FUNDAMENTA
MATHEMATICAE
231 (2015)

Jumps of entropy for C^r interval maps

by

David Burguet (Paris)

Abstract. We study the jumps of topological entropy for C^r interval or circle maps. We prove in particular that the topological entropy is continuous at any $f \in C^r([0,1])$ with $h_{\mathrm{top}}(f) > \frac{\log^+ \|f'\|_\infty}{r}$. To this end we study the continuity of the entropy of the Buzzi–Hofbauer diagrams associated to C^r interval maps.

1. Introduction. In this paper we study the upper semicontinuity of the (topological) entropy in the spaces $C^r([0,1])$ and $C^r(\mathbb{S}^1)$ of C^r interval maps and circle maps endowed with the usual C^r topology where r is a real number with $1 \leq r \leq \infty$. This problem has been investigated in several previous works in different settings [19], [21], [29]. Let us recall the main related results.

Lower semicontinuity of the entropy was proved by M. Misiurewicz and W. Szlenk [24] for interval maps in the C^0 topology and by A. Katok [16] for $C^{1+\alpha}$ surface diffeomorphisms in the C^1 topology. In both cases this follows from the characterization of entropy by horseshoes which are persistent in the above mentioned topologies. In dimension larger than two the entropy may not be lower semicontinuous even in the C^∞ topology [19].

In [21] Misiurewicz investigated upper semicontinuity of the entropy for continuous piecewise monotone maps of the interval. Let $\mathcal{M}_k^r([0,1])$, with $r = 0$ or 1, be the set of C^r interval maps f which admit a partition of $[0,1]$ into k intervals such that f is weakly monotone on each element of this partition.

We say $x \in [0,1]$ is a *turning point* of an interval map f when there exist $0 \leq a < b \leq x \leq c < d \leq 1$ such that f is constant on $[b,c]$ and strictly monotone both on $[a,b]$ and $[c,d]$ but in the opposite sense.

Misiurewicz proved upper semicontinuity (and thus continuity) of entropy in $\mathcal{M}_2^0([0,1])$ in the C^0 topology at all maps at which it is positive.

2010 *Mathematics Subject Classification*: 37A35, 37C05, 37B10, 37B40.
Key words and phrases: entropy, smooth interval maps, Buzzi–Hofbauer diagram.

DOI: 10.4064/fm231-3-5

For all k he also gave a complete description of the possible jumps of entropy in $\mathcal{M}_k^0([0,1])$ at any $f \in \mathcal{M}_k^0([0,1]) \setminus \bigcup_{l=1}^{k-1} \mathcal{M}_l^0([0,1])$:

$$\limsup_{g \xrightarrow{C^0} f,\, g \in \mathcal{M}_k^0([0,1]} h_{\mathrm{top}}(g) = \max(h_{\mathrm{top}}(f), \beta(f)),$$

with

$$\beta(f) := \max\left\{\tfrac{p}{q}\log 2 : \text{there exists a periodic point of } f \text{ of period } q \text{ with } p\ (\leq q) \text{ turning points in its orbit}\right\}.$$

Moreover, Misiurewicz proved continuity of entropy for the C^1 topology for C^1 piecewise monotone maps with a uniform number of pieces [23], i.e. for any positive integer k and for any $f \in \mathcal{M}_k^1([0,1])$ we have

$$\limsup_{g \xrightarrow{C^1} f,\, g \in \mathcal{M}_k^1([0,1]} h_{\mathrm{top}}(g) = h_{\mathrm{top}}(f), \tag{1.1}$$

Axiom A interval maps are open and dense in $C^r([0,1])$ with $r \geq 1$ [17]. We recall that a C^1 interval map f is said to be *Axiom A* if all periodic points are hyperbolic and, with $B(f)$ denoting the union of the basins of attracting periodic points of f, the set $[0,1] \setminus B(f)$ is hyperbolic, that is, there are constants $C > 0$ and $\lambda > 1$ such that $|(f^k)'(x)| \geq C\lambda^k$ for all $x \in [0,1] \setminus B(f)$ and $k \in \mathbb{N}$. By structural stability, entropy is locally constant, hence continuous, on the set of Axiom A interval maps.

For $r > 1$ the set $D^r([0,1])$ of C^r interval maps with no critical point flat up to order r is also open and dense in $C^r([0,1])$ and neither contains the set of Axiom A maps nor is contained in it. The entropy is also continuous on $D^r([0,1])$. This result is due to R. Bowen [1] and Misiurewicz–Szlenk [24] for $r = 2$ and to K. Iwai [15] for larger r (the latter proof is based on a variant of the kneading theory of Milnor and Thurston). We show in the Appendix that it is in fact a direct consequence of Misiurewicz's result (1.1).

Upper semicontinuity was also established by Y. Yomdin [29] for the C^∞ topology on any compact manifold M. In fact he bounds the default of upper semicontinuity of the entropy for C^r maps, $1 \leq r \leq \infty$, as follows:

$$\limsup_{g \xrightarrow{C^r} f} h_{\mathrm{top}}(g) \leq h_{\mathrm{top}}(f) + \frac{d \cdot R(f)}{r}, \tag{1.2}$$

where d is the dimension of M and

$$R(f) := \lim_n \frac{\log^+ \|Df^n\|_\infty}{n}$$

(note that R is upper semicontinuous in the C^1 topology as the limit of a subadditive sequence of continuous functions, i.e. $\limsup_{g \xrightarrow{C^1} f} R(g) \leq R(f)$).

Earlier Misiurewicz [22] proved that upper semicontinuity of the entropy fails for diffeomorphisms in the C^r topology with finite r in dimension larger than or equal to three . For interval maps [6] and for surface diffeomorphisms [7] the only known C^r examples at which the entropy is not upper semicontinuous all satisfy $h_{\text{top}}(f) < R(f)/r$. We prove in this paper that this is always the case in dimension one.

Main Theorem. *Let f be a C^r interval or circle map with $1 \leq r \leq \infty$. Then*

$$\limsup_{g \xrightarrow{C^r} f} h_{\text{top}}(g) \leq \max(h_{\text{top}}(f), R(f)/r).$$

In fact we will prove a stronger statement where the lim sup is taken over g going to f in the C^1 topology and staying in a C^r bounded set (Yomdin's inequality (1.2) also holds true in this setting).

Obviously the statement of the Main Theorem for arbitrarily large r implies the same for $r = \infty$ and we recover in this last case the upper semicontinuity of entropy in the C^∞ topology proved by Y. Yomdin. Note also that the above theorem is trivial for $r = 1$ as $h_{\text{top}}(f)$ is always less than or equal to $dR(f)$ for any C^1 dynamical system on a compact manifold of dimension d.

We conjecture that the Main Theorem should also hold true for surface diffeomorphisms.

Let f be a C^r interval map, $1 < r < \infty$, and let p be a repelling periodic point with period T. Here f may be non-invertible, and the unstable manifold $W^u(\mathcal{O}(p))$ of the orbit $\mathcal{O}(p)$ of p is then defined as the set of points x such that there exists an infinite backward orbit $(x_k)_{k\leq 0}$ through x, i.e. $x_{k+1} = f(x_k)$ for any $k < 0$ and $x = x_l$ for some $l \leq 0$, such that x_{kT} goes to p when k goes to $-\infty$. We say that f has *homoclinic tangency of order r at p* if there exists a critical point $c \in [0, 1]$ flat up to order r, i.e. $f(x) - f(c) = o((x - c)^r)$, such that $c \in W^u(\mathcal{O}(p))$ and $f^k(c) = p$ for some $k > 0$. In dimension one the stable manifold at a repelling periodic point is in general zero-dimensional, and homoclinic tangency may be geometrically interpreted as a point of intersection of the stable and unstable manifold at which the graph of the interval map is tangent to the horizontal axis. Observe that homoclinic tangency of order r is of order s for any $s \leq r$.

Proposition 1.1. *Let f be a C^r interval map, $1 < r < \infty$, with homoclinic tangency of order r at a repelling periodic point p. Then*

$$\limsup_{g \xrightarrow{C^r} f} h_{\text{top}}(g) \geq \max(h_{\text{top}}(f), \lambda(p)/r),$$

where $\lambda(p)$ is the Lyapunov exponent at p. When moreover $\lambda(p) = R(f)$, we *have*

$$\limsup_{g \xrightarrow{C^r} f} h_{\mathrm{top}}(g) = \max(h_{\mathrm{top}}(f), \lambda(p)/r).$$

Note that $R(f)$ is the maximum of the Lyapunov exponents of all invariant measures (and zero). Indeed, firstly the Lyapunov exponent $\lambda(\mu)$ of an f-invariant measure μ satisfies

$$\lambda(\mu) = \int \log|f'|\,d\mu = \int \lim_n \frac{\log|(f^n)'|}{n}\,d\mu \leq R(f).$$

For the converse inequality we may assume $R(f) > 0$. If x_n is such that $|(f^n)'|(x_n) = \|(f^n)'\|_\infty > 1$, we let ν_n be the atomic measure given by

$$\nu_n := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x_n}.$$

Clearly we have

$$\int \log|f'|\,d\nu_n = \frac{\log|(f^n)'|(x_n)}{n}$$

and then any limit ν of $(\nu_n)_n$ in the weak-$*$ topology is f-invariant and satisfies, by upper semicontinuity,

$$\int \log|f'|\,d\nu \geq \lim_n \int \log|f'|\,d\nu_n = R(f).$$

Question 1.2. *Let f be a discontinuity point of entropy in $C^r([0,1])$ with $1 < r < \infty$. Does there exist $f_n \in C^r([0,1])$ with homoclinic tangency of order r at a repelling periodic point p_{f_n} and going to f in the C^r topology when n goes to infinity? Do we have moreover*

$$\limsup_{g \xrightarrow{C^r} f} h_{\mathrm{top}}(g) = \limsup_n \lambda(p_{f_n})/r?$$

The proof of the Main Theorem is based on the study of the Buzzi–Hofbauer diagram and its behaviour under C^r perturbations. This diagram is introduced in Section 3. In Section 2 we investigate the upper semicontinuity of the entropy of topological Markov shifts with countable state sets. The Main Theorem then follows by applying the previous abstract framework to the Buzzi–Hofbauer diagram and by applying the strategy developed in [8], [4] (to prove existence and finiteness of measures of maximal entropy for C^r interval maps f with $h_{\mathrm{top}}(f) > R(f)/r$). Finally we sketch the proof of Proposition 1.1 in the last section, which is based on a classical construction of arbitrarily small horseshoes near a homoclinic tangency by C^r perturbations.

2. Continuity of the entropy of topological Markov shifts with countable state sets. In this section we introduce a notion of convergence for countable Markov shifts and analyze the case of lack of upper semicontinuity of Gurevich entropy. We first recall standard terminology.

Let $\mathcal{G}$ be an oriented graph with a countable set of vertices $V(\mathcal{G})$. For $u, v \in V(\mathcal{G})$, we use the notation $u \to v$ when there is an oriented arrow from u to v. A *closed path* at a vertex u is a sequence of vertices $(u_1, \ldots, u_{p+1})$ with $u_1 = u_{p+1} = u$ and with $u_i \to u_{i+1}$ for $i = 1, \ldots, p$. The integer p is called the *length* of the closed path. The *period* $p(\mathcal{G})$ of the graph $\mathcal{G}$ is the greatest common divisor of the lengths of its closed paths. A closed path $(u_1, \ldots, u_{p+1})$ at u is said to be a *first return* at u if $u_i \neq u$ for $i \neq 1, p+1$. Any closed path γ at u is a concatenation of first returns $(\gamma_i)_{i=1,\ldots,j}$ at u and we write

$$\gamma = \gamma_1 * \cdots * \gamma_j. \tag{2.1}$$

In other words, if $\gamma = (u_1, \ldots, u_{p+1})$ then there exist $1 = k_1 < \cdots < k_i < \cdots < k_{j+1} = p + 1$ such that $\gamma_i = (u_{k_i}, \ldots, u_{k_{i+1}})$ are first returns at u. A graph $\mathcal{G}$ is said to be *admissible* when for any $M \in \mathbb{N}$ and for any vertex $u \in V(\mathcal{G})$ the number of closed paths at u of length M is finite.

We let $\Delta_p(\mathcal{G})$ (resp. $\Delta_p^u(\mathcal{G})$) be the set of closed paths of length p in $\mathcal{G}$ (resp. at u). For any positive integer M and for any $u \in V(\mathcal{G})$ we let $\Delta_{p,M}^u(\mathcal{G})$ be the set of $\gamma \in \Delta_p^u(\mathcal{G})$ such that all first returns γ_i at u appearing in the decomposition (2.1) of γ in $\mathcal{G}$ have length at most M.

Consider $\Sigma(\mathcal{G}) := \{(v_n)_n \in \mathcal{G}^{\mathbb{Z}} : \forall n \in \mathbb{Z}, v_n \to v_{n+1}\}$. The *Markov shift* on $\mathcal{G}$ is the shift $\sigma((v_n)_n) = (v_{n+1})_n$ on $\Sigma(\mathcal{G})$. Obviously there is a correspondence between p-periodic points of $\Sigma(\mathcal{G})$ and closed paths of length p in $\mathcal{G}$: to any closed path $\gamma = (u_1, \ldots, u_{p+1})$ in $\mathcal{G}$ of length p corresponds the p-periodic point $\tilde{\gamma} = (v_n)_n$ of $\Sigma(\mathcal{G})$ with $v_n = u_{n \,(\mathrm{mod}\, p)+1}$ for all n.

If F is a subset of vertices of the graph $\mathcal{G}$ we write $[F] := \{(v_n)_n \in \Sigma(\mathcal{G}) : v_0 \in F\}$. We also let $\mathcal{M}(\Sigma(\mathcal{G}))$ be the set of invariant Borel probability measures on $\Sigma(\mathcal{G})$.

2.1. A notion of convergence. We consider a set E and a family $\underline{E} = (E_L)_{L \in \mathcal{L}}$ of subsets with $E = \bigcup_L E_L$.

A family $(\mathcal{G}_i)_{i \in I}$ of oriented admissible graphs with vertices in E is said to be *uniform* with respect to $\underline{E}$ when for any $L \in \mathcal{L}$ we have

$$\sup_{i \in I} \sharp V(\mathcal{G}_i) \cap E_L < \infty. \tag{2.2}$$

Definition 2.1. A sequence $(\mathcal{G}_n)_{n \in \mathbb{N}}$ of graphs uniform with respect to $\underline{E}$ *converges* to a graph $\mathcal{G}$ when $\forall L\ \forall M\ \exists n_0\ \forall n > n_0$ we have

$$\forall u^n \in V(\mathcal{G}_n) \cap E_L\ \exists u \in V(\mathcal{G}) \text{ such that } \forall p,\ \sharp \Delta_{p,M}^{u^n}(\mathcal{G}_n) \leq \sharp \Delta_p^u(\mathcal{G}).$$

Clearly $(\mathcal{G}_n)_{n \in \mathbb{N}}$ does not determine $\mathcal{G}$ uniquely. For example, if we add some edges to $\mathcal{G}$, then $(\mathcal{G}_n)_n$ also converges to the resulting graph.

Definition 2.2. Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of graphs uniform with respect to $\underline{E}$. We say $(\xi_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{M}(\Sigma(\mathcal{G}_n))$ *goes to infinity* when for any $L \in \mathcal{L}$ we have $\lim_n \xi_n([E_L]) = 0$.

As $\sup_n \sharp V(\mathcal{G}_n) \cap E_L < \infty$ for any L by (2.2), the sequence $(\xi_n)_{n\in\mathbb{N}}$ goes to infinity when for any L the sequence $(\sup_{u^n \in V(\mathcal{G}_n)\cap E_L} \xi_n([\{u^n\}]))_n$ goes to zero.

2.2. Entropy and measure of maximal entropy. The shift space $\Sigma(\mathcal{G})$ is a priori not compact. Following B. Gurevich we define the entropy $h(\mathcal{G})$ as the supremum of $h(\sigma, \xi)$ over all σ-invariant probability measures ξ. A σ-invariant measure ξ is said to be *maximal* if $h(\sigma, \xi) = h(\mathcal{G})$. Such a measure does not always exist. In the following we consider a converging sequence $(\mathcal{G}_n)_n$ of graphs with $\lim_n h(\mathcal{G}_n) > h(\mathcal{G})$. We do not assume that the graphs $\mathcal{G}_n$ or $\mathcal{G}$ admit maximal measures.

We will use the following theorem of Gurevich which enables us to work with finite connected [1] graphs.

Theorem 2.3 ([12, Corollary 1.7]). *Let $\mathcal{G}$ be an oriented graph with a countable set of vertices. Then*

$$h(\mathcal{G}) = \sup_{\mathcal{G}_0} h(\mathcal{G}_0),$$

where $\mathcal{G}_0$ ranges over all finite connected subgraphs of $\mathcal{G}$.

A finite connected graph admits a unique measure of maximal entropy (the so-called *Parry measure*). We now recall the characterization of the Parry measure: periodic orbits equidistribute along this measure.

Theorem 2.4 ([2]). *Let $\mathcal{G}_0$ be a finite connected graph with $h(\mathcal{G}_0) > 0$. Then periodic points equidistribute along the unique maximal measure μ of $(\Sigma(\mathcal{G}_0), \sigma)$, i.e.*

$$\frac{1}{\sharp \Delta_p(\mathcal{G}_0)} \sum_{\gamma \in \Delta_p(\mathcal{G}_0)} \delta_{\tilde{\gamma}} \xrightarrow{p:\, p(\mathcal{G}_0)|p} \mu,$$

where $\delta_{\tilde{\gamma}}$ denotes the Dirac measure at the periodic point $\tilde{\gamma} \in \Sigma(\mathcal{G}_0)$.

Finally, we recall that the Gurevich entropy of a connected oriented graph with a countable set of vertices may be written as the exponential growth rate in p of the number of closed paths of length p at a given vertex u:

Theorem 2.5 ([13]). *Let $\mathcal{G}$ be a connected oriented graph with a countable set of vertices. Then for any $u \in V(\mathcal{G})$,*

$$h(\mathcal{G}) = \lim_{p:\, p(\mathcal{G})|p} \frac{1}{p} \log \sharp \Delta_p^u(\mathcal{G}).$$

[1] A graph is said to be *connected* when any pair of vertices may be joined by a path.

As the Gurevich entropy of an oriented graph $\mathcal{G}$ with a countable set of vertices is the supremum of the Gurevich entropies of its connected components, for any $u \in V(\mathcal{G})$ we always have

$$h(\mathcal{G}) \geq \lim_{p:\, p(\mathcal{G})|p} \frac{1}{p} \log \sharp \Delta_p^u(\mathcal{G}). \tag{2.3}$$

2.3. Main proposition

PROPOSITION 2.6. *Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of graphs uniform with respect to $\underline{E}$ converging to a graph $\mathcal{G}$. Let $(\mathcal{G}'_n)_{n\in\mathbb{N}}$ be a sequence of finite connected graphs with $\mathcal{G}'_n \subset \mathcal{G}_n$ for all n and $\lim_n h(\mathcal{G}'_n) > h(\mathcal{G})$. Let $\mu_n \in \mathcal{M}(\Sigma(\mathcal{G}'_n)) \subset \mathcal{M}(\Sigma(\mathcal{G}_n))$ be the maximal measure of $\Sigma(\mathcal{G}'_n)$. Then $(\mu_n)_{n\in\mathbb{N}}$ goes to infinity.*

Proof. Fix $L\in\mathcal{L}$. Let $\epsilon>0$. Let $\delta>0$ be such that $\lim_n h(\mathcal{G}'_n) > h(\mathcal{G}) + \delta$. We fix M large enough so that $1/M<\epsilon/2$ and $\lim_p e^{-\epsilon\delta p/4} \sum_{0\leq k\leq p/M} \binom{p}{k} = 0$ (it is well known that $\limsup_p \frac{\log \binom{p}{[\alpha p]}}{p} \to 0$ as $\alpha \to 0$). By convergence of $\mathcal{G}_n$ to $\mathcal{G}$ we may fix n_0 large enough such that for $n > n_0$, there exists for all $u^n \in V(\mathcal{G}_n) \cap E_L$ a vertex $u \in V(\mathcal{G})$ such that for all integers p,

$$\sharp \Delta_{p,M}^{u^n}(\mathcal{G}_n) \leq \sharp \Delta_p^u(\mathcal{G}). \tag{2.4}$$

We can also assume that $h(\mathcal{G}'_n) > h(\mathcal{G}) + \delta$ for $n > n_0$. We fix $n > n_0$ and we now prove $\mu_n([\{u^n\}]) \leq \epsilon$ for any $u^n \in V(\mathcal{G}_n) \cap E_L$.

As $[\{u^n\}]$ is a clopen (possibly empty) set in $\Sigma(\mathcal{G}'_n)$, by Theorem 2.4 we have

$$\mu_n([\{u^n\}]) = \lim_{p:\, p(\mathcal{G}'_n)|p} \frac{1}{\sharp \Delta_p(\mathcal{G}'_n)} \sum_{\gamma\in\Delta_p(\mathcal{G}'_n)} \delta_{\tilde{\gamma}}([\{u^n\}]) = \lim_{p:\, p(\mathcal{G}'_n)|p} \frac{\sharp \Delta_p^{u^n}(\mathcal{G}'_n)}{\sharp \Delta_p(\mathcal{G}'_n)}.$$

The number of returns at u^n in a closed path $\gamma = (u^n = u_1, u_2, \ldots, u_{p+1} = u^n)$ of length p at u^n will be denoted by $r(\gamma)$, i.e. $r(\gamma) := \{1 \leq k \leq p : u_k = u^n\}$. It is also the number of closed paths at u^n defining the same periodic orbit in $\Sigma(\mathcal{G}'_n)$. We let $\tilde{\Delta}_p^{u^n}(\mathcal{G}'_n)$ be the subset of closed paths $\gamma \in \Delta_p^{u^n}(\mathcal{G}'_n)$ such that the minimal period of $\tilde{\gamma} \in \Sigma(\mathcal{G}'_n)$ is p. When $\gamma \in \tilde{\Delta}_p^{u^n}(\mathcal{G}'_n)$ there are p distinct closed paths in $\Delta_p(\mathcal{G}'_n)$ (in general not at the vertex u^n) defining the same periodic orbit in $\Sigma(\mathcal{G}'_n)$. Therefore we have, for any p,

$$\frac{\sharp\{\gamma \in \tilde{\Delta}_p^{u^n}(\mathcal{G}'_n) : r(\gamma) < p\epsilon\}}{\sharp \Delta_p(\mathcal{G}'_n)} < \epsilon.$$

It follows immediately from Theorem 2.5 and $h(\mathcal{G}'_n) > 0$ that

$$\lim_{p:\, p(\mathcal{G}'_n)|p} \frac{\sharp(\Delta_p^{u^n}(\mathcal{G}'_n) \setminus \tilde{\Delta}_p^{u^n}(\mathcal{G}'_n))}{\sharp \Delta_p(\mathcal{G}'_n)} \leq \lim_{p:\, p(\mathcal{G}'_n)|p} \frac{\sum_{q|p,\, q\neq p} \sharp \Delta_q^{u^n}(\mathcal{G}'_n)}{\sharp \Delta_p(\mathcal{G}'_n)} = 0.$$

To deduce $\mu_n([\{u^n\}]) \le \epsilon$ (recall n is fixed), it is enough to show that for large p the cardinality of the set of closed paths γ of length p at u^n with $r(\gamma) \ge p\epsilon$ is less than $e^{p(h(\mathcal{G}'_n)-\epsilon\delta/8)}$, because the cardinality of $\Delta_p(\mathcal{G}'_n)$ grows exponentially faster in p by Theorem 2.5 and thus we will finally get

$$\mu_n([\{u^n\}]) = \lim_{p:\,p(\mathcal{G}'_n)|p} \frac{\sharp\{\gamma \in \tilde{\Delta}_p^{u^n}(\mathcal{G}'_n) : r(\gamma) < p\epsilon\}}{\sharp\Delta_p(\mathcal{G}'_n)} \le \epsilon.$$

Therefore Proposition 2.6 will be proved once we show

Claim. *There exists P (depending on n) such that for all $p > P$,*

$$\sharp\{\gamma \in \Delta_p^{u^n}(\mathcal{G}'_n) : r(\gamma) \ge p\epsilon\} < e^{p(h(\mathcal{G}'_n)-\epsilon\delta/8)}.$$

Proof. For $\gamma \in \Delta_p^{u^n}(\mathcal{G}'_n)$ we let γ^- and γ^+ be the closed paths at u obtained by concatenating the first returns at u^n appearing in γ of length less than or equal to M and larger than M, respectively (recall M was fixed earlier and depends only on ϵ and δ). More precisely if $\gamma = \gamma_1 * \cdots * \gamma_j$ is the writing (2.1) of γ in first returns at u, then we let $\gamma^- = \gamma_{i_1} * \cdots * \gamma_{i_k}$ where $\{i_1 < \cdots < i_k\}$ is the set of integers $i \in [1,j]$ such that the length of γ_i is at most M, and we define γ^+ similarly. For $\gamma = (u^n = u_1, u_2, \ldots, u_{p+1} = u^n) \in \Delta_p^{u^n}(\mathcal{G}'_n)$ we also let $i(\gamma)$ be the set of integers $k \in [1,p]$ such that there exists a first return at $u_k = u^n$ of length larger than M. Note that the cardinality of $i(\gamma)$ is no more than $p/M < p\epsilon/2$.

We let $\mathcal{P}_p^l$ be the set of all subsets of $\{1,\ldots,p\}$ whose cardinality is less than or equal to l. We consider the map $\Phi : \Delta_p^{u^n}(\mathcal{G}'_n) \to \bigcup_{0\le q\le p} \Delta_{q,M}^{u^n}(\mathcal{G}_n) \times \Delta_{p-q}^{u^n}(\mathcal{G}'_n) \times \mathcal{P}_p^{p/M}$ which maps any $\gamma \in \Delta_p^{u^n}(\mathcal{G}'_n)$ to the triple $(\gamma^-, \gamma^+, i(\gamma))$. Clearly this map is injective. Now for any $\gamma \in \Delta_p^{u^n}(\mathcal{G}'_n)$ the length of γ^- is at least $r(\gamma) - \sharp i(\gamma)$. By (2.4) it follows that, for all p,

$$\begin{aligned}\sharp\{\gamma \in \Delta_p^{u^n}(\mathcal{G}'_n) : r(\gamma) \ge p\epsilon\} &\le \sum_{p\ge q>p\epsilon/2} \sharp\{\Delta_{q,M}^{u^n}(\mathcal{G}_n) \times \Delta_{p-q}^{u^n}(\mathcal{G}'_n) \times \mathcal{P}_p^{p/M}\} \\ &\le \sum_{p\ge q>p\epsilon/2} \sharp\Delta_q^u(\mathcal{G}) \times \sharp\Delta_{p-q}^{u^n}(\mathcal{G}'_n) \times \sharp\mathcal{P}_p^{p/M},\end{aligned}$$

and then for large p (depending on the fixed n) we finally obtain, by (2.3),

$$\begin{aligned}\sharp\{\gamma \in \Delta_p^{u^n}(\mathcal{G}'_n) : r(\gamma) \ge p\epsilon\} &\le e^{\epsilon\delta p/4} \sum_{p\ge q>p\epsilon/2} \sharp\Delta_q^u(\mathcal{G}) \times \sharp\Delta_{p-q}^{u^n}(\mathcal{G}'_n) \\ &\le e^{\epsilon\delta p/3} \sum_{p\ge q>p\epsilon/2} e^{qh(\mathcal{G})+(p-q)h(\mathcal{G}'_n)} \\ &< e^{p(h(\mathcal{G}'_n)-\epsilon\delta/8)}.\end{aligned}$$

The Claim is established. ∎

As previously mentioned, this concludes the proof of Proposition 2.6. ∎

3. Proof of the Main Theorem via the Buzzi–Hofbauer diagram

3.1. Symbolic dynamics associated to natural partitions and the Hofbauer–Markov diagram. We consider a C^1 interval map f. Let $C(f)$ be the *critical set* of f, i.e. the set of vanishing points of the derivative f'. A *monotone* (resp. *strictly monotone*) branch of f is an open interval I such that $f|_I$ is monotone (resp. strictly monotone). We say I is a *critical monotone branch* if I is a monotone branch (not necessarily strictly) and the two boundary points of I belong to $C(f) \cup \{0,1\}$. A (countable) collection P of disjoint critical monotone branches is called a *natural partition* of f when the union of all monotone branches in P covers any strictly monotone branch of f, i.e. $\{f' \neq 0\} \subset \bigcup_{I \in P} I$.

For a natural partition P of f the two-sided *symbolic dynamic* $\Sigma(f,P)$ associated to f is defined as the shift on the closure in $P^{\mathbb{Z}}$ (for the product topology) of the two-sided sequences $A = (A_n)_n$ such that for all $n \in \mathbb{Z}$ and $l \in \mathbb{N}$ the word $A_n \dots A_{n+l}$ is *admissible*, by which we mean that $\bigcap_{k=0}^{l} f^{-k} A_{n+k}$ is non-empty and the f^{l+1}-image of this open interval is not reduced to a point. The *follower set* of a finite P-word $B_n \dots B_{n+l}$ is $\mathrm{fol}(B_n \dots B_{n+l}) := \{A_{n+l} A_{n+l+1} \dots \in P^{\mathbb{N}} : \exists (A_n) \in \Sigma(f,P) \text{ with } A_n \dots A_{n+l} = B_n \dots B_{n+l}\}$.

Let $\mathcal{P}$ be the set of admissible P-words. We consider the following equivalence relation on $\mathcal{P}$. We say $A_{-n} \dots A_0 \sim B_{-m} \dots B_0$ if there exist $0 \le k \le \min(m,n)$ such that:

- $A_{-k} \dots A_0 = B_{-k} \dots B_0$;
- $\mathrm{fol}(A_{-n} \dots A_0) = \mathrm{fol}(A_{-k} \dots A_0)$;
- $\mathrm{fol}(B_{-m} \dots A_0) = \mathrm{fol}(B_{-k} \dots A_0)$.

We endow the quotient space $\mathcal{D} = \mathcal{D}(f,P) := \mathcal{P}/\!\sim$ with a structure of oriented graph, known as the *Buzzi–Hofbauer diagram*, in the following way [6]: there exists an oriented arrow $\alpha \to \beta$ between two elements α, β of $\mathcal{D}$ if and only if there exists an integer n and $A_{-n} \dots A_0 A_1 \in \mathcal{P}$ such that $\alpha \sim A_{-n} \dots A_0$ and $\beta \sim A_{-n} \dots A_0 A_1$.

The *significant part* of $\alpha \in \mathcal{D}$ is the representative $A_{-n_\alpha} \dots A_0$ of α with the shortest length. Such a word $A_{-n_\alpha} \dots A_0$ is also called *irreducible*; it is the shortest element in its class. In particular $\mathrm{fol}(A_{-n_\alpha} \dots A_0) \neq \mathrm{fol}(A_{-n_\alpha+1} \dots A_0)$ when $n_\alpha > 0$.

We let $\mathcal{D}_N$ be the subset of $\mathcal{D}$ generated by elements of $\bigcup_{k=1}^{N} P^k$, i.e. $\alpha \in \mathcal{D}_N$ if and only if there exist $0 \le k < N$ and $A_{-k} \dots A_0 \in \mathcal{P}$ such that $\alpha \sim A_{-k} \dots A_0$. Thus $\mathcal{D}_N$ is the subset of $\mathcal{D}$ whose significant part has length less than or equal to N.

3.2. Convergence of the Buzzi–Hofbauer diagram. Let E be the union $\bigcup_{f,P} \mathcal{D}(f,P)$ over all C^1 interval maps f and all natural partitions P

of f. For $\alpha \in \mathcal{D}(f,P)$ let $L(\alpha)$ be the length of $f^{N'+1}(\bigcap_{0\le i\le N'} f^{i-N'}A_{-i})$ for some (any) representative $A_{-N'}\ldots A_0 \in \mathcal{P}$ of α. For any $(N,K)\in(\mathbb{N}\setminus\{0\})^2$ we consider the subset $E_{N,K}$ of E defined by

$$E_{N,K} := \Big\{\alpha \in \bigcup_{f,P} \mathcal{D}_N(f,P) : L(\alpha)\ge 1/K\Big\}.$$

Clearly we have $E=\bigcup_{N,K} E_{N,K}$.

We now analyze the convergence of the Buzzi–Hofbauer diagrams associated to a converging sequence of C^1 interval maps. We begin with some preliminary facts.

Fact 0. *Let $\mathcal{F}$ be a C^1 bounded set of C^1 interval maps. For any N,K,M there exist $\tilde N,\tilde K$ depending only on $N,K,M,\sup_{f\in\mathcal{F}}\|f'\|_\infty$ such that for any $f\in\mathcal{F}$ and for any natural partition P of f, any closed path in $\mathcal{D}(f,P)$ of length M at a vertex in $E_{N,K}$ is contained in $E_{\tilde N,\tilde K}$.*

Proof. This is an immediate consequence of the following two properties of the Buzzi–Hofbauer diagram:

- if $\alpha\in\mathcal{D}_N(f,P)$ and $\alpha\to\beta$ then $\beta\in\mathcal{D}_{N+1}(f,P)$;
- if $\alpha\to\beta$ then $L(\beta)\le\|f'\|_\infty L(\alpha)$.

Indeed, just set $\tilde N=N+M$ and $\tilde K=[K\sup_{f\in\mathcal{F}}\|f'\|_\infty^M]+1$. ∎

For any C^1 interval map f, any natural partition P of f and any positive integer m, we denote by $P(m)$ the set of elements of P where $|f'|$ attains $1/m$.

Fact 1. *Let $\mathcal{F}$ be a C^1 bounded set of C^1 interval maps. For any N,K there exists m depending only on $N,K,\sup_{f\in\mathcal{F}}\|f'\|_\infty$ such that for any $f\in\mathcal{F}$ and any natural partition P of f, the set $\mathcal{D}(f,P)\cap E_{N,K}$ is generated by $\bigcup_{k=1}^N P(m)^k$.*

Proof. Let $\alpha\sim A_{-N'}\ldots A_0$ with $N'<N$ and suppose that the length of $f^{N'+1}(\bigcap_{0\le i\le N'} f^{i-N'}A_{-i})$ is larger than or equal to $1/K$. Clearly we have $\prod_{i=0}^{N'}\sup_{x\in A_{-i}}|f'(x)|\ge 1/K$ and thus, for any $0\le i\le N'$,

$$\sup_{x\in A_{-i}}|f'(x)|\ge\frac{1}{K\|f'\|_\infty^{N'}}.$$

Therefore it is enough to take $m>K\max(1,\sup_{f\in\mathcal{F}}\|f'\|_\infty)^N$. ∎

From now on we fix C^1 interval maps f and $(f_n)_{n\in\mathbb{N}}$ such that $(f_n)_n$ converges to f in the C^1 topology, and we consider natural partitions $(P_n)_n$ of $(f_n)_n$.

Fact 2. *For any positive integer m we have*

$$\sup_n \sharp P_n(m)<\infty.$$

Proof. Let $m \in \mathbb{N}$. Assume $\sup_n \sharp P_n(m) = \infty$. Clearly for any fixed n the set $P_n(m)$ is finite. Then up to extracting a subsequence one may find for each n an element (a_n, b_n) of $P_n(m)$ whose length $|a_n - b_n|$ goes to zero as $n \to \infty$. For any n we let $c_n \in (a_n, b_n)$ with $|f_n'(c_n)| \geq 1/m$. We may assume that a_n, b_n are not boundary points of the unit interval, so that $f_n'(a_n) = f_n'(b_n) = 0$. Any accumulation point (a, b, c) of the sequence (a_n, b_n, c_n) satisfies $a = b = c$. Moreover, as $(f_n)_n$ C^1-converges to f, we have $f'(a) = f'(b) = 0$ and $|f'(c)| \geq 1/m$ and we get a contradiction. Therefore $\sup_n \sharp P_n(m) < \infty$ for any given m. ∎

When A is an open interval and n is an integer, we denote by A^n the element (2) of P_n which contains the midpoint of A.

FACT 3. *There exists a subsequence $(n_k)_k$ with the following property. There is a non-decreasing sequence $(R_m)_m$ of finite collections of disjoint critical monotone branches of f such that for any positive integer m:*

- *for all $k \geq m$ we have $P_{n_k}(m) = \{A^{n_k} : A \in R_m\}$;*
- *for any $A \in R_m$ the sequence $(A^{n_k})_{k \geq m}$ goes to A in the Hausdorff topology as $k \to \infty$.*

Moreover the union $R := \bigcup_{m \in \mathbb{N}} R_m$ defines a natural partition of f.

Proof. By Cantor's diagonal argument, one may find a subsequence $(n_k)_k$ such that for any m the cardinality of $(P_{n_k}(m))_k$ is constant for $k \geq m$ and $P_{n_k}(m)$ converges in the Hausdorff topology to a (finite) collection R_m of critical monotone branches of f. We may also assume $P_{n_k}(m)$ is so close to R_m for any $k \geq m$ that $P_{n_k}(m)$ is given by the family $\{A^{n_k} : A \in R_m\}$.

Finally, observe that for any point x in $\{f' \neq 0\}$ there exist an integer m and an open neighbourhood U of x which is contained in a unique element of $P_{n_k}(m)$ for large k. By taking the limit in k we deduce that U is contained in some $A \in R_m \subset P$. Therefore $R = \bigcup_{m \in \mathbb{N}} R_m$ is a natural partition of f. ∎

We let $\mathcal{D}^n$ and $\mathcal{D}$ be the Buzzi–Hofbauer diagrams associated to the natural partitions P_n and R (given by Fact 3) of f_n and f respectively. It follows immediately from Facts 1 and 2 that $(\mathcal{D}^n)_n$ is uniform with respect to $\underline{E} = (E_{N,K})_{N,K}$. From Facts 0 and 1 one also easily sees that all these diagrams are admissible graphs. In fact by Theorem 2.5 admissibility may be deduced more directly from the finiteness of Gurevich entropy, which will follow from the Isomorphism Theorem (Theorem 3.2).

We now prove the convergence of the Buzzi–Hofbauer diagrams by taking a subsequence again.

(2) If it exists; when using the notation A^n, we claim it is well defined.

Lemma 3.1. *There exists a subsequence $(n_{k_l})_l$ such that the Buzzi–Hofbauer diagrams $(\mathcal{D}^{n_{k_l}})_l$ of $(f_{n_{k_l}})_l$ converge to the Buzzi–Hofbauer diagram $\mathcal{D}$ of f.*

Proof. Let N, K, M be positive integers. By Facts 0, 1, 3, there exists an integer m depending only on $N, K, M, \sup_n \|f_n'\|_\infty$ such that for $k \geq m$ any closed path γ_k of length $M' \leq M$ at a vertex $\alpha^{n_k} \in \mathcal{D}^{n_k} \cap E_{N,K}$ is given by a P_{n_k}-word $A^{n_k}_{-N'} \dots A^{n_k}_0 A^{n_k}_1 \dots . A^{n_k}_{M'}$ with $N' < N$ such that

$$\alpha^{n_k} \sim A^{n_k}_{-N'} \dots A^{n_k}_0 \sim A^{n_k}_{-N'} \dots A^{n_k}_0 A^{n_k}_1 \dots A^{n_k}_{M'},$$

where $A_{-N'}, \dots, A_{M'}$ belongs to R_m. Up to extracting a subsequence one may assume by uniform convergence of $(f_{n_k})_k$ to f that $A_{-N'} \dots A_0$ is an admissible word and moreover $A_{-N'} \dots A_0 A_1 \dots A_{M'} \sim A_{-N'} \dots A_0$. This last relation defines a closed path γ of length M' at the class α of $A_{-N'} \dots A_0$ in $\mathcal{D}$. The function $\phi_k : \Delta^{\alpha^{n_k}}_M(\mathcal{D}^{n_k}) \to \Delta^\alpha_M(\mathcal{D})$ mapping γ_k to γ may be extended to $\Delta^{\alpha^{n_k}}_{p,M}(\mathcal{D}^{n_k})$ for all p by concatenating the ϕ_k-images of the first returns at α^{n_k} appearing in the decomposition of a closed path in $\Delta^{\alpha^{n_k}}_{p,M}(\mathcal{D}^{n_k})$. The resulting map, which takes values in $\Delta^\alpha_p(\mathcal{D})$, is injective. Indeed, any closed path γ_k in $\Delta^{\alpha^{n_k}}_{p,M}(\mathcal{D}^{n_k})$ is given by a P_{n_k}-word $A^{n_k}_{-N'} \dots A^{n_k}_0 A^{n_k}_1 \dots . A^{n_k}_p$ with $N' < N$ and with $A_{-N'}, \dots, A_p \in R_m$ such that

$$\alpha^{n_k} \sim A^{n_k}_{-N'} \dots A^{n_k}_0 \sim A^{n_k}_{-N'} \dots A^{n_k}_0 A^{n_k}_1 \dots A^{n_k}_p.$$

Then $\phi_k(\gamma_k)$ is the closed path in $\mathcal{D}$ of length p at α whose qth term is the class of $A_{-N'} \dots A_0 A_1 \dots A_{q-1}$ in $\mathcal{D}$ for any $1 \leq q \leq p$. Since these classes allow us to determine A_q for $1 \leq q \leq p$, two closed paths in $\Delta^{\alpha^{n_k}}_{p,M}$ with the same image under ϕ_k coincide. It follows that

$$\sharp \Delta^{\alpha^{n_k}}_{p,M}(\mathcal{D}^{n_k}) \leq \sharp \Delta^\alpha_p(\mathcal{D}).$$

Finally, by Cantor's diagonal argument we may extract a subsequence $(n_{k_l})_l$ such that this holds true for any N, K, M whenever l is large enough, i.e. the Buzzi–Hofbauer diagrams $(\mathcal{D}^{n_{k_l}})_l$ of $(f_{n_{k_l}})_l$ converge to the Buzzi–Hofbauer diagram $\mathcal{D}$ of f. ∎

3.3. Isomorphism Theorem. Let f be a C^1 interval map and P be a natural partition of f. The symbolic dynamic extends the dynamic on the interval as follows. For any $A = (A_n)_{n \in \mathbb{Z}} \in \Sigma(f, P)$ we let $\pi_0(A) := \bigcap_{k \in \mathbb{N}} \overline{\bigcap_{l=0}^k f^{-l} A_l}$. As f is monotone on each element of P, the set $\bigcap_{l=0}^k f^{-l} A_l$ is an interval for all $k \in \mathbb{N}$. In particular $\pi_0(A)$ is a point or a compact non-trivial interval; but this last possibility occurs only for a countable set of elements $(A_n)_{n \in \mathbb{N}}$. Therefore there is a Borel subset of $\Sigma(f, P)$ of full μ-measure for any σ-invariant measure μ with positive entropy such that the restriction of π_0 to this subset defines a Borel map (in the following, this Borel map will also be denoted by π_0).

Moreover, consider the projection $\pi_1 : \Sigma(\mathcal{D}) \to \Sigma(f, P)$ defined by $\pi_1((\alpha_n)_n) = B_n$ where B_n is the last letter of the word α_n.

We now recall the Isomorphism Theorem for $C^{1+\alpha}$ interval maps obtained in [4] based on previous work of J. Buzzi [6]:

THEOREM 3.2 ([4]). *Let f be a C^r map of the interval with $r > 1$. The Borel map $\pi := \pi_0 \circ \pi_1 : \Sigma(\mathcal{D}) \to [0, 1]$ induces an entropy preserving bijection between ergodic invariant measures with positive entropy of $(\Sigma(\mathcal{D}), \sigma)$ and $([0, 1], f)$.*

REMARK 3.3. In [4], [6] the authors consider the particular natural partition given by the set of connected components of $[0, 1] \setminus D(f)$, where $D(f)$ is the set of points which do not belong to the interior of any strictly monotone branch of f. However the proof of the Isomorphism Theorem applies straightforwardly to general natural partitions. Here we need to work with general natural partitions because Lemma 3.1 is false for the above particular choice used in [4], [6].

3.4. Bound of the entropy at infinity. We first prove the following refinement of [8, Proposition 3]:

LEMMA 3.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of C^1 interval maps converging in the C^1 topology. For any integer n let $\mathcal{D}^n$ be the Buzzi–Hofbauer diagram associated to some natural partition P_n of f_n. Let $(\xi_n)_n$ be a sequence of ergodic σ-invariant measures on $\Sigma(\mathcal{D}^n)$ such that:*

- *$h(\sigma, \xi_n) > 0$ for all n;*
- *$(\xi_n)_n$ goes to infinity.*

Then $\lim_n \xi_n([\mathcal{D}^n_N]) = 0$ for all $N \in \mathbb{N}$.

Proof. For any N, n, m we denote by $\mathcal{D}^n_{N,m}$ the subset of $\mathcal{D}^n_N$ given by irreducible P_n-words $A_{-N'} \ldots A_0$ (note that $N' < N$) with

$$\sup_{x\in\bigcap_{i=0}^{N'} f_n^{-i}A_{-N'+i}} |(f_n^{N'+1})'(x)| > 1/m.$$

Obviously $\mathcal{D}^n_{N,m} \subset \mathcal{D}^n_{N,m'}$ for $m' > m$. We now fix N.

STEP 1: $\xi_n([\mathcal{D}^n_N \setminus \mathcal{D}^n_{N,m}]) \xrightarrow{m\to\infty} 0$ *uniformly in n.* Assume for contradiction that for infinitely many m there exists $n = n(m)$ with $\xi_n([\mathcal{D}^n_N \setminus \mathcal{D}^n_{N,m}]) > a > 0$. Let $\mu_n = \pi^*\xi_n$ be the induced f_n-invariant ergodic measure on $[0, 1]$ and λ_{μ_n} its Lyapunov exponent. By the ergodic theorem one easily gets

$$\lambda_{\mu_n} \le \log^+ \|f_n'\|_\infty - \frac{a \log m}{2N}.$$

Therefore by choosing $m > \max(1, \sup_n \|f_n'\|_\infty)^{2N/a}$ we conclude that the Lyapunov exponent of μ_n is negative and thus its entropy is zero by Ruelle's inequality, $h(f,\mu_n) \le \max(\lambda_{\mu_n},0)$. We get a contradiction with our first assumption $h(\sigma,\xi_n) > 0$, as by the Isomorphism Theorem (Theorem 3.2) we have $h(f,\mu_n) = h(\sigma,\xi_n)$.

STEP 2: *For all m there exists K with $\bigcup_n \mathcal{D}^n_{N,m} \subset E_{N,K}$.* Observe that for any irreducible P_n word $A_{-N'} \dots A_0$ representing $\alpha \in \mathcal{D}^n_{N,m}$ there exists $z \in \bigcap_{i=0}^{N'} f_n^{-i} A_{-N'+i}$ with $|(f_n^{N'+1})'(z)| > 1/m$. There is a positive integer K' such that $|(f_n^{N'+1})'(t)| > 1/(2m)$ for any t in the $1/K'$-neighborhood of z in $[0,1]$ (in particular this neighbourhood is entirely contained in $\bigcap_{i=0}^{N'} f_n^{-i} A_{-N'+i}$). We may also choose K' independent of z by uniform continuity of f_n, but also independent of n by C^1 convergence of $(f_n)_n$. Consequently, $L(\alpha) \ge 1/(2mK')$ and thus α belongs to $E_{N,K}$ for $K = 2mK'$.

STEP 3: *Conclusion.* Let $\epsilon > 0$. By Step 1 we may find m so large that $\xi_n([\mathcal{D}^n_N \setminus \mathcal{D}^n_{N,m}]) < \epsilon/2$ for all n. Then by Step 2 there exists an integer $K = K(m)$ such that for all n we have $\mathcal{D}^n_{N,m} \subset E_{N,K}$ and hence, for all n,

$$\xi_n([\mathcal{D}^n_N \setminus E_{N,K}]) \le \xi_n([\mathcal{D}^n_N \setminus \mathcal{D}^n_{N,m}]) < \epsilon/2.$$

Finally, as $\xi_n \to \infty$, there exists n_0 such that for all $n > n_0$ we have $\xi_n([E_{N,K}]) < \epsilon/2$ and thus

$$\xi_n([\mathcal{D}^n_N]) \le \xi_n([\mathcal{D}^n_N \setminus E_{N,K}]) + \xi_n([E_{N,K}]) < \epsilon. \ \blacksquare$$

The entropy of a sequence of ergodic σ-invariant measures going to infinity may be bounded from above as follows.

PROPOSITION 3.5. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of C^1 interval maps converging in the C^1 topology with $\sup_n \|(f_n)^{(r)}\|_\infty < \infty$. For any integer n let $\mathcal{D}^n$ be the Buzzi–Hofbauer diagram associated to some natural partition P_n of f_n. Let $(\xi_n)_n$ be a sequence of ergodic σ-invariant measures on $\Sigma(\mathcal{D}^n)$ such that $\lim_n \xi_n([\mathcal{D}^n_N]) = 0$ for all $N \in \mathbb{N}$. Then for any weak limit $\mu := \lim_k \mu_{n_k}$ of $(\mu_n)_n := (\pi^*\xi_n)_n$ we have*

$$\limsup_k h(f_{n_k},\mu_{n_k}) \le \frac{\int \log^+ |f'|\, d\mu}{r}.$$

Proposition 3.5 is shown by following straightforwardly the proof of [4, Proposition 4]. In [4] the statement concerns a single map, but the proof works when considering a C^1-converging sequence of C^r maps with uniformly bounded r-derivative. Let us sketch the main ideas of this generalization.

For any $A \in \Sigma(f_n,P_n)$ and any $N \in \mathbb{N}$ we let $r_N(A)$ be the least integer $m \in \mathbb{N} \cup \{\infty\}$ with $m > N$ such that $\mathrm{fol}(A_{-m-1} \dots A_0) \ne \mathrm{fol}(A_{-m} \dots A_0)$. The latter implies that there exists $y \in \partial A_{-m-1}$ which shadows the piece of orbit $A_{-m} \dots A_0$, i.e. $f_n(y) \in \bigcap_{l=0}^{n} f_n^{-l} A_{-m+l}$.

Then, a sequence $(\nu_n)_n \in \prod_m \mathcal{M}(\Sigma(f_n, P_n), \sigma)$ is said to satisfy the *shadowing property* when $\lim_n \nu_n(r_N < \infty) = 1$ for all integers N. It follows easily from the definitions that if $(\xi_n)_n$ is a sequence of σ-invariant ergodic measures on $\Sigma(\mathcal{D}^n)$ with $\lim_n \xi_n([\mathcal{D}^n_N]) = 0$, then the sequence $(\pi_1^*\xi_n)_n$ satisfies the shadowing property.

We now consider a sequence $(\nu_n)_n \in \prod_m \mathcal{M}(\Sigma(f_n, P_n), \sigma)$ satisfying the shadowing property and we explain how to bound the entropy of $(\mu_n)_n := (\pi_0^*\nu_n)_n$. For any N, a typical orbit of length L for μ_n with n large enough may be shadowed at almost any time by at most $k \leq L/N$ critical points of f_n on disjoint orbit segments of length larger than N. Then by using combinatorial arguments one may bound the entropy of μ_n by the exponential growth rate in L of the number of such possible k-tuples of critical points. This last rate may be bounded in terms of the Lyapunov exponent of μ_n as follows. If at a starting time of an orbit segment the $|f_n'|$-value of our μ_n-typical orbit is l then the number of possible shadowing points for this piece is less than $C\|(f_n)^{(r)}\|_\infty l^{1/(r-1)}$ with $C = C(r)$ depending only on r. Indeed, for a C^r interval map g the number of critical strictly monotone branches where $|g'|$ exceeds the value l is at most $C(r)\|g^{(r)}\|_\infty l^{1/(r-1)}$ (see [10, Lemma 4.1]). Then, as $\sup_n \|(f_n)^{(r)}\|_\infty < \infty$, there exists for any $\epsilon > 0$ an integer n_ϵ such that for $n > n_\epsilon$ we have

$$h(f_n, \mu_n) \leq \frac{\int -\log^- |f_n'|\, d\mu_n}{r-1} + \epsilon.$$

Together with the Ruelle inequality, $h(f_n, \mu_n) \leq \max(\int \log |f_n'|\, d\mu_n, 0)$, we get for $n > n_\epsilon$ (we may assume $h(\mu_n) > 0$)

$$\begin{aligned} h(f_n, \mu_n) &\leq \frac{\int -\log |f_n'|\, d\mu_n + \int \log^+ |f_n'|\, d\mu_n}{r-1} + \epsilon \\ &\leq \frac{-h(f_n, \mu_n) + \int \log^+ |f_n'|\, d\mu_n}{r-1} + \epsilon, \end{aligned}$$

and thus for $n > n_\epsilon$ we get

$$h(f_n, \mu_n) \leq \frac{\int \log^+ |f_n'|\, d\mu_n}{r} + \epsilon.$$

We conclude by continuity of the integral in the right member that for any limit $\mu := \lim_k \mu_{n_k}$ of $(\mu_n)_n$,

$$\limsup_k h(f_{n_k}, \mu_{n_k}) \leq \frac{\int \log^+ |f'|\, d\mu}{r}.$$

3.5. Proof of the Main Theorem for interval maps. We can now prove the Main Theorem for interval maps. Let $(f_n)_n$ be a sequence of C^r interval maps converging in the C^1 topology to f with $\sup_n \|(f_n)^{(r)}\|_\infty < \infty$ such that $\limsup_n h_{\mathrm{top}}(f_n) > h_{\mathrm{top}}(f)$. We may assume that $\lim_n h_{\mathrm{top}}(f_n)$

exists and is finite since $\limsup_n h_{\mathrm{top}}(f_n) \le \limsup_n R(f_n) \le R(f)$. Let $(P_n)_n$ be natural partitions of $(f_n)_n$. By Theorem 3.2 and Lemma 3.1 we can suppose by taking a subsequence that there exists a natural partition P of f such that the Buzzi–Hofbauer diagrams $\mathcal{D}^n$ associated to (f_n, P_n) converge to the Buzzi–Hofbauer diagram $\mathcal{D}$ associated to (f, P). By the Isomorphism Theorem 3.2 we have $h(\mathcal{D}^n) = h_{\mathrm{top}}(f_n)$ and $h(\mathcal{D}) = h_{\mathrm{top}}(f)$. Also by Theorem 2.3 one may find finite connected subgraphs $\mathcal{G}_n \subset \mathcal{D}^n$ with $\lim_n h(\mathcal{D}^n) = \lim_n h(\mathcal{G}_n) > h(\mathcal{D})$. The Main Proposition 2.6 states that the measure ξ_n of maximal entropy of $\mathcal{G}_n$ goes to infinity as $n \to \infty$. Then by Lemma 3.4 we get $\lim_n \xi_n([\mathcal{D}^n_N]) = 0$ for all N and thus by Proposition 3.5, for any weak limit $\mu \in \mathcal{M}(f, [0,1])$ of $(\mu_n)_n = (\pi^* \xi_n)$ we have

$$\lim_n h_{\mathrm{top}}(f_n) = \lim_n h(\mathcal{D}^n) = \lim_n h(\mathcal{G}_n) = \lim_n h(\xi_n) = \lim_n h(\mu_n)$$
$$\le \frac{\int \log^+ |f'|\, d\mu}{r} \le \frac{\log^+ \|f'\|_\infty}{r}.$$

Now if m is an integer so large that $R(f) \simeq \frac{\log^+ \|(f^m)'\|_\infty}{m}$, we may apply the previous result to the sequence f_n^m and to f^m since

$$\lim_n h_{\mathrm{top}}(f_n^m) = m \lim_n h_{\mathrm{top}}(f_n) > m h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f^m).$$

Then we get

$$\lim_n h_{\mathrm{top}}(f_n) = \lim_n \frac{h_{\mathrm{top}}(f_n^m)}{m} \le \frac{\log^+ \|(f^m)'\|_\infty}{mr} \lesssim \frac{R(f)}{r}.$$

This concludes the proof of the Main Theorem for interval maps.

3.6. Proof of the Main Theorem for circle maps. For circle maps the theorem is proved by reduction to the case of interval maps as follows. In the assumptions of the Main Theorem one may assume that f has positive entropy. Indeed, when f has zero entropy the inequality in the Main Theorem is just given by Yomdin's inequality (1.2). It is well known that a circle map with positive entropy has a horseshoe [18], in particular there exist a positive integer k and an interval $J \neq \mathbb{S}^1$ such that the closure of J is contained in the f^k-image of the interior of J. This still holds for maps g which are C^0-close to f. For n large enough we let $p_n \in J$ be a k-periodic point of f_n. By extracting a subsequence we may assume that $(p_n)_n$ is converging to a k-periodic point p of f. Then the interval maps $\tilde{f^p}$ and $\tilde{f_n^p}$, obtained from f^p and f_n^p by blowing up the circle at the fixed point p and p_n respectively, are C^r interval maps such that $(\tilde{f_n^p})_n$ converges to $\tilde{f^p}$ in the C^r topology. By applying the Main Theorem for interval maps we get

$$\limsup_n h_{\mathrm{top}}(\tilde{f_n^p}) \le \max(h_{\mathrm{top}}(\tilde{f^p}), R(\tilde{f^p})/r)$$

It is easily checked that

$$h_{\text{top}}(\tilde{f_n^p}) = h_{\text{top}}(f_n^p) = p h_{\text{top}}(f_n),$$
$$h_{\text{top}}(\tilde{f^p}) = h_{\text{top}}(f^p) = p h_{\text{top}}(f),$$
$$R(\tilde{f^p}) = R(f^p) = pR(f),$$

so that

$$\limsup_n h_{\text{top}}(f_n) \le \max(h_{\text{top}}(f), R(f)/r).$$

4. Proof of Proposition 1.1. Similar examples to those below have already been built in [7], [26], [3] and we refer to these papers for the details of the construction. Let f be a C^r interval map with homoclinic tangency of order r at a repelling fixed point p. We denote by c the critical point flat up to order r in the unstable manifold of p with $f^k(c) = p$ for some $k > 0$. We perturb f only on a small neighbourhood $]c - \delta, c + \delta[$ by letting the perturbation g be a sinusoidal of the form $g(x) = a\sin(Nx/\delta) + f(c)$. We may choose $a = C\delta|f'(p)|^{-l}$ for some constant C to get an N-horseshoe for f^l with $l \gg |\log\delta|$. The entropy of this horseshoe is given by $(\log N)/l$, whereas to ensure the C^r-closeness one may take $aN^r = \delta^r/l$. In this way we obtain g_l going to f in the C^r topology such that

$$h_{\text{top}}(g_l) \ge \frac{\log N}{l} \ge \frac{\log(\delta^r/al)}{rl} \ge \frac{\log(\delta^{r-1}|f'(p)|^l/Cl)}{rl} \ge \frac{\lambda(p)}{r} + o(1/l).$$

By lower semicontinuity of entropy we finally get

$$\limsup_l h_{\text{top}}(g_l) \ge \max(h_{\text{top}}(f), \lambda(p)/r).$$

Appendix. As a consequence of Misiurewicz's result (1.1) we give a short proof of the following theorem.

THEOREM ([15]). *For any real $r > 1$, entropy is continuous on the set $D^r([0,1])$ of C^r interval maps with no critical point flat up to order r.*

Proof. If $f \in D^r([0,1])$ then its critical set is finite and thus f is piecewise monotone. In fact for any C^r-bounded set $\mathcal{V}$ there exist an integer k and a C^1 neighbourhood $\mathcal{U}$ of f such that any $g \in \mathcal{U} \cap \mathcal{V}$ belongs to $\mathcal{M}_k([0,1])$:

CLAIM. *Let $r > 1$, $R > 0$ and $f \in D^r([0,1])$. There exist $\epsilon_0 > 0$ and a C^1 neighbourhood $\mathcal{U}$ of f such that any map $g \in \mathcal{U} \cap C^r([0,1])$ with $\|g^{(r)}\|_\infty \le R$ has at most $r-1$ critical points in any ball of radius ϵ_0; in particular g is in $\mathcal{M}_k^1([0,1])$ with $k = [r/\epsilon_0] + 1$.*

Proof. For contradiction assume that for some $r > 1$, $R > 0$, there exists $f \in D^r([0,1])$ such that for any $\delta > 0$ and any $\epsilon > 0$ there exists a C^r map $g_{\delta,\epsilon}$ which is δ-C^1-close to f with $\|(g_{\delta,\epsilon})^{(r)}\|_\infty < R$ and with r critical points of $g_{\delta,\epsilon}$ in an interval $I_{\delta,\epsilon}$ of length ϵ. Let x be an accumulation point of the intervals $(I_{\delta,\epsilon})_{\delta,\epsilon}$ as $\delta, \epsilon \to 0$. In particular for some arbitrarily small δ and ϵ the length of $g_{\delta,\epsilon}(I)$ for any $I \supset I_{\delta,\epsilon}$ is less than $R|I|^r$ by [5, Lemma 3.2]. Thus as $(g_{\delta,\epsilon})_\delta$ converges uniformly to f as $\delta \to 0$, this also holds true for $f(I)$ for any open interval I containing x, which easily implies that f has a critical point flat up to order r at x, contradicting our assumption. ■

Finally, it follows from (1.1) that entropy is continuous at $f \in D^r([0,1])$ for the C^1 topology in any C^r-bounded set (in particular for the C^r topology). ■

References

[1] R. Bowen, *Entropy for maps of the interval*, Topology 16 (1977), 465–467.

[2] R. Bowen, *Some systems with unique equilibrium states*, Math. Systems Theory 8 (1974/75), 193–202.

[3] D. Burguet, *Examples of $\mathcal{C}^r$ interval map with large symbolic extension entropy*, Discrete Contin. Dynam. Systems 26 (2010), 873–899.

[4] D. Burguet, *Existence of measures of maximal entropy for C^r interval maps*, Proc. Amer. Math. Soc. 142 (2014), 957–968.

[5] D. Burguet, G. Liao and J. Yang, *Asymptotic h-expansiveness rate of C^∞ maps*, Proc. London Math. Soc. 111 (2015), 381–419.

[6] J. Buzzi, *Intrinsic ergodicity for smooth interval maps*, Israel J. Math. 100 (1997), 125–161.

[7] J. Buzzi, *C^r surface diffeomorphisms with no maximal entropy measure*, Ergodic Theory Dynam. Sytems 34 (2014), 1770–1793.

[8] J. Buzzi and S. Ruette, *Large entropy implies existence of a maximal entropy measure for interval maps*, Discrete Contin. Dynam. Systems 14 (2006), 673–688.

[9] T. Downarowicz, *Entropy in Dynamical Systems*, New Math. Monogr. 18, Cambridge Univ. Press, Cambridge, 2011.

[10] T. Downarowicz and A. Maass, *Smooth interval maps have symbolic extensions*, Invent. Math. 176 (2009), 617–636.

[11] B. M. Gurevich, *Shift entropy and Markov measures in the path space of a denumerable graph*, Dokl. Akad. Nauk SSSR 192 (1970), 963–965 (in Russian); English transl.: Soviet Math. Dokl. 11 (1970), 744–747.

[12] B. M. Gurevich, *Topological entropy of enumerable Markov chains*, Dokl. Akad. Nauk SSSR 187 (1969), 715–718 (in Russian); English transl.: Soviet Math. Dokl. 10 (1969), 911–915.

[13] B. M. Gurevich and S. V. Savchenko, *Thermodynamic formalism for countable symbolic Markov chains*, Russian Math. Surveys 53 (1998), 245–344.

[14] F. Hofbauer, *On intrinsic ergodicity of piecewise monotone transformations with positive entropy I, II*, Israel J. Math. 34 (1979), 213–237; 38 (1981), 107–115.

[15] K. Iwai, *Continuity of topological entropy of one-dimensional maps with degenerate critical points*, J. Math. Sci. Univ. Tokyo 5 (1998), 19–40.

[16] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. I.H.E.S. 51 (1980), 137–173.

[17] O. Kozlovski, W. Shen, and S. van Strien, *Density of hyperbolicity in dimension one*, Ann. of Math. 166 (2007), 145–182.

[18] M. Misiurewicz, *Horseshoes for continuous mappings of an interval*, in: Dynamical Systems, Liguori, Napoli, 1980, 127–135.
[19] M. Misiurewicz, *On non-continuity of topological entropy*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), 319–320.
[20] M. Misiurewicz, *Topological conditional entropy*, Studia Math. 55 (1976), 175–200.
[21] M. Misiurewicz, *Jumps of entropy in one dimension*, Fund. Math. 132 (1989), 215–226.
[22] M. Misiurewicz, *Diffeomorphism without any measure with maximal entropy*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 903–910.
[23] M. Misiurewicz, *Continuity of entropy revisited*, in: Dynamical Systems and Applications, World Sci. Ser. Appl. Anal. 4, World Sci., River Edge, NJ, 1995, 495–503.
[24] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), 45–63.
[25] S. Newhouse, *Continuity properties of entropy*, Ann. of Math. 129 (1989), 215–235.
[26] S. Ruette, *Mixing $\mathcal{C}^r$ maps of the interval without maximal measure*, Israel J. Math. 127 (2002), 253–277.
[27] D. Vere-Jones, *Geometric ergodicity in denumerable Markov chains*, Quart. J. Math. Oxford Ser. (2) 13 (1962), 7–28.
[28] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, New York, 1982.
[29] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. 57 (1987), 285–300.

David Burguet
LPMA – CNRS UMR 7599
Université Paris 6
75252 Paris Cedex 05, France
E-mail: david.burguet@upmc.fr

Received 12 March 2014;
in revised form 27 January 2015