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On partial orderings having precalibre-$\aleph_1$ and fragments of Martin's axiom

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Abstract. We define a countable antichain condition (ccc) property for partial orderings, weaker than precalibre-$\aleph_1$, and show that Martin's axiom restricted to the class of partial orderings that have the property does not imply Martin's axiom for σ-linked partial orderings. This yields a new solution to an old question of the first author about the relative strength of Martin's axiom for σ-centered partial orderings together with the assertion that every Aronszajn tree is special. We also answer a question of J. Steprāns and S. Watson (1988) by showing that, by a forcing that preserves cardinals, one can destroy the precalibre-$\aleph_1$ property of a partial ordering while preserving its ccc-ness.

Introduction. A question asked in [1] is if MA(σ-centered) plus "Every Aronszajn tree is special" implies MA(σ-linked). The interest in this question originated in the result of Harrington–Shelah [5] showing that if $\aleph_1$ is accessible to reals, i.e., there exists a real number x such that the cardinal $\aleph_1$ in the model $L[x]$ is equal to the real $\aleph_1$, then MA implies that there exists a $\Delta^1_3(x)$ set of real numbers that does not have the Baire property. The hypothesis that $\aleph_1$ is accessible to reals is necessary, for if $\aleph_1$ is inaccessible to reals and MA holds, then $\aleph_1$ is actually weakly compact in L ([5]), and K. Kunen showed that starting from a weakly compact cardinal one can get a model where MA holds and every projective set of reals has the Baire property.

In [1], using Todorčević's ρ-functions [12], it was shown that MA(σ-centered) plus "Every Aronszajn tree is special" is sufficient to produce a $\Delta^1_3(x)$ set of real numbers without the Baire property, assuming $\aleph_1 = \aleph_1^{L[x]}$. Thus, it was natural to ask how weak is MA(σ-centered) plus "Every Aronszajn tree is special" as compared to the full MA, and in particular if it implies MA(σ-linked). The answer is negative, since it has been observed by D. Chodounský and J. Zapletal that a finite-support iteration of σ-centered

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posets combined with the forcing that specializes Aronszajn trees has the Y-c.c. property, and therefore does not add random reals (see [2]).

In the first part of the paper we give a new and stronger negative answer, namely we show that a fragment of MA that includes MA(σ-centered), and even MA(3-Knaster), and implies "Every Aronszajn tree is special", does not imply MA(σ-linked). A partial ordering with the precalibre-$\aleph_1$ property plays the key role in the construction of the model.

In the second part of the paper we answer a question of Steprāns–Watson [9]. They ask if it is possible to destroy the precalibre-$\aleph_1$ property of a partial ordering, while preserving its ccc-ness, in a forcing extension of the set-theoretic universe V that preserves cardinals. This is a natural question considering that, as shown in [9], on the one hand, assuming MA plus the Covering Lemma, every precalibre-$\aleph_1$ partial ordering has precalibre-$\aleph_1$ in every forcing extension of V that preserves cardinals; and on the other hand, the ccc property of a partial ordering having precalibre-$\aleph_1$ can always be destroyed while preserving $\aleph_1$, and consistently even preserving all cardinals.

We answer the Steprāns–Watson question positively, and in a very strong sense. Namely, we show that it is consistent, modulo ZFC, that the Continuum Hypothesis holds and there exist a forcing notion T of cardinality $\aleph_1$ that preserves $\aleph_1$ (and therefore it preserves all cardinals, cofinalities, and the cardinal arithmetic), and two precalibre-$\aleph_1$ partial orderings, such that forcing with T preserves their ccc-ness, but it also forces that their product is not ccc and therefore they do not have precalibre-$\aleph_1$.

1. Preliminaries. Recall that a partially ordered set (or poset) $\mathbb{P}$ is *ccc* if every antichain of $\mathbb{P}$ is countable; it is *productive-ccc* if the product of $\mathbb{P}$ with any ccc poset is also ccc; it is *Knaster* (or has *property* $\mathcal{K}$) if every uncountable subset of $\mathbb{P}$ contains an uncountable subset consisting of pairwise compatible elements. More generally, for $k \geq 2$, $\mathbb{P}$ is *k-Knaster* if every uncountable subset of $\mathbb{P}$ contains an uncountable subset such that any k of its elements have a common lower bound. Thus, Knaster is the same as 2-Knaster. Furthermore, $\mathbb{P}$ has *precalibre-*$\aleph_1$ if every uncountable subset of $\mathbb{P}$ has an uncountable subset such that any finite set of its elements has a common lower bound; it is *σ-linked* (or *σ-2-linked*) if it can be partitioned into countably many pieces so that each piece is pairwise compatible. More generally, for $k \geq 2$, $\mathbb{P}$ is σ-k-linked if it can be partitioned into countably many pieces so that any k elements in the same piece have a common lower bound. Finally, $\mathbb{P}$ is *σ-centered* if it can be partitioned into countably many pieces so that any finite number of elements in the same piece have a common lower bound. We have the following implications, for every $k \geq 2$:

$$\sigma\text{-centered} \Rightarrow \sigma\text{-}k\text{-linked} \Rightarrow k\text{-Knaster} \Rightarrow \text{productive-ccc} \Rightarrow \text{ccc},$$

and

$$\sigma\text{-centered} \;\Rightarrow\; \text{precalibre-}\aleph_1 \;\Rightarrow\; k\text{-Knaster}.$$

These are the only implications that can be proved in ZFC.

For any property Γ of posets that implies the ccc, and an infinite cardinal κ, *Martin's axiom for* Γ *and for families of* κ*-many dense open sets*, denoted by $\mathrm{MA}_\kappa(\Gamma)$, asserts: for every $\mathbb{P}$ that satisfies the property Γ and every family $\{D_\alpha : \alpha < \kappa\}$ of dense open subsets of $\mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ that is *generic* for the family, that is, $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$.

When $\kappa = \aleph_1$ we omit the subscript and write $\mathrm{MA}(\Gamma)$ for $\mathrm{MA}_{\aleph_1}(\Gamma)$. Also, for an infinite cardinal θ, the notation $\mathrm{MA}_{<\theta}(\Gamma)$ means: $\mathrm{MA}_\kappa(\Gamma)$ for all $\kappa < \theta$. The axiom $\mathrm{MA}_{\aleph_0}(\Gamma)$ is provable in ZFC; and it is consistent, modulo ZFC, that the Continuum Hypothesis fails and $\mathrm{MA}_{<2^{\aleph_0}}(\text{ccc})$ holds (see [7], or [6]). *Martin's axiom*, denoted by MA, is MA(ccc).

Thus, we have the following implications, for every $k \geq 2$:

$$\mathrm{MA}_\kappa(\text{ccc}) \;\Rightarrow\; \mathrm{MA}_\kappa(\text{productive-ccc}) \;\Rightarrow$$
$$\Rightarrow\; \mathrm{MA}_\kappa(k\text{-Knaster}) \Rightarrow \mathrm{MA}_\kappa(\sigma\text{-}k\text{-linked}) \Rightarrow \mathrm{MA}_\kappa(\sigma\text{-centered}),$$

and

$$\mathrm{MA}_\kappa(k\text{-Knaster}) \;\Rightarrow\; \mathrm{MA}_\kappa(\text{precalibre-}\aleph_1) \;\Rightarrow\; \mathrm{MA}_\kappa(\sigma\text{-centered}).$$

Again, the arrows cannot be reversed (see [13], [10] for even finer distinctions, and also [11] for Borel examples).

For all the facts mentioned in the rest of the paper without a proof, as well as for all undefined notions and notation, see [6].

2. The property Pr_k**.** Let us consider the following property of partial orderings, weaker than the k-Knaster property.

Definition 1. For $k \geq 2$, let $\mathrm{Pr}_k(\mathbb{Q})$ mean that $\mathbb{Q}$ is a forcing notion such that if $p_\varepsilon \in \mathbb{Q}$, for all $\varepsilon < \aleph_1$, then we can find $\bar{u}$ such that:

(a) $\bar{u} = \langle u_\xi : \xi < \aleph_1\rangle$.
(b) u_ξ is a finite subset of $\aleph_1$.
(c) $u_{\xi_0} \cap u_{\xi_1} = \emptyset$ whenever $\xi_0 \neq \xi_1$.
(d) If $\xi_0 < \cdots < \xi_{k-1}$, then we can find $\varepsilon_l \in u_{\xi_l}$, for $l < k$, such that $\{p_{\varepsilon_l} : l < k\}$ has a common lower bound.

Notice that $\mathrm{Pr}_k(\mathbb{Q})$ implies that $\mathbb{Q}$ is ccc, and that $\mathrm{Pr}_{k+1}(\mathbb{Q})$ implies $\mathrm{Pr}_k(\mathbb{Q})$. Also note that if $\mathbb{Q}$ is k-Knaster, then $\mathrm{Pr}_k(\mathbb{Q})$ holds. For a given subset $\{p_\varepsilon : \varepsilon < \aleph_1\}$ of $\mathbb{Q}$, there exists an uncountable $X \subseteq \aleph_1$ such that $\{p_{\varepsilon_l} : l < k\}$ has a common lower bound for every $\varepsilon_0 < \cdots < \varepsilon_{k-1}$ in X, so we can take u_ξ to be the singleton that contains the ξth element of X. Finally, observe that if $\mathbb{Q}$ has precalibre-$\aleph_1$, then $\mathrm{Pr}_k(\mathbb{Q})$ holds for every $k \geq 2$.

Recall that if T is an Aronszajn tree on ω_1, then the forcing that specializes T consists of finite functions p from ω_1 into ω such that if $\alpha \neq \beta$ are in the domain of p and are comparable in the tree ordering, then $p(\alpha) \neq p(\beta)$. The ordering is the reversed inclusion. It is consistent, modulo ZFC, that the specializing forcing is not productive-ccc, an example being the case when T is a Suslin tree. However, we have the following:

Lemma 2. *If T is an Aronszajn tree and $\mathbb{Q} = \mathbb{Q}_T$ is the forcing that specializes T with finite conditions, then* $\mathrm{Pr}_k(\mathbb{Q})$ *holds for every $k \geq 2$.*

Proof. Without loss of generality, $T = (\omega_1, <_T)$. Let $p_\alpha \in \mathbb{Q}$ for $\alpha < \aleph_1$. By a Δ-system argument we may assume that $\{\mathrm{dom}(p_\alpha) : \alpha < \aleph_1\}$ forms a Δ-system with root r. Moreover, we may assume that for some fixed n, $|\mathrm{dom}(p_\alpha) \setminus r| = n$ for all $\alpha < \omega_1$. Let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be an enumeration of $\mathrm{dom}(p_\alpha) \setminus r$. We may also assume that if $\alpha < \beta$, then the highest level of T that contains some α_i $(1 \leq i \leq n)$ is strictly lower than the lowest level of T that contains some β_j $(1 \leq j \leq n)$.

Fix a uniform ultrafilter D over ω_1. For each $\alpha < \omega_1$ and $1 \leq i, j \leq n$, let

$$D_{\alpha,i,j} := \{\beta > \alpha : \alpha_i <_T \beta_j\}, \quad D_{\alpha,i,0} := \{\beta > \alpha : \alpha_i \not<_T \beta_j \text{ for all } j\}.$$

For every α and every i, there exists $j_{\alpha,i} \leq n$ such that $D_{\alpha,i,j_{\alpha,i}} \in D$. Moreover, for every $1 \leq i \leq n$, there exists $E_i \in D$ such that $j_{\alpha,i}$ is fixed, say with value j_i for all $\alpha \in E_i$. We claim that $j_i = 0$ for all $1 \leq i \leq n$. For suppose i is such that $j_i \neq 0$. Pick $\alpha < \beta < \gamma$ in $E_i \cap D_{\alpha,i,j_i} \cap D_{\beta,i,j_i}$. Then $\alpha_i, \beta_i <_T \gamma_{j_i}$, hence $\alpha_i <_T \beta_i$. This yields an ω_1-chain in T, which is impossible. Now let $E := \bigcap_{1 \leq i \leq n} E_i \in D$.

We claim that for every m and every α we can find $u \in [\omega_1 \setminus \alpha]^m$ such that if $\beta < \gamma$ are in u, then $\beta_i \not<_T \gamma_j$ for every $1 \leq i, j \leq n$. Indeed, given m and α, choose any $\beta^0 \in E \setminus \alpha$. Now given $\beta^0, \ldots, \beta^l$, all in E, let $\beta^{l+1} \in E \cap \bigcap_{1 \leq i \leq n} \bigcap_{l' \leq l} D_{\beta^{l'},i,0}$. Then the set $u := \{\beta^0, \ldots, \beta^{m-1}\}$ is as required.

We can now choose $\langle u_\xi : \xi < \aleph_1 \rangle$ pairwise disjoint, with $|u_\alpha| > k \cdot n$, so that if $\xi_1 < \xi_2$, then $\sup(u_{\xi_1}) < \min(u_{\xi_2})$, and each u_ξ is as above, i.e., if $\beta < \gamma$ are in u_ξ, then $\beta_i \not<_T \gamma_j$ for every $1 \leq i, j \leq n$. We claim that $\langle u_\xi : \xi < \aleph_1 \rangle$ is as required. So, suppose $\xi_0 < \cdots < \xi_{k-1}$. We choose $\alpha^\ell \in u_{\xi_\ell}$ by downward induction on $\ell \in \{0, \ldots, k-1\}$ so that $\{p_{\alpha^\ell} : \ell < k\}$ has a common lower bound. Let α^{k-1} be any element of $u_{\xi_{k-1}}$. Now suppose $\alpha^{\ell+1}, \ldots, \alpha^{k-1}$ have already been chosen and we shall choose α^ℓ. We may assume that for each $\beta \in u_{\xi_\ell}$, p_β is incompatible with $p_{\alpha^{\ell'}}$ for some ℓ' in $\{\ell+1, \ldots, k-1\}$, for otherwise we could take as our α^ℓ any $\beta \in u_{\xi_\ell}$ with p_β compatible with all $p_{\alpha^{\ell'}}$, $\ell' \in \{\ell+1, \ldots, k-1\}$. Thus, for each $\beta \in u_{\xi_\ell}$ there exist $\ell' \in \{\ell+1, \ldots, k-1\}$ and $1 \leq i, j \leq n$ such that $\beta_i <_T \alpha^{\ell'}_j$. So, since $|u_{\xi_\beta}| > k \cdot n$, there must exist $\beta, \beta' \in u_{\xi_\ell}$ and ℓ' such that $\beta_i, \beta_{i'} <_T \alpha_j^{\ell'}$ for some $1 \le i, i', j \le n$ with $\beta_i \ne \beta_{i'}$. But this implies that β_i and $\beta_{i'}$ are $<_T$-comparable, contradicting our choice of u_{ξ_ℓ}. ■

We show next that the property Pr_k for forcing notions is preserved under iterations with finite support, of any length.

LEMMA 3. *For any $k \ge 2$, the property Pr_k is preserved under finite-support forcing iterations. That is, if*

$$\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \lambda,\ \beta < \lambda \rangle$$

is a finite-support iteration of forcing notions such that $\mathrm{Pr}_k(\mathbb{P}_0)$ holds and $\Vdash_{\mathbb{P}_\beta}$ "$\mathrm{Pr}_k(\underset{\sim}{\mathbb{Q}}_\beta)$ holds" for every $\beta < \lambda$, then $\mathrm{Pr}_k(\mathbb{P}_\lambda)$ holds.

Proof. We use induction on $\alpha \le \lambda$. For $\alpha = 0$ it is trivial. If α is a limit ordinal with $\mathrm{cf}(\alpha) \ne \aleph_1$, and $p_\varepsilon \in \mathbb{P}_\alpha$ for all $\varepsilon < \aleph_1$, then either uncountably many p_ε have the same support (in the case $\mathrm{cf}(\alpha) = \omega$) or the support of all p_ε is bounded by some $\alpha' < \alpha$. In either case $\mathrm{Pr}_k(\mathbb{P}_\alpha)$ follows easily from the induction hypothesis.

If $\mathrm{cf}(\alpha) = \aleph_1$, then we may use a Δ-system argument, as in the usual proof of the preservation of the ccc.

So, suppose $\alpha = \beta + 1$. Let $p_\varepsilon \in \mathbb{P}_\alpha$ for all $\varepsilon < \aleph_1$. Without loss of generality, we may assume that $\beta \in \mathrm{dom}(p_\varepsilon)$ for all $\varepsilon < \aleph_1$.

Since $\mathbb{P}_\beta$ is ccc, there is $q \in \mathbb{P}_\beta$ such that

$$q \Vdash_{\mathbb{P}_\beta} \text{“}|\{\varepsilon : p_\varepsilon{\restriction}\beta \in \underset{\sim}{G}_\beta\}| = \aleph_1\text{”}.$$

Let $G \subseteq \mathbb{P}_\beta$ be generic over V and with $q \in G$. In $V[G]$ we observe that $p_\varepsilon(\beta)[G] \in \underset{\sim}{\mathbb{Q}}_\beta[G]$, and $\mathrm{Pr}_k(\underset{\sim}{\mathbb{Q}}_\beta[G])$ holds. So, there is $\langle u_\xi^0 : \xi < \aleph_1 \rangle$ as in Definition 1 for the sequence $\langle p_\varepsilon(\beta)[G] : p_\varepsilon{\restriction}\beta \in G \rangle$. Hence,

$$q \Vdash_{\mathbb{P}_\beta} \text{“}\langle \underset{\sim}{u}_\xi^0 : \xi < \aleph_1 \rangle \text{ is as in Definition 1 for } \langle p_\varepsilon(\beta) : p_\varepsilon{\restriction}\beta \in \underset{\sim}{G}_\beta \rangle\text{”}.$$

For each ξ, let (q_ξ, u_ξ^1) be such that:

- $q_\xi \in \mathbb{P}_\beta$ and $q_\xi \le q$.
- $q_\xi \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{u}_\xi^0 = u_\xi^1$", so u_ξ^1 is finite.
- $q_\xi \le p_\varepsilon{\restriction}\beta$ for every $\varepsilon \in u_\xi^1$. (This can be ensured because if $\varepsilon \in u_\xi^1$, then $q_\xi \Vdash_{\mathbb{P}_\beta}$ "$p_\varepsilon{\restriction}\beta \in \underset{\sim}{G}_\beta$", so we may as well take $q_\xi \le p_\varepsilon{\restriction}\beta$.)

Now apply the induction hypothesis for $\mathbb{P}_\beta$ to obtain $\langle u_\zeta^2 : \zeta < \aleph_1 \rangle$ as in the definition of Pr_k for the sequence $\langle q_\xi : \xi < \aleph_1 \rangle$. We may assume, by refining the sequence if necessary, that $\max(u_\zeta^2) < \min(u_{\zeta'}^2)$ whenever $\zeta < \zeta'$.

Let $u_\zeta^* := \bigcup \{u_\xi^1 : \xi \in u_\zeta^2\}$. We claim that $\bar{u}^* = \langle u_\zeta^* : \zeta < \aleph_1 \rangle$ is as in the definition, for the sequence $\langle p_\varepsilon : \varepsilon < \aleph_1 \rangle$. Clearly, the u_ζ^* are finite and pairwise disjoint. Moreover, given $\zeta_0 < \cdots < \zeta_{k-1}$, we can find $\xi_0 \in$

$u^2_{\zeta_0}, \ldots, \xi_{k-1} \in u^2_{\zeta_{k-1}}$ such that in $\mathbb{P}_\beta$ there is a common lower bound q_* to $\{q_{\xi_0}, \ldots, q_{\xi_k}\}$. Since $q_* \le q_{\xi_0}, \ldots, q_{\xi_{k-1}} \le q$, there are some $q_{**} \le q_*$ and $\varepsilon_l \in u^1_{\xi_l}$, for each $l < k$, such that for some $\mathbb{P}_\beta$-name $\underset{\sim}{p}$,

$$q_{**} \Vdash \text{“}\underset{\sim}{p} \le_{\underset{\sim}{\mathbb{Q}}_\beta} p_{\varepsilon_0}(\beta), \ldots, p_{\varepsilon_{k-1}}(\beta)\text{”}.$$

Then the condition $q_{**} * \underset{\sim}{p}$ is a common lower bound for the conditions $p_{\varepsilon_0}, \ldots, p_{\varepsilon_{k-1}}$. ■

3. On fragments of MA. We shall now prove that MA(Pr_{k+1}) does not imply MA(σ-k-linked), which yields a negative answer to the first question stated in the Introduction. The following is the main lemma.

LEMMA 4. *For $k \ge 2$, there is a forcing notion $\mathbb{P}_* = \mathbb{P}^k_*$ and $\mathbb{P}_*$-names $\underset{\sim}{\mathcal{A}}$ and $\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \mathbb{Q}^k_{\underset{\sim}{\mathcal{A}}}$ such that:*

(1) *$\mathbb{P}_*$ has precalibre-$\aleph_1$ and is of cardinality $\aleph_1$.*
(2) *$\Vdash_{\mathbb{P}_*}$ “$\underset{\sim}{\mathcal{A}} \subseteq [\aleph_1]^{k+1}$”.*
(3) *$\Vdash_{\mathbb{P}_*}$ “$\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \{v \in [\aleph_1]^{<\aleph_0} : [v]^{k+1} \cap \underset{\sim}{\mathcal{A}} = \emptyset\}$, ordered by $\supseteq$, is σ-k-linked”.*
(4) *$\Vdash_{\mathbb{P}_*}$ “$\underset{\sim}{I}_\alpha := \{v \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} : v \not\subseteq \alpha\}$ is dense for all $\alpha < \aleph_1$”.*
(5) *$\Vdash_{\mathbb{P}_*}$ “If $v_\alpha \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$ is such that $v_\alpha \not\subseteq \alpha$ for $\alpha < \aleph_1$, and $u_\xi \in [\aleph_1]^{<\aleph_0}$, for $\xi < \aleph_1$, are non-empty and pairwise disjoint, then there exist $\xi_0 < \cdots < \xi_k$ such that for every $\langle \alpha_\ell : \ell \le k\rangle \in \prod_{\ell \le k} u_{\xi_\ell}$ the set $\bigcup_{\ell \le k} v_{\alpha_\ell}$ does not belong to $\mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$”.*

Proof. We define $\mathbb{P}_*$ by: $p \in \mathbb{P}_*$ if and only if p has the form $(u, A, h) = (u_p, A_p, h_p)$, where

(a) $u \in [\aleph_1]^{<\aleph_0}$,
(b) $A \subseteq [u]^{k+1}$, and
(c) $h : \wp_p \to \omega$, where $\wp_p := \{v \subseteq u : [v]^{k+1} \cap A = \emptyset\}$ is such that if $w_0, \ldots, w_{k-1} \in \wp_p$ and h is constant on $\{w_0, \ldots, w_{k-1}\}$, then $w_0 \cup \cdots \cup w_{k-1} \in \wp_p$.

The order is given by: $p \le q$ if and only if $u_q \subseteq u_p$, $A_q = A_p \cap [u_q]^{k+1}$, and $h_q \subseteq h_p$ (hence $\wp_q = \wp_p \cap \mathcal{P}(u_q)$ and $h_p{\restriction}\wp_q = h_q$).

(1) Clearly, $\mathbb{P}_*$ has cardinality $\aleph_1$, so we show that it has precalibre-$\aleph_1$. Given $\{q_\xi = (u_\xi, A_\xi, h_\xi) : \xi < \aleph_1\} \subseteq \mathbb{P}_*$, and writing $\wp_\xi$ instead of the more cumbersome $\wp_{q_\xi}$, we can find an uncountable $W \subseteq \aleph_1$ such that:

(i) The set $\{u_\xi : \xi \in W\}$ forms a Δ-system with heart u_*.
(ii) The sets $[u_*]^{k+1} \cap A_\xi$ for $\xi \in W$ are all the same. Hence the sets $\wp_\xi \cap \mathcal{P}(u_*)$ for $\xi \in W$ are also all the same.
(iii) The functions $h_\xi{\restriction}(\wp_\xi \cap \mathcal{P}(u_*))$ for $\xi \in W$ are all the same.

(iv) The ranges of h_ξ, for $\xi \in W$, are all the same, say R. So, R is finite.
(v) For each $i \in R$, the sets $\{w \cap u_* : h_\xi(w) = i\}$ for $\xi \in W$ are the same.

We will show that every finite subset of $\{q_\xi : \xi \in W\}$ has a common lower bound. Given $\xi_0, \dots, \xi_m \in W$, let $q = (u_q, A_q, h_q)$ be such that:

- $u_q = \bigcup_{\ell \le m} u_{\xi_\ell}$.
- $A_q = \bigcup_{\ell \le m} A_{\xi_\ell}$. Note that this implies that the $\wp_{\xi_\ell}$ are contained in $\wp_q = \{v \subseteq u_q : [v]^{k+1} \cap A_q = \emptyset\}$. Indeed, if, say, $w \in \wp_{\xi_\ell}$, then $[w]^{k+1} \cap A_{\xi_\ell} = \emptyset$, and we claim that also $[w]^{k+1} \cap A_{\xi_j} = \emptyset$ for $j \le m$. Indeed, if $v \in [w]^{k+1} \cap A_{\xi_j}$ with $j \neq \ell$, then $v \subseteq u_*$, and therefore $v \in [u_*]^{k+1} \cap A_{\xi_j} = [u_*]^{k+1} \cap A_{\xi_\ell}$. Hence, $v \in [w]^{k+1} \cap A_{\xi_\ell}$, which is impossible because $[w]^{k+1} \cap A_{\xi_\ell}$ is empty.
- $h_q : \wp_q \to \omega$ is such that $h_q(v) = h_{\xi_\ell}(v)$ for all $v \in \wp_{\xi_\ell}$, and the $h_q(v)$ are all distinct and greater than $\sup\{h_q(v) : v \in \bigcup_{\ell \le m} \wp_{\xi_\ell}\}$ for $v \notin \bigcup_{\ell \le m} \wp_{\xi_\ell}$. Notice that h_q is well-defined because the restrictions $h_{\xi_\ell} \restriction (\wp_{\xi_\ell} \cap \mathcal{P}(u_*))$ for $\ell \le m$ are all the same.

We claim that $q \in \mathbb{P}_*$. For this, we only need to show that if $\{w_0, \dots, w_{k-1}\} \subseteq \wp_q$ and h_q is constant on $\{w_0, \dots, w_{k-1}\}$, then $[\bigcup_{j<k} w_j]^{k+1} \cap A_q = \emptyset$. So fix a set $\{w_0, \dots, w_{k-1}\} \subseteq \wp_q$ and suppose h_q is constant on it, say with constant value i. By definition of h_q we must have $\{w_0, \dots, w_{k-1}\} \subseteq \bigcup_{\ell \le m} \wp_{\xi_\ell}$. Now suppose, towards a contradiction, that $v \in [\bigcup_{j<k} w_j]^{k+1} \cap A_{\xi_\ell}$ for some $\ell \le m$. Let $s = \{w_j : j < k\} \cap \wp_{\xi_\ell}$, and let $t = \{w_j : j < k\} \setminus s$. Thus, $v \subseteq \bigcup s \cup (\bigcup t \cap u_*)$, for if $\alpha \in v \setminus \bigcup s$, then $\alpha \in \bigcup t$ and $\alpha \in \bigcup \wp_{\xi_{\ell'}}$ for some $\ell' \neq \ell$, hence $\alpha \in u_\xi \cap u_{\xi'} = u_*$.

By (v),

$$\{w \cap u_* : h_{\xi_\ell}(w) = i\} = \{w \cap u_* : h_{\xi_{\ell'}}(w) = i\}$$

for every $\ell' \le m$. So, for every $w_j \in t$, there exists $w'_j \in \wp_{\xi_\ell}$ such that $w_j \cap u_* = w'_j \cap u_*$ and $h_{\xi_\ell}(w'_j) = i$. Let $t' = s \cup \{w'_j : w_j \in t\}$. Note that $t' \subseteq \wp_{\xi_\ell}$ and $t' \subseteq \{w : h_{\xi_\ell}(w) = i\}$. So,

$$v \subseteq \bigcup t' \subseteq \bigcup \{w : h_{\xi_\ell}(w) = i\}.$$

Thus, $v \in [\bigcup\{w : h_{\xi_\ell}(w) = i\}]^{k+1} \cap A_{\xi_\ell}$. But this is impossible because $\bigcup\{w : h_{\xi_\ell}(w) = i\} \in \wp_{\xi_\ell}$ (since h_{ξ_ℓ} satisfies property (c) above), and therefore

$$\Big[\bigcup\{w : h_{\xi_\ell}(w) = i\}\Big]^{k+1} \cap A_{\xi_\ell} = \emptyset.$$

Now one can easily check that $q \le q_{\xi_0}, \dots, q_{\xi_m}$. And this shows that the set $\{q_\xi : \xi \in W\}$ is finite-wise compatible.

(2) Let

$$\underset{\sim}{A} = \{(\check{v}, p) : v \in A_p,\, p \in \mathbb{P}_*\}.$$

Thus, $\underset{\sim}{\mathcal{A}}$ is a name for the set $\bigcup\{A_p : p \in G\}$, where G is the $\mathbb{P}_*$-generic filter. Clearly, (2) holds.

(3) Let

$$\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \{(\check{v}, p) : v \in \wp_p,\, p \in \mathbb{P}_*\}.$$

Thus, $\mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$ is a name for the set $\bigcup\{\wp_p : p \in G\}$, where G is the $\mathbb{P}_*$-generic filter. Clearly, $\Vdash_{\mathbb{P}_*}$ "$\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \{v \in [\aleph_1]^{<\aleph_0} : [v]^{k+1} \cap \underset{\sim}{\mathcal{A}} = \emptyset\}$". Moreover, if G is $\mathbb{P}_*$-generic over V, then, by (c), the function $\bigcup\{h_p : p \in G\}$ witnesses that the interpretation $i_G(\mathbb{Q}_{\underset{\sim}{\mathcal{A}}})$, ordered by $\supseteq$, is σ-k-linked.

(4) Clear.

(5) Suppose that $p \in \mathbb{P}_*$ forces $\dot{v}_\alpha \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$ is such that $\dot{v}_\alpha \not\subseteq \alpha$ for all $\alpha < \aleph_1$; and it also forces $\dot{u}_\xi \in [\aleph_1]^{<\aleph_0}$ for all $\xi < \aleph_1$ are non-empty and pairwise disjoint.

For each $\xi < \aleph_1$, let $q_\xi = (u_\xi, A_\xi, h_\xi) \leq p$ and let $u^*_\xi \in [\aleph_1]^{<\aleph_0}$ and $\bar{v}^*_\xi = \langle v^*_{\xi,\alpha} : \alpha \in u^*_\xi\rangle$, with $v^*_{\xi,\alpha} \in [\aleph_1]^{<\aleph_0}$, be such that

$$q_\xi \Vdash_{\mathbb{P}_*} \text{“}\dot{u}_\xi = u^*_\xi \text{ and } \dot{v}_\alpha = v^*_{\xi,\alpha} \text{ for } \alpha \in u^*_\xi\text{”}.$$

We may assume, by extending q_ξ if necessary, that $u^*_\xi \cup \bigcup_{\alpha\in u^*_\xi} v^*_{\xi,\alpha} \subseteq u_\xi$.

As in (1), we can find an uncountable $W \subseteq \aleph_1$ such that (i)–(v) hold for the set of conditions $\{q_\xi : \xi \in W\}$. Hence $\{q_\xi : \xi \in W\}$ is pairwise compatible (in fact, finite-wise compatible), from which it follows that the set $\{u^*_\xi : \xi \in W\}$ is pairwise disjoint. Now choose $\xi_0 < \cdots < \xi_k$ from W so that:

- the heart u_* of the Δ-system $\{u_\xi : \xi \in W\}$ is an initial segment of u_{ξ_ℓ} for all $\ell \leq k$,
- $\sup(u_{\xi_\ell}) < \inf(u_{\xi_{\ell+1}} \setminus u_*)$ for all $\ell < k$, and
- $u^*_{\xi_\ell} \subseteq u_{\xi_\ell} \setminus u_*$ for all $\ell \leq k$.

For each $\sigma = \langle \alpha_\ell : \ell \leq k\rangle \in \prod_{\ell\leq k} u^*_{\xi_\ell}$, pick $w_\sigma \in [\bigcup_{\ell\leq k} v^*_{\xi_\ell,\alpha_\ell}]^{k+1}$ such that $|w_\sigma \cap (v^*_{\xi_\ell,\alpha_\ell} \setminus \alpha_\ell)| = 1$ for all $\ell \leq k$. This is possible because $v^*_{\xi_\ell,\alpha_\ell} \not\subseteq \alpha_\ell$.

Claim 5. *$w_\sigma \not\subseteq u_{\xi_\ell}$, hence $w_\sigma \notin A_{\xi_\ell}$, for all $\sigma \in \prod_{\ell\leq k} u^*_{\xi_\ell}$ and all $\ell \leq k$.*

Proof. Fix $\sigma = \langle \alpha_\ell : \ell \leq k\rangle$ and $\ell \leq k$, and suppose for a contradiction that $w_\sigma \subseteq u_{\xi_\ell}$. Then $w_\sigma \subseteq u_{\xi_\ell} \setminus u_*$. If $\ell < k$, then as $\sup(u_{\xi_\ell}) < \inf(u_{\xi_{\ell+1}} \setminus u_*) \leq \inf(u^*_{\xi_{\ell+1}}) \leq \alpha_{\ell+1}$, we would have $w_\sigma \setminus \alpha_{\ell+1} = \emptyset$, which contradicts our choice of w_σ. But if $\ell = k$, then since $\sup(v^*_{\xi_{\ell-1},\alpha_{\ell-1}}) \leq \sup(u_{\xi_{\ell-1}}) < \inf(u_{\xi_\ell} \setminus u_*)$, we would have $w_\sigma \cap v^*_{\xi_{\ell-1},\alpha_{\ell-1}} = \emptyset$, which contradicts again our choice of w_σ. ∎

Now define $q = (u_q, A_q, h_q)$ as follows:

- $u_q = \bigcup_{\ell\leq k} u_{\xi_\ell}$.

- $A_q = (\bigcup_{\ell \le k} A_{\xi_\ell}) \cup \{w_\sigma : \sigma \in \prod_{\ell \le k} u^*_{\xi_\ell}\}$. Note that since $w_\sigma \not\subseteq u_{\xi_\ell}$ (Claim 5), we have $w_\sigma \notin \wp_{\xi_\ell}$ for all $\sigma \in \prod_{\ell \le k} u^*_{\xi_\ell}$ and $\ell \le k$. Hence, $\wp_{\xi_\ell} \subseteq \wp_q$ for all $\ell \le k$.
- $h_q : \wp_q \to \omega$ is such that $h_q(v) = h_{\xi_\ell}(v)$ for $v \in \wp_{\xi_\ell}$, for all $\ell \le k$, and the $h_q(v)$ are all distinct and greater than $\sup\{h_q(v) : v \in \bigcup_{\ell \le k} \wp_{\xi_\ell}\}$ for $v \notin \bigcup_{\ell \le k} \wp_{\xi_\ell}$.

As in (1), we can now check that $q \in \mathbb{P}_*$. Moreover, by Claim 5, $A_{\xi_\ell} = A_q \cap [u_{\xi_\ell}]^{k+1}$. Hence, $q \le q_{\xi_\ell}$ for all $\ell \le k$, and so

$$q \Vdash_{\mathbb{P}_*} \text{“}\dot{u}_{\xi_\ell} = u^*_{\xi_\ell} \text{ and } \dot{v}_\alpha = v^*_{\xi_\ell,\alpha} \text{ for } \alpha \in u^*_{\xi_\ell}\text{”}.$$

And since $w_\sigma \in [\bigcup_{\ell \le k} v^*_{\alpha_\ell}]^{k+1} \cap A_q$ for every $\sigma \in \prod_{\ell \le k} u^*_{\xi_\ell}$, we have

$$q \Vdash_{\mathbb{P}_*} \text{“}\bigcup_{\ell \le k} \dot{v}_{\alpha_\ell} \notin \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} \text{ for all } \langle \alpha_\ell : \ell \le k\rangle \in \prod_{\ell \le k} \dot{u}_{\xi_\ell}\text{”}.$$

This finishes the proof of Lemma 4. ∎

Lemma 6. *Let $k \ge 2$ and let $\mathbb{P}_*$ be as in Lemma 4. Suppose $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_*$-name for a forcing notion that satisfies* Pr_{k+1}. *Then*

$$\Vdash_{\mathbb{P}_* * \underset{\sim}{\mathbb{Q}}} \text{“There is no directed } G \subseteq \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} \text{ such that } \underset{\sim}{I}_\alpha \cap G \ne \emptyset \text{ for all } \alpha < \aleph_1\text{”},$$

where $\underset{\sim}{I}_\alpha$ is a name for the dense open set $\{v \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} : v \not\subseteq \alpha\}$.

Proof. Suppose for a contradiction that $p * \dot{q} \in \mathbb{P}_* * \underset{\sim}{\mathbb{Q}}$ and

$$p * \dot{q} \Vdash_{\mathbb{P}_* * \underset{\sim}{\mathbb{Q}}} \text{“There exists } G \subseteq \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} \text{ directed with } \underset{\sim}{I}_\alpha \cap G \ne \emptyset \text{ for all } \alpha < \aleph_1\text{”}.$$

Suppose $G_0 \subseteq \mathbb{P}_*$ is a filter generic over V with $p \in G_0$. So, in $V[G_0]$, letting $q = i_{G_0}(\dot{q})$ and $\mathbb{Q} = i_{G_0}(\underset{\sim}{\mathbb{Q}})$, we see that for some $\mathbb{Q}$-name $\underset{\sim}{G}$,

$$q \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{G} \subseteq \mathbb{Q}_{\mathcal{A}} \text{ is directed and } I_\alpha \cap \underset{\sim}{G} \ne \emptyset \text{ for all } \alpha < \aleph_1\text{”}.$$

For each $\alpha < \aleph_1$, let $q_\alpha \le q$, and let $v_\alpha \in [\aleph_1]^{<\aleph_0}$ be such that

$$q_\alpha \Vdash_{\mathbb{Q}} \text{“}\check{v}_\alpha \in I_\alpha \cap \underset{\sim}{G}\text{”}.$$

Thus, $v_\alpha \not\subseteq \alpha$ for all $\alpha < \aleph_1$.

Since $\mathbb{Q}$ satisfies Pr_{k+1}, there exists $\bar{u} = \langle u_\xi : \xi < \aleph_1\rangle$ such that:

(a) u_ξ is a finite subset of $\aleph_1$ for all $\xi < \aleph_1$,
(b) $u_{\xi_0} \cap u_{\xi_1} = \emptyset$ whenever $\xi_0 \ne \xi_1$, and
(c) if $\xi_0 < \cdots < \xi_k$, then we can find $\alpha_\ell \in u_{\xi_\ell}$ for $\ell \le k$ such that $\{q_{\alpha_\ell} : \ell \le k\}$ have a common lower bound.

By Lemma 4, we can find $\xi_0 < \cdots < \xi_k$ such that for every $\langle \alpha_\ell : \ell \le k\rangle$ in $\prod_{\ell \le k} u_{\xi_\ell}$ the set $\bigcup_{\ell \le k} v_{\alpha_\ell}$ does not belong to $\mathbb{Q}_{\mathcal{A}}$.

By (c), let $\alpha_\ell \in u_{\xi_\ell}$ for $\ell \le k$ be such that $\{q_{\alpha_\ell} : \ell \le k\}$ have a common lower bound, say r. Then r forces that $\{\check{v}_{\alpha_\ell} : \ell \le k\} \subseteq \underset{\sim}{G}$. And since r forces that $\underset{\sim}{G}$ is directed, it also forces that $\bigcup_{\ell \le k} v_{\alpha_\ell} \in \mathbb{Q}_{\mathcal{A}}$, a contradiction. ∎

All elements are now in place to prove the main result of this section.

Theorem 7. *Let $k \ge 2$. Assume $\lambda = \lambda^{<\theta}$, where $\theta = \mathrm{cf}(\theta) > \aleph_1$. Then there is a finite-support iteration*

$$\bar{\mathbb{P}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \lambda,\, \beta < \lambda \rangle,$$

where:

(1) $\mathbb{P}_0$ *is the forcing* $\mathbb{P}_*$ *from Lemma* 4.
(2) $\Vdash_{\mathbb{P}_\beta}$ "$\mathrm{Pr}_{k+1}(\underset{\sim}{\mathbb{Q}}_\beta)$" *for every* $0 < \beta < \lambda$.
(3) *In* $V^{\mathbb{P}_\lambda}$ *the axiom* $\mathrm{MA}_{<\theta}(\mathrm{Pr}_{k+1})$ *holds, hence in particular* (*Lemma* 2) *every Aronszajn tree on* ω_1 *is special.*
(4) $\underset{\sim}{\mathbb{Q}}_{\mathcal{A}}$ *witnesses that* MA(*σ-k-linked*) *fails in* $V^{\mathbb{P}_\lambda}$.

Proof. To obtain (3), we proceed in the standard way as in all iterations forcing (some fragment of) MA, that is, we iterate all posets with the Pr_{k+1} property and having cardinality $< \theta$, which are given by some fixed bookkeeping function (see [6] or [7] for details).

Since after forcing with $\mathbb{P}_0$ the rest of the iteration $\bar{\mathbb{P}}$ has the property Pr_{k+1} (Lemma 3), (4) follows immediately from Lemma 6. ∎

Corollary 8. *For every $k \ge 2$, ZFC plus* $\mathrm{MA}(\mathrm{Pr}_{k+1})$ *does not imply* MA(*σ-k-linked*).

Thus, since $\mathrm{MA}(\mathrm{Pr}_{k+1})$ implies both MA(σ-centered) and "Every Aronszajn tree is special", the corollary answers in the negative and in a strong way the question from [1]: Does MA(σ-centered) plus "Every Aronszajn tree is special" imply MA(σ-linked)?

4. On destroying precalibre-$\aleph_1$ while preserving the ccc. We turn now to the second question stated in the Introduction (Steprāns–Watson [9]): Is it consistent that there exists a precalibre-$\aleph_1$ poset which is ccc but does not have precalibre-$\aleph_1$ in some forcing extension that preserves cardinals?

Note that the forcing extension cannot be ccc, since ccc forcing preserves the precalibre-$\aleph_1$ property. Also, as shown in [9], assuming MA plus the Covering Lemma, every forcing that preserves cardinals also preserves the precalibre-$\aleph_1$ property. Moreover, the examples provided in [9] of cardinal-preserving forcing notions that destroy precalibre-$\aleph_1$ do so by actually destroying the ccc property.

A positive answer to the Steprāns–Watson question is provided by the following theorem. Before stating it, let us recall a strong form of Jensen's diamond principle, *diamond-star relativized to a stationary set S*, which is also due to Jensen. For S a stationary subset of ω_1, let

$\diamondsuit_S^*$: There exists a sequence $\langle \mathcal{S}_\alpha : \alpha \in S \rangle$, where $\mathcal{S}_\alpha$ is a countable set of subsets of α, such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$ for every $\alpha \in C \cap S$.

The principle $\diamondsuit_S^*$ holds in the constructible universe L, for every stationary $S \subseteq \omega_1$ (see [3, 3.5] for a proof in the case $S = \omega_1$, which can be easily adapted to any stationary S). Also, $\diamondsuit_S^*$ can be forced by a σ-closed forcing notion (see [7, Chapter VII, Exercises H18 and H20], where it is shown how to force the even stronger form of diamond known as $\diamondsuit_S^+$).

Theorem 9. *It is consistent, modulo ZFC, that the CH holds and there exist:*

(1) *A forcing notion T of cardinality $\aleph_1$ that preserves cardinals.*
(2) *Two posets $\mathbb{P}_0$ and $\mathbb{P}_1$ of cardinality $\aleph_1$ that have precalibre-$\aleph_1$ and are such that*

$$\Vdash_T \text{“}\mathbb{P}_0, \mathbb{P}_1 \textit{ are } \text{ccc}, \textit{ but } \mathbb{P}_0 \times \mathbb{P}_1 \textit{ is not } \text{ccc”}.$$

Hence $\Vdash_T$ “$\mathbb{P}_0$ and $\mathbb{P}_1$ do not have precalibre-$\aleph_1$”.

Proof. Let $\{S_1, S_2\}$ be a partition of $\Omega := \{\delta < \omega_1 : \delta \text{ limit}\}$ into two stationary sets. By a preliminary forcing, we may assume that $\diamondsuit_{S_1}^*$ holds. So, there exists $\langle \mathcal{S}_\alpha : \alpha \in S_1 \rangle$, where $\mathcal{S}_\alpha$ is a countable set of subsets of α, such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$ for every $\alpha \in C \cap S_1$. In particular, the CH holds. Using $\diamondsuit_{S_1}^*$, we can build an S_1-*oracle*, i.e., an $\subset$-increasing sequence $\bar{M} = \langle M_\delta : \delta \in S_1 \rangle$ with M_δ countable and transitive, $\delta \in M_\delta$, $M_\delta \models$ “$\mathrm{ZFC}^- + \delta$ is countable”, and such that for every $A \subseteq \omega_1$ there is a club $C_A \subseteq \omega_1$ such that $A \cap \delta \in M_\delta$ for every $\delta \in C_A \cap S_1$. (For the latter, one simply needs to require that $\mathcal{S}_\delta \subseteq M_\delta$ for all $\delta \in S_1$.) Moreover, we can build $\bar{M}$ so that it has the following additional property:

($*$) For every regular uncountable cardinal χ and a well-ordering $<_\chi^*$ of $H(\chi)$, the set of all (universes of) countable $N \preceq \langle H(\chi), \in, <_\chi^* \rangle$ such that the Mostowski collapse of N belongs to M_δ, where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$.

Property ($*$) will be needed to prove that the tree partial ordering T (defined below) has many branches, and also to prove that the product partial ordering $\mathbb{Q} \times T$ (defined below) is S_1-proper (Claim 10 later on), and so it does not collapse $\aleph_1$.

To ensure $(*)$, take a large enough regular cardinal λ and define the sequence $\bar{M}$ so that, for every $\delta \in S_1$, M_δ is the Mostowski collapse of a countable elementary substructure X of $H(\lambda)$ that contains $\bar{M}\restriction\delta$, for all ordinals $\leq \delta$, and all elements of $\mathcal{S}_\delta$. To see that $(*)$ holds, fix a regular uncountable cardinal χ, a well-ordering $<^*_\chi$ of $H(\chi)$, and a club $E \subseteq [H(\chi)]^{\aleph_0}$. Let $\bar{N} = \langle N_\alpha : \alpha < \aleph_1 \rangle$ be an $\subset$-increasing and $\in$-increasing continuous chain of elementary substructures of $\langle H(\chi), \in, <^*_\chi \rangle$ with the universe of N_α in E for all $\alpha < \aleph_1$. We shall find $\delta \in S_1$ such that the transitive collapse of N_δ belongs to M_δ, where $\delta = N_\delta \cap \omega_1$.

Fix a bijection $h : \aleph_1 \to \bigcup_{\alpha<\aleph_1} N_\alpha$, and let $\Gamma : \aleph_1 \times \aleph_1 \to \aleph_1$ be the standard pairing function (cf. [6, Chapter 3]). Observe that the set

$$D := \{\delta < \aleph_1 : \delta \text{ is closed under } \Gamma \text{ and } h \text{ maps } \delta \text{ onto } N_\delta\}$$

is a club. Now let

$$\begin{aligned} X_1 &:= \{\Gamma(i,j) : h(i) \in h(j)\}, \\ X_2 &:= \{\Gamma(\alpha,i) : h(i) \in N_\alpha\}, \\ X_3 &:= \{\Gamma(i,j) : h(i) <^*_\chi h(j)\}, \\ X &:= \{3j + i : j \in X_i \text{ and } i \in \{1,2,3\}\}. \end{aligned}$$

The set $S_1' := \{\delta \in S_1 : X \cap \delta \in M_\delta\}$ is stationary. Thus, since the set $C := \{\delta < \aleph_1 : \delta = N_\delta \cap \omega_1\}$ is a club, we can pick $\delta \in C \cap D \cap S_1'$. Since $\delta \in D$, the structure

$$Y := \big\langle X_2 \cap \delta, \{\langle i,j \rangle : \Gamma(i,j) \in X_1 \cap \delta\}, \{\langle i,j \rangle : \Gamma(i,j) \in X_3 \cap \delta\} \big\rangle$$

is isomorphic to N_δ, and therefore Y and N_δ have the same transitive collapse; and Y belongs to M_δ, because $\delta \in S_1'$. Hence, since $M_\delta \models \mathrm{ZFC}^-$, the transitive collapse of Y belongs to M_δ. Finally, since $\delta \in C$, $\delta = N_\delta \cap \omega_1$.

We shall now define the forcing T. Let us write $\aleph_1^{<\aleph_1}$ for the set of all countable sequences of countable ordinals. Let

$$\begin{aligned} T := \{\eta \in \aleph_1^{<\aleph_1} : {} & \mathrm{Range}(\eta) \subset S_1,\ \eta \text{ is increasing and continuous,} \\ & \text{of successor length, and if } \varepsilon < \mathrm{lh}(\eta), \text{ then } \eta\restriction\varepsilon \in M_{\eta(\varepsilon)}\}. \end{aligned}$$

Let $\leq_T$ be the partial order on T given by end-extension. Thus, $(T, \leq_T)$ is a tree. Note that, since $\delta \in M_\delta$ for every $\delta \in S_1$, if $\eta \in T$, then η in $M_{\sup \mathrm{Range}(\eta)}$. Also notice that if $\eta \in T$, then $\eta^\frown\langle\delta\rangle \in T$ for every $\delta \in S_1$ greater than $\sup \mathrm{Range}(\eta)$. In particular, every node of T of finite length has $\aleph_1$-many extensions of any greater finite length. Now suppose $\alpha < \omega_1$ is a limit, and suppose inductively that for every successor $\beta < \alpha$, every node of T of length β has $\aleph_1$-many extensions of every higher successor length below α.

We claim that every $\eta \in T$ of length less than α has $\aleph_1$-many extensions in T of length $\alpha + 1$ (and in fact, the set of their suprema is stationary).

For every $\delta < \omega_1$, let $T_\delta := \{\eta \in T : \sup \mathrm{Range}(\eta) < \delta\}$. Notice that T_δ is countable: otherwise, uncountably many $\eta \in T_\delta$ would have the same $\sup \mathrm{Range}(\eta)$, and therefore they would all belong to the model $M_{\sup \mathrm{Range}(\eta)}$, which is impossible because it is countable. Now fix a node $\eta \in T$ of length less than α, and let $B := \{b_\gamma : \gamma < \omega_1\}$ be an enumeration of all the *branches* (i.e., linearly ordered subsets of T closed under predecessors) b of T that contain η and have length α (i.e., $\bigcup\{\mathrm{dom}(\eta') : \eta' \in b\} = \alpha$). For a club C of δ the set $\{b_\gamma : \gamma < \delta\}$ belongs to M_δ.

We shall next build a sequence $B^* := \langle b^*_\xi : \xi < \omega_1\rangle$ of branches from B so that the set $\sup B^* := \langle \sup \mathrm{Range}(\bigcup b^*_\xi) : \xi < \omega_1\rangle$ is the increasing enumeration of a club. To this end, start by fixing an increasing sequence $\langle \alpha_n : n < \omega\rangle$ of successor ordinals converging to α, with α_0 greater than the length of η. Then let $b^*_0 := b_0$. Given b^*_ξ, let γ be the least ordinal such that $\bigcup b_\gamma(\alpha_0) > \sup \mathrm{Range}(\bigcup b^*_\xi)$, and let $b^*_{\xi+1} := b_\gamma$. Finally, given b^*_ξ for all $\xi < \delta$, where $\delta < \omega_1$ is a limit ordinal, pick an increasing sequence $\langle \xi_n : n < \omega\rangle$ converging to δ. By construction, the sequence $\langle \sup \mathrm{Range}(\bigcup b^*_{\xi_n}) : n < \omega\rangle$ is increasing. Now let $f : \alpha \to \aleph_1$ be such that $f\restriction[0,\alpha_0] = \bigcup b^*_{\xi_0}\restriction[0,\alpha_0]$, and $f\restriction(\alpha_n,\alpha_{n+1}] = \bigcup b^*_{\xi_{n+1}}\restriction(\alpha_n,\alpha_{n+1}]$ for all $n < \omega$. Then set $b^*_\delta := \{f\restriction\beta : \beta < \alpha$ is a successor$\}$. One can easily check that b^*_δ is a branch of T of length α with $\sup \mathrm{Range}(\bigcup b^*_\delta) = \sup\{\sup \mathrm{Range}(\bigcup b^*_\xi) : \xi < \zeta\}$. Finally, notice that if $\delta \in S_1 \cap C$ is greater than α and belongs to the club enumerated by $\sup B^*$, then since $M_\delta \models$ "δ is countable", we can pick the sequences $\langle \alpha_n : n < \omega\rangle$ and $\langle \xi_n : n < \omega\rangle$ in M_δ. Then the sequence $\langle b^*_{\xi_n} : n < \omega\rangle$ belongs to M_δ, and therefore $(\bigcup b^*_\delta)^\frown\langle\delta\rangle \in T$.

By $(*)$ the set of all countable $N \preceq \langle H(\aleph_2), \in, <^*_{\aleph_2}\rangle$ that contain B^* and $\langle \alpha_n : n < \omega\rangle$, with $\alpha \subseteq N$, and such that the Mostowski collapse of N belongs to M_δ, where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$. So, since the set $\mathrm{Lim}(\sup B^*)$ of limit points of $\sup B^*$ is a club, there is such an N with $\delta := N \cap \omega_1 \in \mathrm{Lim}(\sup B^*)$. If $\bar N$ is the transitive collapse of N, we deduce that $B^*\restriction\delta \in \bar N \in M_\delta$, and so in M_δ we can build, as above, the branch b^*_δ. Therefore, since $\delta = \sup \mathrm{Range}(\bigcup b^*_\delta)$, we see that $\bigcup b^*_\delta \cup \{\langle \alpha, \delta\rangle\}$ is in T and extends η. We have thus shown that η has $\aleph_1$-many extensions in T of length $\alpha + 1$. Even more, the set $\{\sup \mathrm{Range}(\bigcup b) : b$ is a branch of length $\alpha + 1$ that extends $\eta\}$ is stationary.

Note however that since the complement of S_1 is stationary, T has no branch of length ω_1, because the range of such a branch would be a club contained in S_1. But since every $\eta \in T$ has extensions of length $\alpha + 1$ for every α greater than or equal to the length of η, forcing with $(T, \geq_T)$ yields a branch of T of length ω_1.

In order to obtain the forcing notions $\mathbb{P}_0$ and $\mathbb{P}_1$ claimed by the theorem, we need first to force with the forcing $\mathbb{Q}$ which we define as follows. For u a subset of T, let $[u]^2_T$ be the set of all pairs $\{\eta,\nu\} \subseteq u$ such that $\eta \neq \nu$ and η and ν are $<_T$-comparable. Let

$$\mathbb{Q} := \{p : [u]^2_T \to \{0,1\} : u \text{ is a finite subset of } T\},$$

ordered by reversed inclusion.

It is easily seen that $\mathbb{Q}$ is ccc and it has cardinality $\aleph_1$, so forcing with $\mathbb{Q}$ does not collapse cardinals, does not change cofinalities, and preserves cardinal arithmetic. (In fact, $\mathbb{Q}$ is equivalent, as a forcing notion, to the poset for adding $\aleph_1$ Cohen reals, which is σ-centered, but we shall not make use of this fact.)

Notice that if $G \subseteq \mathbb{Q}$ is a generic filter over V, then $\bigcup G : [T]^2_T \to \{0,1\}$.

Recall that, for $S \subseteq \aleph_1$ stationary, a forcing notion $\mathbb{P}$ is called *S-proper* if for all (some) large enough regular cardinals χ and all (stationarily many) countable $\langle N, \in\rangle \preceq \langle H(\chi), \in\rangle$ that contain $\mathbb{P}$ and are such that $N \cap \aleph_1 \in S$, and all $p \in \mathbb{P} \cap N$, there is a condition $q \leq p$ that is $(N,\mathbb{P})$-generic. If $\mathbb{P}$ is S-proper, then it does not collapse $\aleph_1$. (See [8] or [4] for details.)

Claim 10. *The forcing $\mathbb{Q}\times T$ is S_1-proper, hence it does not collapse $\aleph_1$.*

Proof. Let χ be a large enough regular cardinal, and let $<^*_\chi$ be a well-ordering of $H(\chi)$. Let $N \preceq \langle H(\chi), \in, <^*_\chi\rangle$ be countable and such that $\mathbb{Q}\times T$ belongs to N, $\delta := N \cap \aleph_1 \in S_1$, and the Mostowski collapse of N belongs to M_δ. Fix $(q_0,\eta_0) \in (\mathbb{Q}\times T) \cap N$. It will be sufficient to find a condition $\eta_* \in T$ such that $\eta_0 \leq_T \eta_*$ and (q_0,η_*) is $(N, \mathbb{Q}\times T)$-generic.

Let

$$\mathbb{Q}_\delta := \{p \in \mathbb{Q} : \text{if } \{\eta,\nu\} \in \mathrm{dom}(p), \text{ then } \eta,\nu \in T_\delta\}.$$

Thus, $\mathbb{Q}_\delta$ is countable. Moreover, notice that $T_\delta = T \cap N$, and therefore $\mathbb{Q}_\delta = \mathbb{Q} \cap N$. Hence, T_δ and $\mathbb{Q}_\delta$ are the Mostowski collapses of T and $\mathbb{Q}$, respectively, and so they belong to M_δ.

In M_δ, let $\langle (p_n, D_n) : n < \omega\rangle$ list all pairs (p, D) such that $p \in \mathbb{Q}_\delta$ and D is a dense open subset of $\mathbb{Q}_\delta \times T_\delta$ that belongs to the Mostowski collapse of N. That is, D is the Mostowski collapse of a dense open subset of $\mathbb{Q}\times T$ that belongs to N.

Also in M_δ, fix an increasing sequence $\langle \delta_n : n < \omega\rangle$ converging to δ, and let

$$D'_n := \{(p,\nu) \in D_n : \mathrm{lh}(\nu) > \delta_n\}.$$

Clearly, D'_n is dense open.

Note that, as the Mostowski collapse of N belongs to M_δ, we find that $<^*_\chi \restriction (\mathbb{Q}_\delta \times T_\delta) = (<^*_\chi \restriction (\mathbb{Q}\times T)) \cap N \in M_\delta$.

Now, still in M_δ, and starting with (q_0,η_0), we inductively choose a sequence $\langle (q_n,\eta_n) : n < \omega\rangle$ with $q_n \in \mathbb{Q}_\delta$ and $\eta_n \in T_\delta$, and such that if $n = m+1$, then:

(a) $p_n \geq q_n$ and $\eta_m <_T \eta_n$.
(b) $(q_n, \eta_n) \in D'_n$.
(c) (q_n, η_n) is the $<^*_\chi$-least such that (a) and (b) hold.

Then $\eta_* := (\bigcup_n \eta_n) \cup \{\langle \delta, \delta \rangle\} \in T$ and $\eta^* \in M_\delta$, hence $(q_0, \eta_*) \in \mathbb{Q} \times T$. Clearly, $(q_0, \eta_*) \leq (q_0, \eta_0)$. So, we need only check that (q_0, η_*) is $(N, \mathbb{Q} \times T)$-generic.

Fix an open dense $E \subseteq \mathbb{Q} \times T$ that belongs to N. We need to see that $E \cap N$ is predense below (q_0, η_*). So, fix $(r, \nu) \leq (q_0, \eta_*)$. Since $\mathbb{Q}$ is ccc, q_0 is $(N, \mathbb{Q})$-generic, so we can find $r' \in \{p : (p, \eta) \in E$ for some $\eta\} \cap N$ that is compatible with r. Let n be such that $p_n = r'$ and D_n is the Mostowski collapse of E. Then (p_n, η_n) belongs to the transitive collapse of E, hence to $E \cap N$, and is compatible with (r, ν), as $(p_n, \eta_*) \leq (p_n, \eta_n)$. ■

We thus conclude that if $G \subseteq \mathbb{Q}$ is a filter generic over V, then in $V[G]$ the forcing T does not collapse $\aleph_1$, and therefore, being of cardinality $\aleph_1$, it preserves cardinals, cofinalities, and the cardinal arithmetic.

We shall now define the $\mathbb{Q}$-names for the forcing notions $\underset{\sim}{\mathbb{P}}_\ell$, for $\ell \in \{0, 1\}$, as follows: in $V^{\mathbb{Q}}$, let $\underset{\sim}{b} = \bigcup \underset{\sim}{G}$, where $\underset{\sim}{G}$ is the standard $\mathbb{Q}$-name for the $\mathbb{Q}$-generic filter over V. Then let

$$\underset{\sim}{\mathbb{P}}_\ell := \{(w, c) : w \subseteq T \text{ is finite, } c \text{ is a function from } w \text{ into } \omega \text{ such that}$$
$$\text{if } \{\eta, \nu\} \in [w]^2_T \text{ and } \underset{\sim}{b}(\{\eta, \nu\}) = \ell, \text{ then } c(\eta) \neq c(\nu)\}.$$

A condition (w, c) is stronger than a condition (v, d) if and only if $w \supseteq v$ and $c \supseteq d$.

We shall show that if G is $\mathbb{Q}$-generic over V, then in the extension $V[G]$, the partial orderings $\mathbb{P}_\ell = \underset{\sim}{\mathbb{P}}_\ell[G]$, for $\ell \in \{0, 1\}$, and the forcing T are as required.

Claim 11. *In $V[G]$, $\mathbb{P}_\ell$ has precalibre-$\aleph_1$.*

Proof. Assume $p_\alpha = (w_\alpha, c_\alpha) \in \mathbb{P}_\ell$ for $\alpha < \omega_1$. We shall find an uncountable $S \subseteq \aleph_1$ such that $\{p_\alpha : \alpha \in S\}$ is finite-wise compatible. For each $\delta \in S_2$, let

$$s_\delta := \{\eta \restriction (\gamma+1) : \eta \in w_\delta, \text{ and } \gamma \text{ is maximal such that } \gamma < \mathrm{lh}(\eta) \wedge \eta(\gamma) < \delta\}.$$

As η is an increasing and continuous sequence of ordinals from S_1, hence disjoint from S_2, the set s_δ is well-defined. Notice that s_δ is a finite subset of $T_\delta := \{\eta \in T : \sup \mathrm{Range}(\eta) < \delta\}$, which is countable.

Let $s^1_\delta := w_\delta \cap T_\delta$. Note that $s^1_\delta \subseteq s_\delta$.

Let $f : S_2 \to \omega_1$ be given by $f(\delta) = \max\{\sup \mathrm{Range}(\eta) : \eta \in s_\delta\}$. Thus, f is regressive, hence constant on a stationary $S_3 \subseteq S_2$. Let δ_0 be the constant value of f on S_3. Then $s_\delta \subseteq T_{\delta_0}$ for every $\delta \in S_3$. So, since T_{δ_0} is countable, there exist $S_4 \subseteq S_3$ stationary and s_* such that $s_\delta = s_*$ for every $\delta \in S_4$. Further, there is a stationary $S_5 \subseteq S_4$ and s^1_* and c_* such that for all $\delta \in S_5$,

$$s^1_\delta = s^1_*, \quad c_\delta \restriction s^1_* = c_*, \quad \text{and} \quad \forall \alpha < \delta (w_\alpha \subseteq T_\delta).$$

Hence, if $\delta_1 < \delta_2$ are from S_5, then not only $w_{\delta_1} \cap w_{\delta_2} = s^1_*$, but also if $\eta_1 \in w_{\delta_1} - s^1_*$ and $\eta_2 \in w_{\delta_2} - s^1_*$, then η_1 and η_2 are $<_T$-incomparable. Indeed, suppose otherwise, say $\eta_1 <_T \eta_2$. If $\gamma+1 = \mathrm{lh}(\eta_1)$, then $\eta_2 \restriction (\gamma+1) = \eta_1 <_T \eta_2$, and $\eta_2(\gamma) = \eta_1(\gamma) < \delta_2$, by choice of S_5. Hence, by the definition of s_{δ_2}, $\eta_2 \restriction (\gamma+1) = \eta_1$ is an initial segment of some member of $s_{\delta_2} = s_*$, and so it belongs to T_{δ_1}, hence $\eta_1 \in s^1_*$, contradicting the assumption that $\eta_1 \notin s^1_*$.

So, $\{p_\delta : \delta \in S_5\}$ is as required. ∎

It only remains to show that forcing with T over $V[G]$ preserves the ccc-ness of $\mathbb{P}_0$ and $\mathbb{P}_1$, but makes their product not ccc.

Claim 12. *If G_T is T-generic over $V[G]$, then in the generic extension $V[G][G_T]$, the forcing $\mathbb{P}_\ell$ is ccc.*

Proof. First notice that, by the Product Lemma (see [6, 15.9]), G is $\mathbb{Q}$-generic over $V[G_T]$, and $V[G][G_T] = V[G_T][G]$. Now suppose that $\underset{\sim}{A} = \{(\underset{\sim}{w}_\alpha, \underset{\sim}{c}_\alpha) : \alpha < \omega_1\} \in V[G_T]$ is a $\mathbb{Q}$-name for an uncountable subset of $\mathbb{P}_\ell$. For each $\alpha < \omega_1$, let $p_\alpha \in \mathbb{Q}$ and (w_α, c_α) be such that $p_\alpha \Vdash$ "$(\underset{\sim}{w}_\alpha, \underset{\sim}{c}_\alpha) = (w_\alpha, c_\alpha)$". Let u_α be such that $\mathrm{dom}(p_\alpha) = [u_\alpha]^2_T$. By extending p_α if necessary, we may assume that $w_\alpha \subseteq u_\alpha$ for all $\alpha < \omega_1$. We shall find $\alpha \neq \beta$ and a condition p that extends both p_α and p_β and forces that (w_α, c_α) and (w_β, c_β) are compatible. For this, first extend (w_α, c_α) to (u_α, d_α) by letting d_α give different values in $\omega \setminus \mathrm{Range}(c_\alpha)$ to all $\eta \in u_\alpha \setminus w_\alpha$. We may assume that the set $\{u_\alpha : \alpha < \omega_1\}$ forms a Δ-system with root r. Moreover, we may assume that p_α restricted to $[r]^2_T$ is the same for all $\alpha < \omega_1$, and also that d_α restricted to r is the same for all $\alpha < \omega_1$. Now pick $\alpha \neq \beta$ and let $p : [u_\alpha \cup u_\beta]^2_T \to \{0,1\}$ be such that $p \restriction [u_\alpha]^2_T = p_\alpha$, $p \restriction [u_\beta]^2_T = p_\beta$, and $p(\{\eta, \nu\}) \neq \ell$ for all other pairs in $[u_\alpha \cup u_\beta]^2_T$. Then p extends both p_α and p_β, and forces that (u_α, d_α) and (u_β, d_β) are compatible, hence it forces that (w_α, c_α) and (w_β, c_β) are compatible. ∎

But in $V[G][G_T]$, the product $\mathbb{P}_0 \times \mathbb{P}_1$ is not ccc. Indeed, let $\eta^* = \bigcup G_T$. For every $\alpha < \omega_1$, let $p^\ell_\alpha := (\{\eta^* \restriction (\alpha+1)\}, c^\ell_\alpha) \in \mathbb{P}_\ell$, where $c^\ell_\alpha(\eta^* \restriction (\alpha+1)) = 0$. Then the set $\{(p^0_\alpha, p^1_\alpha) : \alpha < \omega_1\}$ is an uncountable antichain.

This finishes the proof of Theorem 9. ∎

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