COLLOQUIUM MATHEMATICUM

VOL. 142 2016 NO. 2

# RAMIFICATION OF THE GAUSS MAP OF COMPLETE MINIMAL SURFACES IN $\mathbb{R}^m$ ON ANNULAR ENDS

BY

GERD DETHLOFF (Brest), PHAM HOANG HA (Hanoi)
and PHAM DUC THOAN (Hanoi)

**Abstract.** We study the ramification of the Gauss map of complete minimal surfaces in $\mathbb{R}^m$ on annular ends. This is a continuation of previous work of Dethloff–Ha (2014), which we extend here to targets of higher dimension.

**1. Introduction.** In 1988, H. Fujimoto [4] proved Nirenberg's conjecture that if $M$ is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most four points, and the bound is sharp. Later, he also extended that result to minimal surfaces in $\mathbb{R}^m$. He proved that the Gauss map of a non-flat complete minimal surface can omit at most $m(m+1)/2$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position [6]. He also gave an example to show that the number $m(m+1)/2$ is the best possible when $m$ is odd [6].

In 1993, M. Ru [14] refined these results by studying the Gauss maps of minimal surfaces in $\mathbb{R}^m$ with ramification. Using the notation which will be introduced in §3, the result of Ru can be stated as follows.

THEOREM A. *Let $M$ be a non-flat complete minimal surface in $\mathbb{R}^m$. Assume that the (generalized) Gauss map $g$ of $M$ is $k$-non-degenerate (that is, $g(M)$ is contained in a $k$-dimensional linear subspace in $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension), $1 \le k \le m-1$. Let $\{H_j\}_{j=1}^q$ be hyperplanes in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ such that $g$ is ramified over $H_j$ with multiplicity at least $m_j$ for each $j$. Then*

$$\sum_{j=1}^{q}\left(1-\frac{k}{m_j}\right) \le (k+1)\left(m-\frac{k}{2}-1\right)+m.$$

*In particular if there are $q$ ($q > m(m+1)/2$) hyperplanes $\{H_j\}_{j=1}^q$ in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ such that $g$ is ramified over $H_j$ with multiplicity at least*

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2010 *Mathematics Subject Classification*: Primary 53A10; Secondary 53C42, 30D35, 32H30.
*Key words and phrases*: minimal surface, Gauss map, ramification, value distribution theory.

DOI: 10.4064/cm142-2-1

*$m_j$ for each $j$, then*

$$\sum_{j=1}^{q}\left(1-\frac{m-1}{m_j}\right) \leq \frac{m(m+1)}{2}.$$

On the other hand, in 1991, S. J. Kao [10] used the ideas of H. Fujimoto [4] to show that the Gauss map of an end of a non-flat complete minimal surface in $\mathbb{R}^3$ that is conformally an annulus $\{z : 0 < 1/r < |z| < r\}$ must also assume every value, with at most four exceptions. In 2007, L. Jin and M. Ru [9] extended Kao's result to minimal surfaces in $\mathbb{R}^m$. They proved:

THEOREM B. *Let $M$ be a non-flat complete minimal surface in $\mathbb{R}^m$ and let $A$ be an annular end of $M$ which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where $z$ is a conformal coordinate. Then the restriction to $A$ of the (generalized) Gauss map of $M$ cannot omit more than $m(m+1)/2$ hyperplanes in general position in $\mathbb{P}^{m-1}(\mathbb{C})$.*

Recently, the first two authors [3] gave an improvement of the theorem of Kao. Moreover they also gave an analogous result for the case $m = 4$. In this paper we will consider the corresponding problem for the (generalized) Gauss map for non-flat complete minimal surfaces in $\mathbb{R}^m$ for all $m \geq 3$. In this general situation we obtain the following:

MAIN THEOREM. *Let $M$ be a non-flat complete minimal surface in $\mathbb{R}^m$ and let $A$ be an annular end of $M$ which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where $z$ is a conformal coordinate. Assume that the generalized Gauss map $g$ of $M$ is $k$-non-degenerate on $A$ (that is, $g(A)$ is contained in a $k$-dimensional linear subspace in $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension), $1 \leq k \leq m-1$. If there are $q$ hyperplanes $\{H_j\}_{j=1}^q$ in $N$-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ $(N \geq m-1)$ such that $g$ is ramified over $H_j$ with multiplicity at least $m_j$ on $A$ for each $j$, then*

$$\sum_{j=1}^{q}\left(1-\frac{k}{m_j}\right) \leq (k+1)\left(N-\frac{k}{2}\right)+(N+1). \tag{1.1}$$

*Moreover, (1.1) still holds if for all $j = 1, \dots, q$ we replace $m_j$ by the limit inferior of the orders of the zeros of the function $(g, H_j) := \overline{c}_{j0}g_1 + \cdots + \overline{c}_{jm-1}g_{m-1}$ on $A$ (where $g = (g_0 : \cdots : g_{m-1})$ is a reduced representation and, for all $1 \leq j \leq q$, the hyperplane $H_j$ in $\mathbb{P}^{m-1}(\mathbb{C})$ is given by $H_j : \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jm-1}\omega_{m-1} = 0$, where we assume that $\sum_{i=0}^{m-1}|c_{ji}|^2 = 1$) or by $\infty$ if $g$ intersects $H_j$ only a finite number of times on $A$.*

COROLLARY 1. *Let $M$ be a non-flat complete minimal surface in $\mathbb{R}^m$ and let $A$ be an annular end of $M$ which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where $z$ is a conformal coordinate. If there are $q$ hyperplanes $\{H_j\}_{j=1}^q$ in*

*$N$-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ $(N \geq m-1)$ such that the generalized Gauss map $g$ of $M$ is ramified over $H_j$ with multiplicity at least $m_j$ on $A$ for each $j$, then*

$$\sum_{j=1}^{q}\left(1-\frac{m-1}{m_j}\right) \leq m\left(N-\frac{m-1}{2}\right)+(N+1). \tag{1.2}$$

*In particular if the hyperplanes $\{H_j\}_{j=1}^{q}$ are in general position in $\mathbb{P}^{m-1}(\mathbb{C})$, then*

$$\sum_{j=1}^{q}\left(1-\frac{m-1}{m_j}\right) \leq \frac{m(m+1)}{2}. \tag{1.3}$$

*Moreover,* (1.2) *and* (1.3) *still hold if for all $j = 1, \ldots, q$ we replace $m_j$ by the limit inferior of the orders of the zeros of the function $(g, H_j) := \bar{c}_{j0}g_1 + \cdots + \bar{c}_{jm-1}g_{m-1}$ on $A$ (where $g = (g_0 : \cdots : g_{m-1})$ is a reduced representation and, for all $1 \leq j \leq q$, the hyperplane $H_j$ in $\mathbb{P}^{m-1}(\mathbb{C})$ is given by $H_j : \bar{c}_{j0}\omega_0 + \cdots + \bar{c}_{jm-1}\omega_{m-1} = 0$, where we assume that $\sum_{i=0}^{m-1} |c_{ji}|^2 = 1$) or by $\infty$ if $g$ intersects $H_j$ only a finite number of times on $A$.*

Our Corollary 1 gives the following improvement of Theorem B of Jin–Ru:

Corollary 2. *If the* (*generalized*) *Gauss map $g$ on an annular end of a non-flat complete minimal surface in $\mathbb{R}^m$ assumes $m(m+1)/2$ hyperplanes in general position only finitely often, then it takes any other hyperplane in general position* (*with respect to the previous hyperplanes*) *infinitely often with ramification at most $m-1$.*

Remark. It is well known that the image of the (generalized) Gauss map $g : M \to \mathbb{P}^{m-1}$ is contained in the hyperquadric $Q_{m-2} \subset \mathbb{P}^{m-1}$, and that $Q_1(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C})$ and that $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. So the results in Dethloff–Ha [3] which only treat the cases $m = 3$ and $m = 4$ are better than a result which holds for any $m \geq 3$ can be if restricted to the special cases $m = 3, 4$. The easiest way to see the difference is to observe that six lines in $\mathbb{P}^2$ in general position may have only four points of intersection with the quadric $Q_1 \subset \mathbb{P}^2$.

The main idea to prove the Main Theorem is to construct and compare explicit singular flat and negatively curved complete metrics with ramification on these annular ends. This generalizes previous work of Dethloff–Ha [3] (which itself was a refinement of ideas of Ru [14]) to targets of higher dimensions, which needs among other things to combine these explicit singular metrics with the use of technics from hyperplanes in subgeneral position and with the use of intermediate contact functions. After that we use arguments similar to those used by Kao [10] and Fujimoto [4]–[7] to finish the proofs.

**2. Preliminaries.** Let $f$ be a linearly non-degenerate holomorphic map of $\Delta_R := \{z \in \mathbb{C} : |z| < R\}$ into $\mathbb{P}^k(\mathbb{C})$, where $0 < R \leq \infty$. Take a reduced representation $f = (f_0 : \cdots : f_k)$. Then $F := (f_0, \ldots, f_k) : \Delta_R \to \mathbb{C}^{k+1} \setminus \{0\}$ is a holomorphic map with $\mathbb{P}(F) = f$. Consider the holomorphic map

$$F_p = (F_p)_z := F^{(0)} \wedge \cdots \wedge F^{(p)} : \Delta_R \to \bigwedge^{p+1}\mathbb{C}^{k+1}$$

for $0 \leq p \leq k$, where $F^{(0)} := F = (f_0, \ldots, f_k)$ and $F^{(l)} = (F^{(l)})_z := (f_0^{(l)}, \ldots, f_k^{(l)})$ for each $l = 0, \ldots, k$, and where the $l$-th derivatives $f_i^{(l)} = (f_i^{(l)})_z$, $i = 0, \ldots, k$, are taken with respect to $z$. (Here and for the rest of this paper the subscript $z$ means that the corresponding term is defined by using differentiation with respect to the variable $z$, and in order to keep notation simple, we usually drop this subscript if no confusion is possible.) The norm of $F_p$ is given by

$$|F_p| := \Big( \sum_{0 \leq i_0 < \cdots < i_p \leq k} |W(f_{i_0}, \ldots, f_{i_p})|^2 \Big)^{1/2},$$

where $W(f_{i_0}, \ldots, f_{i_p}) = W_z(f_{i_0}, \ldots, f_{i_p})$ denotes the Wronskian of $f_{i_0}, \ldots, f_{i_p}$ with respect to $z$.

Proposition 1 ([7, Proposition 2.1.6]). *For two holomorphic local coordinates $z$ and $\xi$ and a holomorphic function $h : \Delta_R \to \mathbb{C}$, the following hold:*

(a) $W_\xi(f_0, \ldots, f_p) = W_z(f_0, \ldots, f_p) \cdot \big(\frac{dz}{d\xi}\big)^{p(p+1)/2}$.
(b) $W_z(hf_0, \ldots, hf_p) = W_z(f_0, \ldots, f_p) \cdot (h)^{p+1}$.

Proposition 2 ([7, Proposition 2.1.7]). *For holomorphic functions $f_0, \ldots, f_p : \Delta_R \to \mathbb{C}$ the following conditions are equivalent:*

(i) *$f_0, \ldots, f_p$ are linearly dependent over $\mathbb{C}$.*
(ii) *$W_z(f_0, \ldots, f_p) \equiv 0$ for some (or all) holomorphic local coordinate $z$.*

We now take a hyperplane $H$ in $\mathbb{P}^k(\mathbb{C})$ given by

$$H : \bar{c}_0\omega_0 + \cdots + \bar{c}_k\omega_k = 0$$

with $\sum_{i=0}^k |c_i|^2 = 1$. We set

$$F_0(H) := F(H) := \bar{c}_0 f_0 + \cdots + \bar{c}_k f_k$$

and

$$|F_p(H)| = |(F_p)_z(H)| := \Big( \sum_{0 \leq i_1 < \cdots < i_p \leq k} \Big| \sum_{l \neq i_1, \ldots, i_p} \bar{c}_l W(f_l, f_{i_1}, \ldots, f_{i_p}) \Big|^2 \Big)^{1/2}$$

for $1 \leq p \leq k$. We note that by using Proposition 1, $|(F_p)_z(H)|$ is multiplied by a factor $\big|\frac{dz}{d\xi}\big|^{p(p+1)/2}$ if we choose another holomorphic local coordinate $\xi$, and it is multiplied by $|h|^{p+1}$ if we choose another reduced representation

$f = (hf_0 : \cdots : hf_k)$ with a nowhere zero holomorphic function $h$. Finally, for $0 \le p \le k$, define the $p$-th *contact function* of $f$ for $H$ to be

$$\phi_p(H) := \frac{|F_p(H)|^2}{|F_p|^2} = \frac{|(F_p)_z(H)|^2}{|(F_p)_z|^2}.$$

We next consider $q$ hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^k(\mathbb{C})$ given by

$$H_j : \langle \omega, A_j \rangle \equiv \bar{c}_{j0}\omega_0 + \cdots + \bar{c}_{jk}\omega_k \quad (1 \le j \le q),$$

where $A_j := (c_{j0}, \dots, c_{jk})$ with $\sum_{i=0}^{k} |c_{ji}|^2 = 1$.

Assume now $N \ge k$ and $q \ge N+1$. For $R \subseteq Q := \{1, \dots, q\}$, denote by $d(R)$ the dimension of the vector subspace of $\mathbb{C}^{k+1}$ generated by $\{A_j : j \in R\}$.

The hyperplanes $H_1, \dots, H_q$ are said to be *in $N$-subgeneral position* if $d(R) = k+1$ for all $R \subseteq Q$ with $\#(R) \ge N+1$, where $\#(A)$ means the number of elements of a set $A$. In the particular case $N = k$, these are said to be *in general position*.

Theorem 3 ([7, Theorem 2.4.11]). *For given hyperplanes $H_1, \dots, H_q$ $(q > 2N - k + 1)$ in $\mathbb{P}^k(\mathbb{C})$ in $N$-subgeneral position, there are some rational numbers $\omega(1), \dots, \omega(q)$ and $\theta$ satisfying the following conditions:*

(i) $0 < \omega(j) \le \theta \le 1$ $(1 \le j \le q)$,
(ii) $\sum_{j=1}^{q} \omega(j) = k + 1 + \theta(q - 2N + k - 1)$,
(iii) $\frac{k+1}{2N-k+1} \le \theta \le \frac{k+1}{N+1}$,
(iv) *if* $R \subset Q$ *and* $0 < \#(R) \le n+1$, *then* $\sum_{j \in R} \omega(j) \le d(R)$.

Constants $\omega(j)$ $(1 \le j \le q)$ and $\theta$ with the properties of Theorem 3 are called *Nochka weights* and a *Nochka constant* for $H_1, \dots, H_q$ respectively. Related to Nochka weights, we have the following.

Proposition 4 ([7, Proposition 2.4.15]). *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in $N$-subgeneral position and let $\omega(1), \dots, \omega(q)$ be Nochka weights for them, where $q > 2N - k + 1$. For each $R \subseteq Q := \{1, \dots, q\}$ with $0 < \#(R) \le N+1$ and real constants $E_1, \dots, E_q$ with $E_j \ge 1$, there is some $R' \subseteq R$ such that $\#(R') = d(R) = d(R')$ and*

$$\prod_{j \in R} E_j^{\omega(j)} \le \prod_{j \in R'} E_j.$$

We need the following three results of Fujimoto combining the previously introduced concept of contact functions with Nochka weights:

Theorem 5 ([7, Theorem 2.5.3]). *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in $N$-subgeneral position and let $\omega(j)$ $(1 \le j \le q)$ and $\theta$ be Nochka weights and a Nochka constant for these hyperplanes. For every $\epsilon > 0$ there exist some positive numbers $\delta$ $(> 1)$ and $C$, depending only on $\epsilon$ and $H_j$,*

$1 \le j \le q$, *such that*

$$\text{(2.4)} \qquad dd^c \log \frac{\prod_{p=0}^{k-1} |F_p|^{2\epsilon}}{\prod_{1\le j\le q,\, 0\le p\le k-1} \log^{2\omega(j)}(\delta/\phi_p(H_j))} \ge C\biggl(\frac{|F_0|^{2\theta(q-2N+k-1)}|F_k|^2}{\prod_{j=1}^{q}\bigl(|F(H_j)|^2 \prod_{p=0}^{k-1}\log^2(\delta/\phi_p(H_j))\bigr)^{\omega(j)}}\biggr)^{\frac{2}{k(k+1)}} dd^c|z|^2.$$

PROPOSITION 6 ([7, Proposition 2.5.7]). *Set* $\sigma_p = p(p+1)/2$ *for* $0 \le p \le k$ *and* $\tau_k = \sum_{p=0}^{k} \sigma_p$. *Then*

$$\text{(2.5)} \qquad dd^c \log(|F_0|^2 \cdots |F_{k-1}|^2) \ge \frac{\tau_k}{\sigma_k}\biggl(\frac{|F_0|^2 \cdots |F_k|^2}{|F_0|^{2\sigma_{k+1}}}\biggr)^{1/\tau_k} dd^c|z|^2.$$

PROPOSITION 7 ([7, Lemma 3.2.13]). *Let $f$ be a non-degenerate holomorphic map of a domain in $\mathbb{C}$ into $\mathbb{P}^k(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_k)$, and let $H_1, \ldots, H_q$ be hyperplanes in $N$-subgeneral position $(q > 2N - k + 1)$ with Nochka weights $\omega(1), \ldots, \omega(q)$ respectively. Then*

$$\nu_\phi + \sum_{j=1}^{q} \omega(j) \cdot \min(\nu_{(f,H_j)}, k) \ge 0, \quad \textit{where} \quad \phi = \frac{|F_k|}{\prod_{j=1}^{q} |F(H_j)|^{\omega(j)}}.$$

We will also use

LEMMA 8 (Generalized Schwarz's Lemma [1]). *Let $v$ be a non-negative real-valued continuous subharmonic function on $\Delta_R$. If $v$ satisfies the inequality $\Delta \log v \ge v^2$ in the sense of distributions, then*

$$v(z) \le \frac{2R}{R^2 - |z|^2}.$$

LEMMA 9. *Let $f = (f_0 : \cdots : f_k) : \Delta_R \to \mathbb{P}^k(\mathbb{C})$ be a non-degenerate holomorphic map, $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in $N$-subgeneral position ($N \ge k$ and $q > 2N - k + 1$), and $\omega(j)$ be their Nochka weights. If*

$$\gamma := \sum_{j=1}^{q} \omega(j)\biggl(1 - \frac{k}{m_j}\biggr) - (k+1) > 0$$

*and $f$ is ramified over $H_j$ with multiplicity at least $m_j \ge k$ for each $j$ $(1 \le j \le q)$, then for any positive $\epsilon$ with $\gamma > \epsilon\sigma_{k+1}$ there exists a positive constant $C$, depending only on $\epsilon, H_j, m_j, \omega(j)$ $(1 \le j \le q)$, such that*

$$|F|^{\gamma - \epsilon\sigma_{k+1}} \frac{|F_k|^{1+\epsilon} \prod_{j=1}^{q} \prod_{p=0}^{k-1} |F_p(H_j)|^{\epsilon/q}}{\prod_{j=1}^{q} |F(H_j)|^{\omega(j)(1-k/m_j)}} \le C\biggl(\frac{2R}{R^2 - |z|^2}\biggr)^{\sigma_k + \epsilon\tau_k}.$$

*Proof.* For an arbitrary holomorphic local coordinate $z$ and $\delta$ $(> 1)$ chosen as in Theorem 5 we set

$$\eta_z := \left( \frac{|F|^{\gamma-\epsilon\sigma_{k+1}} \cdot |F_k| \cdot \prod_{p=0}^{k} |F_p|^{\epsilon}}{\prod_{j=1}^{q} \bigl( |F(H_j)|^{(1-k/m_j)} \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \bigr)^{\omega(j)}} \right)^{\frac{1}{\sigma_k+\epsilon\tau_k}},$$

and define the pseudometric $d\tau_z^2 := \eta_z^2 |dz|^2$. Using Proposition 1 we can see that

$$\begin{aligned} d\tau_\xi &:= \left( \frac{|F|^{\gamma-\epsilon\sigma_{k+1}} \cdot |(F_k)_\xi| \cdot \prod_{p=0}^{k} |(F_p)_\xi|^{\epsilon}}{\prod_{j=1}^{q} \bigl( |F(H_j)|^{(1-k/m_j)} \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \bigr)^{\omega(j)}} \right)^{\frac{1}{\sigma_k+\epsilon\tau_k}} |d\xi| \\ &= \left( \frac{|F|^{\gamma-\epsilon\sigma_{k+1}} \cdot |(F_k)_z| \left|\frac{dz}{d\xi}\right|^{\sigma_k} \cdot \prod_{p=0}^{k} |(F_p)_z|^{\epsilon} \cdot \left|\frac{dz}{d\xi}\right|^{\sum_{j=0}^{k} \epsilon \frac{j(j+1)}{2}}}{\prod_{j=1}^{q} \bigl( |F(H_j)|^{(1-k/m_j)} \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \bigr)^{\omega(j)}} \right)^{\frac{1}{\sigma_k+\epsilon\tau_k}} \\ &\quad \times \left| \frac{d\xi}{dz} \right| |dz| \\ &= \left( \frac{|F|^{\gamma-\epsilon\sigma_{k+1}} \cdot |(F_k)_z| \cdot \prod_{p=0}^{k} |(F_p)_z|^{\epsilon} \cdot \left|\frac{dz}{d\xi}\right|^{\sigma_k+\epsilon\tau_k}}{\prod_{j=1}^{q} \bigl( |F(H_j)|^{(1-k/m_j)} \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \bigr)^{\omega(j)}} \right)^{\frac{1}{\sigma_k+\epsilon\tau_k}} \left| \frac{d\xi}{dz} \right| |dz| \\ &= d\tau_z. \end{aligned}$$

Thus $d\tau_z^2$ is independent of the choice of the local coordinate $z$. We will denote $d\tau_z^2$ by $d\tau^2$ for convenience.

We now show that $d\tau$ is continuous on $\Delta_R$. Indeed, it is easy to see that $d\tau$ is continuous at every point $z_0$ with $\prod_{j=1}^{q} F(H_j)(z_0) \neq 0$. Now we take a point $z_0$ such that $\prod_{j=1}^{q} F(H_j)(z_0) = 0$. We have

$$\begin{aligned} \nu_{d\tau}(z_0) &\geq \frac{1}{\sigma_k + \epsilon\tau_k} \left( \nu_{F_k}(z_0) - \sum_{j=1}^{q} \omega(j) \nu_{F(H_j)}(z_0) \left( 1 - \frac{k}{m_j} \right) \right) \\ &= \frac{1}{\sigma_k + \epsilon\tau_k} \left( \nu_{F_k}(z_0) - \sum_{j=1}^{q} \omega(j) \nu_{F(H_j)}(z_0) + \sum_{j=1}^{q} \omega(j) \frac{k}{m_j} \nu_{F(H_j)}(z_0) \right). \end{aligned}$$

Combining this with Proposition 7 we get

$$\begin{aligned} &\nu_{d\tau}(z_0) \\ &\quad \geq \frac{1}{\sigma_k + \epsilon\tau_k} \left( - \sum_{j=1}^{q} \omega(j) \min\{\nu_{F(H_j)}(z_0), k\} + \sum_{j=1}^{q} \omega(j) \frac{k}{m_j} \nu_{F(H_j)}(z_0) \right). \end{aligned}$$

By assumption, $\nu_{F(H_j)}(z_0) \geq m_j \geq k$ or $\nu_{F(H_j)}(z_0) = 0$, so $\nu_{d\tau}(z_0) \geq 0$. This concludes the proof that $d\tau$ is continuous on $\Delta_R$.

Using Proposition 6, Theorem 5 and noting that $dd^c \log |F_k| = 0$, we have

$$\begin{aligned}
dd^c \log \eta_z &= \frac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k} dd^c \log |F| + \frac{\epsilon}{4(\sigma_k + \epsilon\tau_k)} dd^c \log(|F_0|^2 \cdots |F_{k-1}|^2) \\
&\quad + \frac{1}{2(\sigma_k + \epsilon\tau_k)} dd^c \log \frac{\prod_{p=0}^{k-1} |F_p|^{2\epsilon/2}}{\prod_{j=1}^{q} \prod_{p=0}^{k-1} \log^{2\omega(j)}(\delta/\phi_p(H_j))} \\
&\geq \frac{\epsilon}{4(\sigma_k + \epsilon\tau_k)} \frac{\tau_k}{\sigma_k} \Big( \frac{|F_0|^2 \cdots |F_k|^2}{|F_0|^{2\sigma_{k+1}}} \Big)^{1/\tau_k} dd^c |z|^2 \\
&\quad + C_0 \bigg( \frac{|F_0|^{2\theta(q-2N+k-1)} |F_k|^2}{\prod_{j=1}^{q} \big( |F(H_j)|^2 \prod_{p=0}^{k-1} \log^2(\delta/\phi_p(H_j)) \big)^{\omega(j)}} \bigg)^{\frac{2}{k(k+1)}} dd^c |z|^2 \\
&\geq \min\Big\{ \frac{1}{4\sigma_k(\sigma_k + \epsilon\tau_k)}, \frac{C_0}{\sigma_k} \Big\} \bigg( \epsilon\tau_k \Big( \frac{|F_0|^2 \cdots |F_k|^2}{|F_0|^{2\sigma_{k+1}}} \Big)^{1/\tau_k} \\
&\quad + \sigma_k \bigg( \frac{|F_0|^{2\theta(q-2N+k-1)} |F_k|^2}{\prod_{j=1}^{q} \big( |F(H_j)|^2 \prod_{p=0}^{k-1} \log^2(\delta/\phi_p(H_j)) \big)^{\omega(j)}} \bigg)^{1/\sigma_k} \bigg) dd^c |z|^2,
\end{aligned}$$

where $C_0$ is the positive constant. So, by using the basic inequality

$$\alpha A + \beta B \geq (\alpha + \beta) A^{\alpha/(\alpha+\beta)} B^{\beta/(\alpha+\beta)} \qquad \text{for all } \alpha, \beta, A, B > 0,$$

we can find a positive constant $C_1$ satisfying

$$\begin{aligned}
dd^c \log \eta_z &\geq C_1 \bigg( \frac{|F|^{\theta(q-2N+k-1)-\epsilon\sigma_{k+1}} \cdot |F_k| \cdot \prod_{p=0}^{k} |F_p|^{\epsilon}}{\prod_{j=1}^{q} \big( |F(H_j)| \cdot \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \big)^{\omega(j)}} \bigg)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2 \\
&= C_1 \bigg( \frac{|F|^{\sum_{j=1}^{q} \omega(j) - k - 1 - \epsilon\sigma_{k+1}} \cdot |F_k| \cdot \prod_{p=0}^{k} |F_p|^{\epsilon}}{\prod_{j=1}^{q} \big( |F(H_j)| \cdot \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \big)^{\omega(j)}} \bigg)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2 \\
&\qquad\qquad \text{(by Theorem 3)} \\
&= C_1 \bigg( \frac{|F|^{\gamma - \epsilon\sigma_{k+1}} \cdot |F_k| \cdot \prod_{p=0}^{k} |F_p|^{\epsilon} \prod_{j=1}^{q} \big( \frac{|F|}{|F(H_j)|} \big)^{\frac{k}{m_j}\omega(j)}}{\prod_{j=1}^{q} \big( |F(H_j)|^{(1-k/m_j)} \cdot \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \big)^{\omega(j)}} \bigg)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2.
\end{aligned}$$

On the other hand,

$$\Big( \frac{|F(H_j)|}{|F|} \Big)^{\frac{k}{m_j}\omega(j)} \leq 1 \qquad \text{for all } j = 1, \ldots, q,$$

so we get

$$\begin{aligned}
&dd^c \log \eta_z \\
&\quad \geq C_1 \bigg( \frac{|F|^{\gamma - \epsilon\sigma_{k+1}} \cdot |F_k| \cdot \prod_{p=0}^{k} |F_p|^{\epsilon}}{\prod_{j=1}^{q} \big( |F(H_j)|^{(1-k/m_j)} \cdot \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)) \big)^{\omega(j)}} \bigg)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2 \\
&\quad = C_1 \eta_z^2 dd^c |z|^2.
\end{aligned}$$

We now use Lemma 8 to show that

$$\Big(\frac{|F|^{\gamma-\epsilon\sigma_{k+1}}\cdot|F_k|\cdot\prod_{p=0}^{k}|F_p|^{\epsilon}}{\prod_{j=1}^{q}\big(|F(H_j)|^{(1-k/m_j)}\cdot\prod_{p=0}^{k-1}\log(\delta/\phi_p(H_j))\big)^{\omega(j)}}\Big)^{\frac{1}{\sigma_k+\epsilon\tau_k}}\le C_2\frac{2R}{R^2-|z|^2}.$$

Then

$$\Big(\frac{|F|^{\gamma-\epsilon\sigma_{k+1}}\cdot|F_k|^{1+\epsilon}\cdot\prod_{j=1}^{q}\prod_{p=0}^{k-1}|F_p(H_j)|^{\epsilon/q}}{\prod_{j=1}^{q}|F(H_j)|^{(1-\frac{k}{m_j})\omega(j)}\cdot\prod_{j=1}^{q}\prod_{p=0}^{k-1}\big((\frac{|F_p(H_j)|}{|F_p|})^{\epsilon/q}\log^{\omega(j)}(\delta/\phi_p(H_j))\big)}\Big)^{\frac{1}{\sigma_k+\epsilon\tau_k}}$$
$$\le C_2\frac{2R}{R^2-|z|^2}.$$

Moreover, combining this with

$$\sup_{0<x\le 1} x^{\epsilon/q}\log^{\omega(j)}(\delta/x^2)<\infty,$$

we get

$$\Big(\frac{|F|^{\gamma-\epsilon\sigma_{k+1}}\cdot|F_k|^{1+\epsilon}\cdot\prod_{j=1}^{q}\prod_{p=0}^{k-1}|F_p(H_j)|^{\epsilon/q}}{\prod_{j=1}^{q}|F(H_j)|^{(1-k/m_j)\omega(j)}}\Big)^{\frac{1}{\sigma_k+\epsilon\tau_k}}\le C\frac{2R}{R^2-|z|^2},$$

where $C$ is the positive constant depending, by Theorem 5 and by our construction, only on $\epsilon, H_j, m_j, \omega(j)$ $(1\le j\le q)$. This implies Lemma 9. ■

We finally need the following result on completeness of open Riemann surfaces with conformally flat metrics due to Fujimoto:

LEMMA 10 ([7, Lemma 1.6.7]). *Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface $M$. Then for every point $p\in M$, there is a holomorphic and locally biholomorphic map $\Phi$ of a disk (possibly with radius $\infty$) $\Delta_{R_0}:=\{w:|w|<R_0\}$ $(0<R_0\le\infty)$ onto an open neighborhood of $p$ with $\Phi(0)=p$ such that $\Phi$ is a local isometry, namely the pull-back $\Phi^*(d\sigma^2)$ is equal to the standard (flat) metric on $\Delta_{R_0}$, and for some point $a_0$ with $|a_0|=1$, the $\Phi$-image of the curve*

$$L_{a_0}: w:=a_0\cdot s\quad (0\le s<R_0)$$

*is divergent in $M$ (i.e., for any compact set $K\subset M$, there exists an $s_0<R_0$ such that the $\Phi$-image of the curve $L_{a_0}: w:=a_0\cdot s$ $(s_0\le s<R_0)$ does not intersect $K$).*

**3. The proof of the Main Theorem.** For the convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^m$. Let $M$ be a complete immersed minimal surface in $\mathbb{R}^m$. Take an immersion $x=(x_0,\dots,x_{m-1}): M\to\mathbb{R}^m$. Then $M$ has the structure of a Riemann surface, and any local isothermal coordinate $(x,y)$ of $M$ gives a local holomorphic coordinate $z=x+\sqrt{-1}\,y$. The *generalized Gauss map* of

$x$ is defined to be

$$g : M \to \mathbb{P}^{m-1}(\mathbb{C}), \qquad g = \mathbb{P}\left(\frac{\partial x}{\partial z}\right) = \left(\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}\right).$$

Since $x : M \to \mathbb{R}^m$ is immersed,

$$G = G_z := (g_0, \ldots, g_{m-1}) = ((g_0)_z, \ldots, (g_{m-1})_z) = \left(\frac{\partial x_0}{\partial z}, \ldots, \frac{\partial x_{m-1}}{\partial z}\right)$$

is a (local) reduced representation of $g$, and since for another local holomorphic coordinate $\xi$ on $M$ we have $G_\xi = G_z \cdot \left(\frac{dz}{d\xi}\right)$, $g$ is well defined (independently of the (local) holomorphic coordinate). Moreover, if $ds^2$ is the metric on $M$ induced by the standard metric on $\mathbb{R}^m$, then

$$ds^2 = 2|G_z|^2|dz|^2. \tag{3.6}$$

Finally since $M$ is minimal, $g$ is a holomorphic map.

Since by hypothesis of the Main Theorem, $g$ is $k$-non-degenerate $(1 \leq k \leq m-1)$, without loss of generality we may assume that $g(M) \subset \mathbb{P}^k(\mathbb{C})$; then

$$g : M \to \mathbb{P}^k(\mathbb{C}), \qquad g = \mathbb{P}\left(\frac{\partial x}{\partial z}\right) = \left(\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_k}{\partial z}\right),$$

is linearly non-degenerate in $\mathbb{P}^k(\mathbb{C})$ (so in particular $g$ is not constant) and the other facts mentioned above still hold.

Let $H_j$ $(j = 1, \ldots, q)$ be $q$ $(\geq N+1)$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in $N$-subgeneral position $(N \geq m-1 \geq k)$. Then $H_j \cap \mathbb{P}^k(\mathbb{C})$ $(j = 1, \ldots, q)$ are $q$ hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in $N$-subgeneral position. Let each $H_j \cap \mathbb{P}^k(\mathbb{C})$ be represented as

$$H_j \cap \mathbb{P}^k(\mathbb{C}) : \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jk}\omega_k = 0$$

with $\sum_{i=0}^k |c_{ji}|^2 = 1$.

Set

$$G(H_j) = G_z(H_j) := \overline{c}_{j0}g_0 + \cdots + \overline{c}_{jk}g_k.$$

We will now, for each contact function $\phi_p(H_j)$ for each of our hyperplanes $H_j$, choose one of the components of the numerator $|((G_z)_p)_z(H_j)|$ which is not identically zero: More precisely, for each $j, p$ $(1 \leq j \leq q,\ 1 \leq p \leq k)$, we can choose $i_1, \ldots, i_p$ with $0 \leq i_1 < \cdots < i_p \leq k$ such that

$$\psi(G)_{jp} = (\psi(G_z)_{jp})_z := \sum_{l \neq i_1, \ldots, i_p} \overline{c}_{jl} W_z(g_l, g_{i_1}, \ldots, g_{i_p}) \not\equiv 0.$$

(Indeed, otherwise we have $\sum_{l \neq i_1, \ldots, i_p} \overline{c}_{jl} W(g_l, g_{i_1}, \ldots, g_{i_p}) \equiv 0$ for all $i_1, \ldots, i_p$, so $W(\sum_{l \neq i_1, \ldots, i_p} \overline{c}_{jl} g_l, g_{i_1}, \ldots, g_{i_p}) \equiv 0$ for all $i_1, \ldots, i_p$, which contradicts the non-degeneracy of $g$ in $\mathbb{P}^k(\mathbb{C})$. Alternatively we can simply observe that in our situation none of the contact functions vanishes identically.) We still set $\psi(G)_{j0} = \psi(G_z)_{j0} := G(H_j)$ $(\not\equiv 0)$, and we also note that $\psi(G)_{jk} = ((G_z)_k)_z$. Since the $\psi(G)_{jp}$ are holomorphic, they have only isolated zeros.

Finally, for later use we write down the transformation formulas for all the terms defined above, which are obtained by using Proposition 1. For local holomorphic coordinates $z$ and $\xi$ on $M$ we have

$$G_\xi = G_z \cdot \left(\frac{dz}{d\xi}\right), \tag{3.7}$$

$$G_\xi(H) = G_z(H) \cdot \left(\frac{dz}{d\xi}\right), \tag{3.8}$$

$$((G_\xi)_k)_\xi = ((G_z)_k)_z \cdot \left(\frac{dz}{d\xi}\right)^{k+1+k(k+1)/2} = ((G_z)_k)_z \cdot \left(\frac{dz}{d\xi}\right)^{\sigma_{k+1}}, \tag{3.9}$$

$$\begin{aligned}(\psi(G_\xi)_{jp})_\xi &= (\psi(G_z)_{jp})_z \cdot \left(\frac{dz}{d\xi}\right)^{p+1+p(p+1)/2} \\ &= (\psi(G_z)_{jp})_z \cdot \left(\frac{dz}{d\xi}\right)^{\sigma_{p+1}} \qquad (0 \le p \le k).\end{aligned} \tag{3.10}$$

Moreover, we will also need the following transformation formulas for mixed variables:

$$((G_\xi)_k)_\xi = ((G_\xi)_k)_z \cdot \left(\frac{dz}{d\xi}\right)^{k(k+1)/2} = ((G_\xi)_k)_z \cdot \left(\frac{dz}{d\xi}\right)^{\sigma_k}, \tag{3.11}$$

$$\begin{aligned}(\psi(G_\xi)_{jp})_\xi &= (\psi(G_\xi)_{jp})_z \cdot \left(\frac{dz}{d\xi}\right)^{p(p+1)/2} \\ &= (\psi(G_\xi)_{jp})_z \cdot \left(\frac{dz}{d\xi}\right)^{\sigma_p} \qquad (0 \le p \le k).\end{aligned} \tag{3.12}$$

Now we prove the Main Theorem in four steps:

STEP 1. We will fix notation on the annular end $A \subset M$. Moreover, by passing to a subannular end of $A \subset M$ we simplify the geometry of the Main Theorem.

Let $A \subset M$ be an annular end of $M$, that is, $A = \{z : 0 < 1/r < |z| < r < \infty\}$, where $z$ is a (global) conformal coordinate of $A$. Since $M$ is complete with respect to $ds^2$, we may assume that the restriction of $ds^2$ to $A$ is complete on the set $\{z : |z| = r\}$, i.e., this set is at infinite distance from any point of $A$.

Let $m_j$ be the limit inferior of the orders of the zeros of the functions $G(H_j)$ on $A$, or $m_j = \infty$ if $G(H_j)$ has only a finite number of zeros on $A$.

All the $m_j$ are increasing if we only consider the zeros which the functions $G(H_j)$ take on a subset $B \subset A$. So without loss of generality we may prove our theorem only on a subannular end, i.e., a subset $A_t := \{z : 0 < t \le |z| < r < \infty\} \subset A$ with some $t$ such that $1/r < t < r$. (We trivially observe that for $c := tr > 1$, $s := r/\sqrt{c}$, $\xi := z/\sqrt{c}$, we have $A_t = \{\xi : 0 < 1/s \leq |\xi| < s < \infty\}$.)

By passing to such a subannular end we will be able to extend the construction of a metric in Step 2 below to the set $\{z : |z| = 1/r\}$, and moreover we may assume that for all $j = 1, \ldots, q$,

(3.13) $g$ omits $H_j$ ($m_j = \infty$) or takes $H_j$ infinitely often with ramification $m_j < \infty$ and is ramified over $H_j$ with multiplicity at least $m_j$.

We next observe that we may also assume

$$m_j > k, \qquad j = 1, \ldots, q. \tag{3.14}$$

In fact, if this does not hold for all $j = 1, \ldots, q$, we just drop the $H_j$ for which it does not hold, and remain with $\tilde{q} < q$ such hyperplanes. If $\tilde{q} \geq N + 1$, they are still in $N$-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$ and we prove our Main Theorem for $\tilde{q}$ instead of $q$; if $\tilde{q} < N + 1$, the assertion (1.1) of our Main Theorem trivially holds. In both cases, since by passing from $\tilde{q}$ to $q$ again the right hand side of (1.1) does not change, however the left hand side only becomes possibly smaller, the inequality (1.1) still holds if we (re)consider all the $q$ hyperplanes, and we are done.

Step 2. On the annular end $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$ minus a discrete subset $S \subset A$ we construct a flat metric $d\tau^2$ on $A \setminus S$ which is complete on the set $\{z : |z| = r\} \cup S$, i.e., this set is at infinite distance from any point of $A \setminus S$. We may assume that

$$\sum_{j=1}^{q} \left(1 - \frac{k}{m_j}\right) > (k+1)\left(N - \frac{k}{2}\right) + (N+1), \tag{3.15}$$

otherwise our Main Theorem is already proved. By (3.15), we get

$$\sum_{j=1}^{q} \left(1 - \frac{k}{m_j}\right) - 2N + k - 1 > \frac{(2N-k+1)k}{2} > 0, \tag{3.16}$$

and by (3.14) this implies in particular that

$$q > 2N - k + 1 \geq N + 1 \geq k + 1. \tag{3.17}$$

By Theorem 3, (3.17) and (3.16), we have

$$(q - 2N + k - 1)\theta = \sum_{j=1}^{q} \omega(j) - k - 1$$

and

$$\theta \geq \omega(j) > 0, \qquad \theta \geq \frac{k+1}{2N-k+1},$$

so

$$2\biggl(\sum_{j=1}^{q}\omega(j)\Bigl(1-\frac{k}{m_j}\Bigr)-k-1\biggr)=\frac{2(\sum_{j=1}^{q}\omega(j)-k-1)\theta}{\theta}-2\sum_{j=1}^{q}\frac{k\omega(j)\theta}{\theta m_j}$$
$$=2(q-2N+k-1)\theta-2\sum_{j=1}^{q}\frac{k\omega(j)\theta}{\theta m_j}\geq 2(q-2N+k-1)\theta-2\sum_{j=1}^{q}\frac{k\theta}{m_j}$$
$$=2\theta\biggl(\sum_{j=1}^{q}\Bigl(1-\frac{k}{m_j}\Bigr)-2N+k-1\biggr)\geq 2\frac{(k+1)\bigl(\sum_{j=1}^{q}\bigl(1-\frac{k}{m_j}\bigr)-2N+k-1\bigr)}{2N-k+1}.$$

Thus, we can now conclude using (3.16) that

$$2\biggl(\sum_{j=1}^{q}\omega(j)\Bigl(1-\frac{k}{m_j}\Bigr)-k-1\biggr)>k(k+1),$$

so

$$\sum_{j=1}^{q}\omega(j)\Bigl(1-\frac{k}{m_j}\Bigr)-k-1-\frac{k(k+1)}{2}>0. \tag{3.18}$$

By (3.18), we can choose a number $\epsilon$ $(>0)\in\mathbb{Q}$ such that

$$\frac{\sum_{j=1}^{q}\omega(j)(1-k/m_j)-(k+1)-k(k+1)/2}{\tau_{k+1}}$$
$$>\epsilon>\frac{\sum_{j=1}^{q}\omega(j)(1-k/m_j)-(k+1)-k(k+1)/2}{1/q+\tau_{k+1}}.$$

So

$$h:=\sum_{j=1}^{q}\omega(j)\Bigl(1-\frac{k}{m_j}\Bigr)-(k+1)-\epsilon\sigma_{k+1}>\frac{k(k+1)}{2}+\epsilon\tau_k \tag{3.19}$$

and

$$\frac{\epsilon}{q}>\sum_{j=1}^{q}\omega(j)\Bigl(1-\frac{k}{m_j}\Bigr)-(k+1)-\frac{k(k+1)}{2}-\epsilon\tau_{k+1}. \tag{3.20}$$

We now consider the number

$$\rho:=\frac{1}{h}\biggl(\frac{k(k+1)}{2}+\epsilon\tau_k\biggr)=\frac{1}{h}(\sigma_k+\epsilon\tau_k). \tag{3.21}$$

Then, by (3.19), we have

$$0<\rho<1. \tag{3.22}$$

Set

$$\rho^*:=\frac{1}{(1-\rho)h}=\frac{1}{\sum_{j=1}^{q}\omega(j)(1-k/m_j)-(k+1)-k(k+1)/2-\epsilon\tau_{k+1}}. \tag{3.23}$$

From (3.20) we get

$$\epsilon\rho^*/q > 1. \tag{3.24}$$

Consider the open subset

$$A_1 = \mathrm{Int}(A) - \bigcup_{j=\overline{1,q},\, p=\overline{0,k}} \{z : \psi(G)_{jp}(z) = 0\}$$

of $A$. Using the global holomorphic coordinate $z$ on $A \supset A_1$ we define a new pseudometric

$$d\tau^2 = \left( \frac{\prod_{j=1}^q |G_z(H_j)|^{\omega(j)(1-k/m_j)}}{|((G_z)_k)_z|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(G_z)_{jp})_z|^{\epsilon/q}} \right)^{2\rho^*} |dz|^2 \tag{3.25}$$

on $A_1$. We note that by the transformation formulas (3.7)–(3.10) for a local holomorphic coordinate $\xi$ we have

$$\begin{aligned} &\left( \frac{\prod_{j=1}^q |G_z(H_j)|^{\omega(j)(1-k/m_j)}}{|((G_z)_k)_z|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(G_z)_{jp})_z|^{\epsilon/q}} \right)^{2\rho^*} |dz|^2 \\ &\qquad = \left( \frac{\prod_{j=1}^q |G_\xi(H_j)|^{\omega(j)(1-k/m_j)}}{|((G_\xi)_k)_\xi|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(G_\xi)_{jp})_\xi|^{\epsilon/q}} \right)^{2\rho^*} |d\xi|^2, \end{aligned} \tag{3.26}$$

so the pseudometric $d\tau$ is in fact defined independently of the choice of the coordinate. Moreover, it is also easy to see that $d\tau$ is flat.

Next we observe that for any point $z \in A$, we have

$$\left( \nu_{G_k} - \sum_{j=1}^q \omega(j)\nu_{G(H_j)}\left(1 - \frac{k}{m_j}\right)\right)(z) \geq 0. \tag{3.27}$$

In fact, set

$$\phi := \frac{|G_k|}{\prod_{j=1}^q |G(H_j)|^{\omega(j)}}.$$

Observing that by (3.14) for all $j = 1, \ldots, q$ and all $z \in A$ we have either $\nu_{G(H_j)}(z) = 0$ or $\nu_{G(H_j)}(z) \geq m_j \geq k$, we get

$$\frac{k}{m_j}\nu_{G(H_j)} \geq \min\{\nu_{G(H_j)}, k\}.$$

So by Lemma 7 we have

$$\begin{aligned} \nu_{G_k} - \sum_{j=1}^q \omega(j)\nu_{G(H_j)}\left(1 - \frac{k}{m_j}\right) &= \nu_\phi + \sum_{j=1}^q \omega(j)\frac{k}{m_j}\nu_{G(H_j)} \\ &\geq \nu_\phi + \sum_{j=1}^q \omega(j)\min\{\nu_{G(H_j)}, k\} \geq 0. \end{aligned}$$

Now it is easy to see that $d\tau$ is continuous and nowhere vanishing on $A_1$. Indeed, for $z_0 \in A_1$ with $\prod_{j=1}^{q} G(H_j)(z_0) \neq 0$, $d\tau$ is continuous and not vanishing at $z_0$. Now assume that there exists $z_0 \in A_1$ such that $G(H_i)(z_0) = 0$ for some $i$. But by (3.27) and (3.14) we then get $\nu_{G_k}(z_0) > 0$, which contradicts $z_0 \in A_1$.

The key point is now to prove the following claim.

Claim 1. *The pseudometric $d\tau$ is complete on the set $\{z : |z| = r\} \cup \bigcup_{j=\overline{1,q},\, p=\overline{0,k}} \{z : \psi(G)_{jp}(z) = 0\}$, i.e., this set is at infinite distance from any interior point in $A_1$.*

First, assume that $\prod_{p=0}^{k} \prod_{j=1}^{q} |\psi(G)_{jp}|(z_0) = 0$. Then using (3.27) we get

$$\begin{aligned}\nu_{d\tau}(z_0) &= -\Big(\nu_{G_k}(z_0) - \sum_{j=1}^{q} \omega(j)\nu_{G(H_j)}(z_0)(1 - k/m_j) \\ &\qquad\qquad + \epsilon\nu_{G_k}(z_0) + \frac{\epsilon}{q}\sum_{j=1}^{q}\sum_{p=0}^{k-1} \nu_{\psi(G)_{jp}}(z_0)\Big)\rho^* \\ &\leq -\epsilon\rho^*\nu_{G_k}(z_0) - \frac{\epsilon\rho^*}{q}\sum_{j=1}^{q}\sum_{p=0}^{k-1} \nu_{\psi(G)_{jp}}(z_0) \leq -\frac{\epsilon\rho^*}{q}.\end{aligned}$$

Thus we can find a positive constant $C$ such that

$$|d\tau| \geq \frac{C}{|z - z_0|^{\epsilon\rho^*/q}}|dz|$$

in a neighborhood of $z_0$ and then, by combining with (3.24), $d\tau$ is complete on $\bigcup_{j=\overline{1,q},\, p=\overline{0,k}} \{z : \psi(G)_{jp}(z) = 0\}$.

Now assume that $d\tau$ is not complete on $\{z : |z| = r\}$. Then there exists $\gamma : [0,1) \to A_1$, where $\gamma(1) \in \{z : |z| = r\}$, such that $|\gamma| < \infty$. Furthermore, we may also assume that $\operatorname{dist}(\gamma(0); \{z : |z| = 1/r\}) > 2|\gamma|$. Consider a small disk $\varDelta$ with center at $\gamma(0)$. Since $d\tau$ is flat, $\varDelta$ is isometric to an ordinary disk in the plane (cf. e.g. Lemma 10). Let $\varPhi : \{w : |w| < \eta\} \to \varDelta$ be this isometry. Extend $\varPhi$, as a local isometry into $A_1$, to the largest disk $\{w : |w| < R\} = \varDelta_R$ possible. Then $R \leq |\gamma|$. The reason that $\varPhi$ cannot be extended to a larger disk is that the image goes to the outside boundary $\{z : |z| = r\}$ of $A_1$ (it cannot go to points $z$ of $A$ with $\prod_{j=\overline{1,q},\, p=\overline{0,k}} \psi(G)_{jp}(z) = 0$ since we have already shown the completeness of $A_1$ with respect to these points). More precisely, there exists a point $w_0$ with $|w_0| = R$ such that $\varPhi(\overline{0, w_0}) = \varGamma_0$ is a divergent curve on $A$.

Since we want to use Lemma 9 to finish up Step 2, for the rest of this step we consider $G_z = ((g_0)_z, \ldots, (g_k)_z)$ as a *fixed globally defined reduced representation of $g$* by means of the global coordinate $z$ of $A \supset A_1$. (We remark that we then of course lose the invariance of $d\tau^2$ under coordinate changes (3.26), but since $z$ is a global coordinate this will be no problem and we will not need this invariance for the application of Lemma 9.) If again $\Phi : \{w : |w| < R\} \to A_1$ is our maximal local isometry, it is in particular holomorphic and locally biholomorphic. So $f := g \circ \Phi : \{w : |w| < R\} \to \mathbb{P}^k(\mathbb{C})$ is a linearly non-degenerate holomorphic map with fixed global reduced representation

$$F := G_z \circ \Phi = ((g_0)_z \circ \Phi, \dots, (g_k)_z \circ \Phi) = (f_0, \dots, f_k).$$

Since $\Phi$ is locally biholomorphic, the metric on $\Delta_R$ induced from $ds^2$ (cf. (3.6)) through $\Phi$ is given by

$$\Phi^* ds^2 = 2|G_z \circ \Phi|^2 |\Phi^* dz|^2 = 2|F|^2 \left|\frac{dz}{dw}\right|^2 |dw|^2. \tag{3.28}$$

On the other hand, $\Phi$ is locally isometric, so

$$\begin{aligned} |dw| &= |\Phi^* d\tau| \\ &= \left( \frac{\prod_{j=1}^q |G_z(H_j) \circ \Phi|^{\omega(j)(1-k/m_j)}}{|((G_z)_k)_z \circ \Phi|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(G_z)_{jp})_z \circ \Phi|^{\epsilon/q}} \right)^{\rho^*} \left|\frac{dz}{dw}\right| |dw|. \end{aligned}$$

By (3.11) and (3.12) we have

$$((G_z)_k)_z \circ \Phi = ((G_z \circ \Phi)_k)_w \cdot \left(\frac{dw}{dz}\right)^{\sigma_k} = (F_k)_w \cdot \left(\frac{dw}{dz}\right)^{\sigma_k},$$

$$(\psi(G_z)_{jp})_z \circ \Phi = (\psi(G_z \circ \Phi)_{jp})_w \cdot \left(\frac{dw}{dz}\right)^{\sigma_p} = (\psi(F)_{jp})_w \cdot \left(\frac{dw}{dz}\right)^{\sigma_p} \qquad (0 \le p \le k).$$

Hence, by the definition of $\rho$ in (3.21),

$$\begin{aligned} \left|\frac{dw}{dz}\right| &= \left( \frac{\prod_{j=1}^q |G_z(H_j) \circ \Phi|^{\omega(j)(1-k/m_j)}}{|((G_z)_k)_z \circ \Phi|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(G_z)_{jp})_z \circ \Phi|^{\epsilon/q}} \right)^{\rho^*} \\ &= \left( \frac{\prod_{j=1}^q |F(H_j)|^{\omega(j)(1-k/m_j)}}{|(F_k)_w|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(F)_{jp})_w|^{\epsilon/q}} \right)^{\rho^*} \frac{1}{\left|\frac{dw}{dz}\right|^{h\rho\rho^*}}. \end{aligned}$$

So by the definition of $\rho^*$ in (3.23), we get

$$\begin{aligned} \left|\frac{dz}{dw}\right| &= \left( \frac{|(F_k)_w|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(F)_{jp})_w|^{\epsilon/q}}{\prod_{j=1}^q |F(H_j)|^{\omega(j)(1-k/m_j)}} \right)^{\frac{\rho^*}{1+h\rho\rho^*}} \\ &= \left( \frac{|(F_k)_w|^{1+\epsilon} \prod_{p=0}^{k-1} \prod_{j=1}^q |(\psi(F)_{jp})_w|^{\epsilon/q}}{\prod_{j=1}^q |F(H_j)|^{\omega(j)(1-k/m_j)}} \right)^{1/h}. \end{aligned}$$

Moreover, $|(\psi(F)_{jp})_w| \le |(F_p)_w(H_j)|$ by the definitions, so

$$\left|\frac{dz}{dw}\right| \le \left(\frac{|(F_k)_w|^{1+\epsilon}\prod_{p=0}^{k-1}\prod_{j=1}^{q}|(F_p)_w(H_j)|^{\epsilon/q}}{\prod_{j=1}^{q}|F(H_j)|^{\omega(j)(1-k/m_j)}}\right)^{1/h}. \tag{3.29}$$

From (3.28) and (3.29), we have

$$\Phi^* ds \le \sqrt{2}\,|F|\left(\frac{|(F_k)_w|^{1+\epsilon}\prod_{p=0}^{k-1}\prod_{j=1}^{q}|(F_p)_w(H_j)|^{\epsilon/q}}{\prod_{j=1}^{q}|F(H_j)|^{\omega(j)(1-k/m_j)}}\right)^{1/h}|dw|.$$

By (3.17) and (3.19) all the assumptions of Lemma 9 are satisfied, so

$$\Phi^* ds \le C\left(\frac{2R}{R^2-|w|^2}\right)^{\rho}|dw|.$$

Since by (3.22) we have $0<\rho<1$, it then follows that

$$d_{\Gamma_0} \le \int_{\Gamma_0} ds = \int_{\overline{0,w_0}} \Phi^*\, ds \le C\int_0^R\left(\frac{2R}{R^2-|w|^2}\right)^{\rho}|dw| < \infty$$

(where $d_{\Gamma_0}$ denotes the length of the divergent curve $\Gamma_0$ in $M$), contradicting the assumption of completeness of $M$. Claim 1 is proved.

Step 3. We will "symmetrize" the metric $d\tau^2$ constructed in Step 2 so that it will become a complete and flat metric on $\mathrm{Int}(A)\setminus(S\cup\tilde S)$ (with $\tilde S$ another discrete subset).

We introduce a new coordinate $\xi(z) := 1/z$ on $A=\{z : 1/r \le |z| < r\}$. By (3.10) we have

$$S=\Big\{z : \prod_{p=0}^{k}\prod_{j=1}^{q}(\psi(G_z)_{jp})_z(z)=0\Big\}=\Big\{z : \prod_{p=0}^{k}\prod_{j=1}^{q}(\psi(G_\xi)_{jp})_\xi(z)=0\Big\}$$

(where the zeros are taken with the same multiplicities), and since by (3.26), $d\tau^2$ is independent of the coordinate $z$, the change of coordinate $\xi(z)=1/z$ yields an isometry of $A\setminus S$ onto the set $\tilde A\setminus\tilde S$, where $\tilde A := \{z : 1/r < |z| \le r\}$ and $\tilde S := \{z : \prod_{p=0}^{k}\prod_{j=1}^{q}(\psi(G_z)_{jp})_z(1/z)=0\}$. In particular we have

$$\begin{aligned} d\tau^2 &= \left(\frac{\prod_{j=1}^{q}|G_\xi(H_j)(1/z)|^{\omega(j)(1-k/m_j)}}{|((G_\xi)_k)_\xi(1/z)|^{1+\epsilon}\prod_{p=0}^{k-1}\prod_{j=1}^{q}|(\psi(G_\xi)_{jp})_\xi(1/z)|^{\epsilon/q}}\right)^{2\rho^*}|d(1/z)|^2\\ &= \left(\frac{\prod_{j=1}^{q}|G_z(H_j)(1/z)|^{\omega(j)(1-k/m_j)}}{|((G_z)_k)_z(1/z)|^{1+\epsilon}\prod_{p=0}^{k-1}\prod_{j=1}^{q}|(\psi(G_z)_{jp})_z(1/z)|^{\epsilon/q}}\right)^{2\rho^*}|dz|^2. \end{aligned}$$

We now define $d\tilde\tau^2=\lambda^2(z)|dz|^2$, where

$$\lambda(z)=\left(\frac{\prod_{j=1}^{q}|G_z(H_j)(z)G_z(H_j)(1/z)|^{\omega(j)(1-k/m_j)}}{|((G_z)_k)_z(z)((G_z)_k)_z(1/z)|^{1+\epsilon}\prod_{p=0}^{k-1}\prod_{j=1}^{q}|(\psi(G_z)_{jp})_z(z)(\psi(G_z)_{jp})_z(1/z)|^{\epsilon/q}}\right)^{\rho^*},$$

on $\tilde{A}_1 := \{z : 1/r < |z| < r\} \setminus \{z : \prod_{p=0}^{k} \prod_{j=1}^{q} (\psi(G_z)_{jp})_z(z)(\psi(G_z)_{jp})_z(1/z) = 0\}$. Then $d\tilde{\tau}^2$ is complete on $\tilde{A}_1$. In fact by what we showed above we have: Towards any point of the boundary $\partial\tilde{A}_1 := \{z : |z| = 1/r\} \cup \{z : |z| = r\} \cup \{z : \prod_{p=0}^{k} \prod_{j=1}^{q} (\psi(G_z)_{jp})_z(z)(\psi(G_z)_{jp})_z(1/z) = 0\}$ of $\tilde{A}_1$, one of the factors of $\lambda^2(z)$ is bounded below away from zero, and the other factor is the one of a complete metric with respect to this part of the boundary. Moreover by the corresponding properties of the two factors of $\lambda^2(z)$ it is trivial that $d\tilde{\tau}^2$ is a continuous nowhere vanishing and flat metric on $\tilde{A}_1$.

Step 4. We produce a contradiction by applying Lemma 10 to the open Riemann surface $(\tilde{A}_1, d\tilde{\tau}^2)$. In fact, we apply Lemma 10 to any point $p \in \tilde{A}_1$. Since $d\tilde{\tau}^2$ is complete, there cannot exist a divergent curve from $p$ to the boundary $\partial\tilde{A}_1$ with finite length with respect to $d\tilde{\tau}^2$. Since $\Phi : \Delta_{R_0} \to \tilde{A}_1$ is a local isometry, we necessarily have $R_0 = \infty$. So $\Phi : \mathbb{C} \to \tilde{A}_1 \subset \{z : |z| < r\}$ is a non-constant holomorphic map, which contradicts Liouville's theorem. So our assumption (3.15) was wrong. This proves the Main Theorem.

*Proof of Corollaries 1 and 2.* We first observe that the inequality (1.1) in the Main Theorem is equivalent to

$$\ell(k) := \frac{k^2}{2} - k\left(\sum_{j=1}^{q} \frac{1}{m_j} + N - \frac{1}{2}\right) \leq 2N - q + 1, \tag{3.30}$$

where $\ell$ is a function defined on $\mathbb{N} \cap [1, m-1]$. Observing that $m - 1 \leq N$, it is easy to see that $\ell$ is decreasing, so if (3.30) is satisfied for some $1 \leq k \leq m-1$, it is also satisfied for $k = m-1$. This proves Corollary 1. To prove Corollary 2, we apply the inequality (1.3) of Corollary 1 to the $q := \sigma_m + 1$ hyperplanes $H_1, \ldots, H_q$ assuming that $g$ meets the first $q-1$ of these hyperplanes only finitely often. Then we get $1 - (m-1)/m_q \leq 0$, which is equivalent to $m_q \leq m - 1$. ∎

**Acknowledgments.** A part of this work was completed during a stay of the first two authors at the Vietnam Institute for Advanced Study in Mathematics (VIASM). The research of the second author was partially supported by a NAFOSTED grant of Vietnam (Grant No. 101.04-2014.48).

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Gerd Dethloff
Université Européenne de Bretagne
France
and
Université de Brest
Laboratoire de Mathématiques
de Bretagne Atlantique –
UMR CNRS 6205
6, avenue Le Gorgeu, BP 452
29275 Brest Cedex, France
E-mail: Gerd.Dethloff@univ-brest.fr

Pham Hoang Ha
Department of Mathematics
Hanoi National University of Education
136 XuanThuy St.
Hanoi, Vietnam
E-mail: ha.ph@hnue.edu.vn

Pham Duc Thoan
Department of Information Technology
National University of Civil Engineering
55 Giai Phong St.
Hanoi, Vietnam
E-mail: ducthoan.hh@gmail.com

*Received 16 December 2014* (6478)