γ. Let z be a quadratic surd over K. Let $\{z_n\}_{n=0}^{\infty}$ be an iteration sequence for z such that the corresponding sequence $\{a_n\}_{n=0}^{\infty}$ of partial quotients is contained in Γ. Let $\{p_n\}, \{q_n\}$ be the $\mathcal{Q}$-pair corresponding to $\{a_n\}_{n=0}^{\infty}$. Suppose that:*

(i) *there exists $\alpha > 0$ such that $|z_n| > 1 + \alpha$ for all $n \geq 1$;*
(ii) *$|q_{n-1}| < |q_n|$ for all $n \geq 1$.*

Then the set $\{\zeta \in \mathbb{C} \mid \zeta = z_n \text{ for some } n\}$ is finite. Consequently, if $\{z_n\}_{n=0}^{\infty}$ is an iteration sequence associated with an algorithm then $\{a_n\}_{n=0}^{\infty}$ is eventually periodic.

Proof. Let $a, b, c \in K$, with $a \neq 0$, be such that $az^2 + bz + c = 0$. Since K is the quotient field of Γ, we may without loss of generality assume that $a, b, c \in \Gamma$. By Proposition 2.1(iii) we have

$$z = \frac{z_{n+1}p_n + p_{n-1}}{z_{n+1}q_n + q_{n-1}} \quad \text{for all } n \geq 0,$$

and hence

$$a\left(\frac{z_{n+1}p_n + p_{n-1}}{z_{n+1}q_n + q_{n-1}}\right)^2 + b\left(\frac{z_{n+1}p_n + p_{n-1}}{z_{n+1}q_n + q_{n-1}}\right) + c = 0.$$

For all $n \geq 0$ let

$$A_n = ap_n^2 + bp_nq_n + cq_n^2, \qquad C_n = A_{n-1},$$
$$B_n = 2ap_np_{n-1} + b(p_nq_{n-1} + q_np_{n-1}) + 2cq_nq_{n-1}.$$

Then $A_n, B_n, C_n \in \Gamma$ for all n, and the above equation can be readily simplified to $A_nz_{n+1}^2 + B_nz_{n+1} + C_n = 0$. The polynomial $a\zeta^2 + b\zeta + c$ has no root in K, and hence it now follows that $A_n \neq 0$ for all n. Now, we have $A_n = (ap_n^2 + bp_nq_n + cq_n^2) - q_n^2(az^2 + bz + c)$, and the latter expression can be rewritten as $(p_n - zq_n)(a(p_n - zq_n) + (2az + b)q_n)$. Under the conditions in the hypothesis, by Proposition 2.2 we have $|q_nz - p_n| \leq \alpha^{-1}|q_n|^{-1}$. Therefore by substitution we get

$$|A_n| \leq \alpha^{-1}|q_n|^{-1}(a\alpha^{-1}|q_n|^{-1} + |2az + b|\,|q_n|) = \alpha^{-1}|2az + b| + \alpha^{-2}a|q_n|^{-2}.$$

Since $|q_n| \geq 1$ for all n, the above observation implies that $\{A_n \mid n \geq 0\}$ is a bounded set, and since $A_n \in \Gamma$ for all n, it further follows that $\{A_n \mid n \geq 0\}$ is finite. Since $C_n = A_{n-1}$ for all $n \geq 1$, $\{C_n \mid n \geq 0\}$ is also finite. An easy computation shows that $B_n^2 - 4A_nC_n = b^2 - 4ac$ for all $n \geq 0$. It follows that $\{A_n\zeta^2 + B_n\zeta + C_n \mid n \geq 0\}$ is a finite collection of polynomials. Since each z_n is a root of one of these polynomials, the set $\{z_n \mid n \geq 0\}$ is finite.

Now suppose that $\{z_n\}_{n=0}^{\infty}$ is an iteration sequence associated with an algorithm. By the first part, there exist $m \geq 0$ and $k \geq 1$ such that $z_{m+k} = z_m$. Since $\{z_n\}_{n=0}^{\infty}$ are determined algorithmically, this implies that $z_{n+k} = z_n$ for all $n \geq m$. In turn we get $a_{n+k} = a_n$ for all $n \geq m$, that is, $\{a_n\}_{n=0}^{\infty}$ is eventually periodic. ∎

The following corollary (together with Proposition 4.1) gives a generalization of the classical Lagrange theorem on quadratic irrationals.

COROLLARY 4.3. *Let Γ be a discrete subring of $\mathbb{C}$ and K be the quotient field of Γ. Let $f : \mathbb{C} \to \Gamma$ be a Γ-valued algorithm such that the fundamental set of f is contained in a ball of radius $0 < r < 1$. Let z be a quadratic surd over K. Let $\{a_n\}_{n=0}^{\infty} \subset \Gamma$ be the sequence of partial quotients with respect to f and let $\{p_n\}, \{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Suppose that $\{a_n\}$ satisfies Condition $\mathcal{C}$. Then $\{a_n\}_{n=0}^{\infty}$ is eventually periodic.*

Proof. This follows from Theorem 4.2: Condition (i) in the theorem is satisfied since the fundamental set of f is contained in a ball of radius $r<1$. Under Condition $\mathcal{C}$, by Theorem 3.2, $\{|q_n|\}$ is strictly increasing, so condition (ii) also holds. ∎

5. Continued fractions for Eisenstein integers. We shall now apply the results of the preceding sections to a class of algorithms with values in the ring $\mathfrak{E}=\{x+y\omega \mid x,y\in\mathbb{Z}\}$ of Eisenstein integers, where ω is a primitive cube root of unity, which we shall realize as $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$. Let $\rho=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (which is a primitive 6th root of unity). Then $\rho=\omega+1$, and every $z\in\mathfrak{E}$ can also be expressed as $x+y\rho$ with $x,y\in\mathbb{Z}$. For convenience we shall also use the notation j for $\sqrt{3}\,i$. Then every $z\in\mathfrak{E}$ can be expressed as $\frac{1}{2}(x+yj)$ with $x+y\in 2\mathbb{Z}$, that is, $x+y$ is an even integer. We shall write the 6th roots of unity as ρ^k with $k\in\mathbb{Z}$, the integer k being understood to be modulo 6.

Given a $\mathfrak{E}$-valued algorithm f we shall denote by Φ_f its fundamental set, and by $C_f(a)$, for $a\in\mathfrak{E}$, the set $\{z\in\mathbb{C}\mid f(z)=a\}$.

THEOREM 5.1. *Let $\mathfrak{E}$ be the ring of Eisenstein integers, let $f:\mathbb{C}\to\mathfrak{E}$ be a $\mathfrak{E}$-valued algorithm and let $\Phi=\Phi_f$. Suppose that:*

(a) $\Phi\subset B(0,r)$ *for some* $0<r<1$;
(b) $|f(\zeta)|>1$ *for all* $\zeta\in\Phi^{-1}$;
(c) *for* $0\le k\le 5$ *and* $t\in\{-1+j,j,1+j\}$, *the sets* $\rho^{-k}t+(C_f(\rho^k j))^{-1}$ *and* $C_f(\rho^{-k}t)\cap\Phi^{-1}$ *are disjoint.*

Let K be the subfield generated by $\mathfrak{E}$. Let $z\in\mathbb{C}\setminus K$, $\{a_n\}_{n=0}^{\infty}$ be the sequence of partial quotients of z corresponding to the algorithm f, and $\{p_n\},\{q_n\}$ be the $\mathcal{Q}$-pair corresponding to $\{a_n\}_{n=0}^{\infty}$. Then:

(i) $|q_n|>|q_{n-1}|$ *for all* $n\ge 1$, *and in particular* $q_n\ne 0$ *for all* n;
(ii) $p_n/q_n\to z$ *as* $n\to\infty$, *and moreover* $|z-p_n/q_n|\le\frac{r}{1-r}|q_n|^{-2}$ *for all* n;
(iii) *z is a quadratic surd over K if and only if $\{a_n\}$ is eventually periodic.*

Proof. Let $\{z_n\}_{n=0}^{\infty}$ denote the corresponding iteration sequence for z with respect to f. From (b) it follows that $|a_n|>1$ for all $n\ge 1$.

We shall show that $\{a_n\}$ satisfies Condition $\mathcal{C}$. For this we first note that for $a\in\mathfrak{E}$, if $1<|a|<2$ then $|a|=\sqrt{3}$, and $a=\rho^k j$ for some $k\in\mathbb{Z}$. Hence we need to show that for $n\ge 1$, if $a_{n+1}=\rho^k j$, $k\in\mathbb{Z}$, then $|2a_n-\rho^{-k}j|\ge 3$.

Let if possible $n\ge 1$ be such that $a_{n+1}=\rho^k j$, $k\in\mathbb{Z}$, and $|2a_n-\rho^{-k}j|<3$. We write a_n as $\frac{1}{2}\rho^{-k}(x+yj)$ with $x+y\in 2\mathbb{Z}$. Then by the above condition we have

$$3>|2a_n-\rho^{-k}j|=|2a_n\rho^k-j|=|x+yj-j|,$$

and hence $x^2+3(y-1)^2 < 9$. Also, since $|a_n| > 1$ we have $x^2+3y^2 \geq 12$. The only common solutions to this, with $x+y$ even, are $x=0$ or ± 2 with $y=2$. Thus $a_n \in \rho^{-k}\{-1+j, j, 1+j\}$. We have $z_n \in \Phi^{-1}$ (as $n \geq 1$), $z_n \in C_f(a_n)$, and also $z_n = a_n + z_{n+1}^{-1} \in a_n + (C_f(\rho^k j))^{-1}$. Since $a_n \in \rho^{-k}\{-1+j, j, 1+j\}$, this contradicts (b) for $t = \rho^k a_n$.

Assertion (i) now follows from Theorem 3.2, and together with condition (a) it implies assertions (ii) and (iii), in view of Theorem 2.4 and Corollary 4.3 respectively. ■

Corollary 5.2. *Let $\mathfrak{E}$ be the ring of Eisenstein integers and let $f : \mathbb{C} \to \mathfrak{E}$ be a $\mathfrak{E}$-valued algorithm such that:*

(a) *$C_f(a)$ is contained in $B(a, \frac{1}{2}(\sqrt{5}-1))$ for all $a \in \mathfrak{E}$;*

(b) *for $0 \leq k \leq 5$, $C_f(\rho^k j) \subset B(\rho^k j, \sqrt{\lambda})$, where $\lambda = \frac{1}{4}(5-\sqrt{13})$.*

Then statements (i)–(iii) *as in Theorem* 5.1 *are satisfied. In particular they are satisfied for the nearest integer algorithm.*

Proof. Condition (a) as in Theorem 5.1 is evidently satisfied for any f as above. We show that (b) and (c) are also satisfied. By (a) we have $\Phi_f \subset B(0,r)$ for $r = \frac{1}{2}(\sqrt{5}-1)$. Hence for $\zeta \in \Phi_f^{-1}$ we have $|\zeta| > r^{-1} = 1+r$, and since $f(\zeta) \in B(\zeta, r)$ this shows that $|f(\zeta)| > 1$, proving (b).

Now let $0 \leq k \leq 5$ and $t \in \{-1+j, j, 1+j\}$. To begin, consider any $r > 0$ such that $C_f(\rho^k j) \subset B(0,r)$; we shall show that condition (c) of Theorem 5.1 holds when $r < \sqrt{\lambda}$. Setting $\sigma = (3-r^2)^{-1}$ (as temporary notation for convenience), we have

$$\rho^{-k}t + (C_f(\rho^k j))^{-1} \subset \rho^{-k}t + B(\rho^k j, r)^{-1} = \rho^{-k}t + \rho^{-k}B(-\sigma j, \sigma r),$$

which is the same as $\rho^{-k}B(t-\sigma j, \sigma r)$. Since Φ_f^{-1} is complementary to $B(0, r^{-1})$, to prove (c) it now suffices to show that $B(t-\sigma j, \sigma r) \subset B(0, r^{-1})$ for all $t \in \{j-1, j, j+1\}$; the condition is now independent of k. For $t = j$ it suffices to note that

$$|t-\sigma j| + \sigma r = (1-\sigma)\sqrt{3} + \sigma r = \sqrt{3} - \sigma(\sqrt{3}-r) = \sqrt{3} - (\sqrt{3}+r)^{-1},$$

on substituting for σ. The last expression is less than r^{-1} when $\sqrt{3}r^2 + r - \sqrt{3} < 0$, viz. if $r < (\sqrt{13}-1)/2\sqrt{3} \approx 0.752\ldots$, so it holds in particular for $r < \sqrt{\lambda} \approx 0.590\ldots$ as in the hypothesis.

It remains to consider the case of $t = \pm 1 + j$, and by symmetry it suffices to consider the case $t = 1+j$. We need to verify that $|1+(1-\sigma)j| + \sigma r < r^{-1}$, or equivalently

$$1 + 3(1-\sigma)^2 < (r^{-1} - r\sigma)^2 = r^{-2}(1-r^2\sigma)^2.$$

Substituting $\sigma = (3-r^2)^{-1}$ and eliminating the denominators we reduce the condition to

$$(3-r^2)^2 r^2 + 3(2-r^2)^2 r^2 - (3-2r^2)^2 < 0.$$

Let $s = r^2$ and $P(s) = 4s^3 - 22s^2 + 33s - 9$; the above expression then coincides with $P(r^2)$. Now we see that

$$P(s) = (s-3)(4s^2 - 10s + 3) = (s-3)(s-\lambda)(s-\mu),$$

where λ is as in the hypothesis and $\mu = \frac{1}{4}(5+\sqrt{13})$ is its quadratic conjugate. Hence $P(s) < 0$ for $s < \lambda$, and so $P(r) < 0$ for $r < \sqrt{\lambda}$. Thus (c) holds, and therefore by Theorem 5.1 assertions (i)–(iii) as in the theorem hold. For the nearest integer algorithm the fundamental set is a regular hexagon contained in $\bar{B}(0, 1/\sqrt{3}) \subset B(0, \sqrt{\lambda})$ and so the assertions hold as a particular case. ∎

Example 5.3. Let P denote the closed parallelogram with vertices at 0, 1, ρ and $1+\rho$. Then $\mathbb{C}$ is tiled by $\{a + P\}_{a\in\mathfrak{E}}$ and it suffices to define the algorithm on each $a + P$, $a \in P$; the points on the boundaries may be assigned a specific tile $a + P$ by some convention. Let $0 < r < 1$ and $V = \{0, 1, \rho, 1 + \rho\}$. Let P_v, $v \in V$, be disjoint subsets of P such that $P_v \subset B(v, r)$ and $P = \bigcup_{v\in V} P_v$. It may be seen that such partitions exist for $r > 1/\sqrt{3}$. Then we can define an algorithm $f : \mathbb{C} \to \mathfrak{E}$ by setting $f(a + \zeta) = a + v$ for any $a \in \mathfrak{E}$ and $\zeta \in P_v$. (The choice of the partition as above may also be made dependent on a.) Then $C_f(a) \subset B(a, r)$ for all $a \in \mathfrak{E}$. If we choose $r \le \frac{1}{2}\sqrt{(5 - \sqrt{13})}$, then the conditions in Corollary 5.2 are satisfied, and therefore the statements in the conclusion hold for such an algorithm.

6. Exponential growth of $\{|q_n|\}$**.** It is known in the case of various algorithms over the ring of Gaussian integers that the sequence $\{|q_n|\}$ increases exponentially (see [1]). We shall show that an analogous assertion also holds in the case of Eisenstein integers. For simplicity we shall restrict to the nearest integer algorithm; extension to some of the algorithms as in the second half of Theorem 5.1 seems feasible but involves some cumbersome computations, which do not seem worthwhile for the present.

Theorem 6.1. *Let $\mathfrak{E}$ be the ring of Eisenstein integers, and K the subfield generated by $\mathfrak{E}$. Let $z \notin K$ and let $\{a_n\}$ be the sequence of partial quotients of z corresponding to the nearest integer algorithm. Let $\{p_n\}, \{q_n\}$ be the $\mathcal{Q}$-pair corresponding to $\{a_n\}$. Then $|q_{n+1}/q_{n-1}| > 3/2$ for all $n \ge 1$.*

We first prove the following.

Proposition 6.2. *Let the notation be as in Theorem 6.1. Then for all $n \ge 1$:*

(i) *if $a_n = j\rho^k$, $k \in \mathbb{Z}$, then $a_{n+1}\rho^k = \frac{1}{2}(x + yj)$ with $x + y \in 2\mathbb{Z}$ such that $\left|\frac{1}{2}x\right| \le 2 - \frac{3}{2}y$;*

(ii) *if $a_n = 2\rho^k$, $k \in \mathbb{Z}$, then $a_{n+1}\rho^k = \frac{1}{2}(x + yj)$ with $x + y \in 2\mathbb{Z}$ and $x \ge -2$.*

Proof. Let $\{z_n\}$ be the iteration sequence of z (with respect to the nearest integer algorithm). Let H be the hexagon with vertices at $\frac{1}{3}\rho^k j$, $0 \le k \le 5$, the fundamental set of the algorithm.

Let $n \ge 1$ be such that $a_n = j\rho^k$ for some k. Since $n \ge 1$ we have $z_n \in H^{-1}$, and so $z_n \notin \bigcup_{m\in\mathbb{Z}} B(\rho^m, 1)$. Hence $z_n - a_n \notin \bigcup_{m\in\mathbb{Z}} B(\rho^m - j\rho^k, 1)$. We have $B(\rho^m - j\rho^k, 1) = \rho^k B(\rho^{m-k} - j, 1)$, and when $m-k = 1$ and 2, we see that $\rho^{m-k} - j = \rho^{-1}$ and ρ^{-2} respectively. Thus in particular $(z_n - a_n)\rho^{-k} \notin B(\rho^{-1}, 1) \cup B(\rho^{-2}, 1)$. Hence

$$z_{n+1}\rho^k = ((z_n - a_n)\rho^{-k})^{-1} \notin B(\rho^{-1}, 1)^{-1} \cup B(\rho^{-2}, 1)^{-1}.$$

The complements of $B(\rho^{-1}, 1)^{-1}$ and $B(\rho^{-2}, 1)^{-1}$ may be seen to be $\{\sigma + \tau i \mid \sigma + \sqrt{3}\,\tau \le 1\}$ and $\{\sigma + \tau i \mid -\sigma + \sqrt{3}\,\tau \le 1\}$ respectively (σ and τ understood to be real). When $z_{n+1}\rho^k$ belongs to the wedge shaped intersection of these two sets, $a_{n+1}\rho^k$ has to belong to the intersection of $\{\sigma + \tau i \in \mathbb{C} \mid \sigma + \sqrt{3}\,\tau \le 2\}$ and $\{\sigma + \tau i \in \mathbb{C} \mid -\sigma + \sqrt{3}\,\tau \le 2\}$. With $a_{n+1}\rho^k$ written as $\frac{1}{2}(x + yj)$, $x + y$ even, this condition yields $\left|\frac{1}{2}x\right| \le 2 - \frac{3}{2}y$, proving (i).

Let $n \ge 1$ be such that $a_n = 2\rho^k$ for some k. Arguing as above we deduce that $z_n - a_n \notin \rho^k B(\rho^{m-k} - 2, 1)$ for any m, and in particular choosing $m = k$ we get $(z_n - a_n)\rho^{-k} \notin B(-1, 1)$. Hence $z_{n+1}\rho^k \notin B(-1, 1)^{-1}$. The complement of $B(-1, 1)^{-1}$ is $\{\sigma + \tau i \mid \sigma \ge -1/2\}$, and we see that when $z_{n+1}\rho^k$ belongs to it, $a_{n+1}\rho^k$ belongs to $\{\sigma + \tau i \mid \sigma \ge -1\}$. Writing $a_{n+1}\rho^k$ as $\frac{1}{2}(x + yj)$, $x + y$ even, we get $x \ge -2$. This proves (ii). ∎

In the proof of Theorem 6.1 we use the following simple observation, which may be of independent interest.

REMARK 6.3. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$ and let $\{p_n\}, \{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Then for all $n \ge 1$ we have

$$q_{n+1} = a_{n+1}q_n + q_{n-1} = a_n a_{n+1} q_{n-1} + a_{n+1} q_{n-2} + q_{n-1},$$

and hence if $|q_{n-2}| \le |q_{n-1}|$ then

$$\left|\frac{q_{n+1}}{q_{n-1}}\right| = \left|a_n a_{n+1} + 1 + a_{n+1}\frac{q_{n-2}}{q_{n-1}}\right| \ge |a_n a_{n+1} + 1| - |a_{n+1}|.$$

Proof of Theorem 6.1. In view of Remark 6.3 it would suffice to show that $|a_n a_{n+1} + 1| > |a_{n+1}| + 3/2$ for all $n \ge 1$. We have

$$|a_n a_{n+1} + 1| - |a_{n+1}| \ge |a_n a_{n+1}| - 1 - |a_{n+1}| = (|a_n| - 1)|a_{n+1}| - 1.$$

If $|a_n| > 2$ then $|a_n| \ge \sqrt{7}$, and since $|a_{n+1}| \ge \sqrt{3}$, we get $|a_n a_{n+1} + 1| - |a_{n+1}| \ge (\sqrt{7} - 1)\sqrt{3} - 1 > 3/2$. It remains to consider the cases $|a_n| = \sqrt{3}$ or 2.

Suppose that $|a_n| = \sqrt{3}$, so $a_n = j\rho^k$ with $k \in \mathbb{Z}$. Then by Proposition 6.2 we have $a_{n+1}\rho^k = \frac{1}{2}(x+yj)$ with $x+y \in 2\mathbb{Z}$ such that $\left|\frac{1}{2}x\right| \le 2 - \frac{3}{2}y$. The last part implies that $y \le 1$, and when $y = 1$ it further implies, together with $x+y$

being even, that $x = \pm 1$, which however is not possible since $|a_{n+1}| \geq \sqrt{3}$. Hence $y \leq 0$. If $y = 0$ then $|x| = 4$ and $|a_n a_{n+1} + 1| = \sqrt{13} > |a_{n+1}| + 3/2$. Now,

$$|a_n a_{n+1} + 1| = \left|j\rho^k \cdot \tfrac{1}{2}(x + yj)\rho^{-k} + 1\right| = \left|\tfrac{1}{2}(xj - 3y) + 1\right|.$$

Therefore for $y \leq -1$ we have

$$|a_n a_{n+1} + 1|^2 = \tfrac{1}{4}\{3x^2 + (2 - 3y)^2\} \geq \tfrac{3}{4}(x^2 + 3y^2) + 4 = 3|a_{n+1}|^2 + 4.$$

We note that $3|a_{n+1}|^2 + 4 \geq (|a_{n+1}| + \sqrt{8/3})^2$, as may be seen by considering the discriminant of the quadratic difference expression. Thus

$$|a_n a_{n+1} + 1| \geq |a_{n+1}| + \sqrt{8/3} > |a_{n+1}| + 3/2,$$

which settles the case at hand.

Now suppose that $|a_n| = 2$, so $a_n = 2\rho^k$ for some $k \in \mathbb{Z}$. Then by Proposition 6.2 we have $a_{n+1}\rho^k = \frac{1}{2}(x + yj)$ with $x + y \in 2\mathbb{Z}$ and $x \geq -2$. Hence $|a_n a_{n+1} + 1| = |x + yj + 1|$. Suppose first that $x \geq 0$. Then

$$\begin{aligned}|a_n a_{n+1} + 1|^2 &= |x + yj + 1|^2 \\ &= (x + 1)^2 + 3y^2 > 4\left|\tfrac{1}{2}(x + yj)\right|^2 = 4|a_{n+1}|^2.\end{aligned}$$

Hence $|a_n a_{n+1} + 1| - |a_{n+1}| \geq |a_{n+1}| \geq \sqrt{3} > 3/2$, as desired.

The only possibilities that remain are $x = -2$ or -1. We note that since $x^2 + 3y^2 \geq 12$, if $x = -2$ then $|y| \geq 2$ and if $x = -1$ then $|y| \geq 3$. Now, $|a_n a_{n+1} + 1|^2 = |x + yj + 1|^2 = (x + 1)^2 + 3y^2 < 4y^2$ and $|a_{n+1}|^2 = |\frac{1}{2}(x + yj)|^2 = \frac{1}{4}x^2 + \frac{3}{4}y^2 \leq y^2$. Hence $|a_n a_{n+1} + 1| + |a_{n+1}| < 3|y|$.

Suppose $x = -2$. Then $|a_n a_{n+1} + 1|^2 - |a_{n+1}|^2 = 1 + 3y^2 - \frac{1}{4}(4 + 3y^2) = \frac{9}{4}y^2$, and dividing by the expression estimated above we get $|a_n a_{n+1} + 1| - |a_{n+1}| > \frac{3}{4}|y| \geq \frac{3}{2}$, since $|y| \geq 2$, as desired.

Finally suppose $x = -1$. Then $|a_n a_{n+1} + 1|^2 - |a_{n+1}|^2 = 3y^2 - \frac{1}{4}(1 + 3y^2) > 2y^2$, and hence $|a_n a_{n+1} + 1| - |a_{n+1}| > \frac{2}{3}|y| \geq 2$, since $|y| \geq 3$ in this case. Thus $|a_n a_{n+1} + 1| - |a_{n+1}| > 3/2$ in this case too. This proves the theorem. ∎

Remark 6.4. The constant 3/2 involved in Theorem 6.1 is not optimal; it can be improved upon with some detailed computations in various special cases. We shall however not concern ourselves here with improving it.

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