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# Analytic sets in the theory of commutative semigroups

by

Robert Kaufman (Urbana, IL)

**Abstract.** A problem about representations of countable, commutative semigroups leads to an analytic non-Borel set.

**1. Introduction.** A *semicharacter* of a commutative semigroup $H$ is a multiplicative map from $H$ to the closed unit disk in the complex plane. The set $H^\wedge$ of all semicharacters of $H$ is a compact Hausdorff space in the product topology, metrizable when $H$ is countable. $H^\wedge$ provides a representation of the complex homomorphisms of a certain commutative Banach algebra introduced by Hewitt and Zuckerman [HZ] and is the basis of extensive analysis based on this algebra; unlike [HZ] we include the identically 0 map.

The theory of semicharacters diverges from the theory of characters of commutative groups in the question of extensions to larger semigroups. Let $H_1$ be a subsemigroup (ssg) of $H$; then $R(H, H_1)$ denotes the set of restrictions to $H_1$ of the set $H^\wedge$. A perfectly rational test for $R(H, H_1)$ was found in [R]: an element $\phi$ of $H_1^\wedge$ belongs to $R(H, H_1)$ if and only if $|\phi(a)| \le |\phi(b)|$ for every pair $a, b \in H_1$ of elements such that $b$ divides $a$ in $H$, i.e. $a = b + h$, where $h$ belongs to $H$. Clearly $R(H, H_1)$ is a closed subset of $H_1^\wedge$.

Matters are much less transparent for the class $H_*^\wedge$ of semicharacters of $H$ omitting the value 0 and the class $R_*(H, H_1)$ of their restrictions to $H_1$. A solution was proposed in [H] (which contains references to several earlier works).

Henceforth we suppose that $H$ is countable, whence $H_*^\wedge$ is a $G_\delta$ in $H^\wedge$, and therefore a Polish space. If $R_*(H, H_1)$ admitted a characterization analogous to the theorem of Ross, it is plausible that it would be a Borel set. It is clearly an analytic set (see [Ku, Chap. 39] and [Ke, Chap. III]), and that is sometimes the correct measure of its complexity. This is our main theorem.

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**2. Main theorems.** Before stating it we define a certain countable semigroup $H$ and ssg $H_1$. Let $G$ be the free abelian group with basis $(w_1, w_2, w_3, \dots)$, $E_1 = (w_1, w_3, w_5, \dots)$, $E_2 = (w_2, w_4, w_6, \dots)$; $H_1$ the ssg generated by $E_1$, $H_2$ that generated by $E_2$. Let $\psi$ be a map of $E_1$ onto the *set* $H_2$ and finally let $H$ be the ssg generated by the set

$$S = E_1 \cup E_2 \cup \{\psi(v) - v : v \in E_1\}.$$

THEOREM 2.1. *Let $A$ be an analytic set in a Polish space $X$. Then there is a continuous map $\theta$ of $X$ into $H_1^\wedge$ such that $\theta^{-1}(R_*(H, H_1)) = A$.*

Theorem 2.1 can be summarized by the statement that $R_*(H, H_1)$ is a complete analytic set in $H_1^\wedge$ . In Theorem 2.1 the 'reduction' of $A$ is effected by a continuous (not merely Borel) map $\theta$. A few extra lines would yield an embedding of $X$.

The proof of Theorem 2.1 is circuitous. First we state an analogue of Theorem 2.1, containing no reference to restrictions; then we show that this theorem is 'encoded' in Theorem 2.1 through our construction of $H$, and finally we prove the intermediate result, using mostly tools from convexity.

Let $K$ be the set of maps of $H_2$ into $[0, 1]$. Let $M$ be the set of elements $g$ in $K$ such that $0 < |\phi| \le g$ for some semicharacter $\phi$ of $H_2$. ($M$ is the set of *majorants*). Then $K$ is a Polish space and $M$ is analytic.

THEOREM 2.2. *$M$ is a complete analytic set in $K$.*

*Encoding.* To explain how this theorem is encoded in the main theorem we associate to each element $g$ of $K$ a semicharacter $\gamma$ of $H_1$ by the formula $\gamma = g \circ \psi$, that is,

$$\gamma(w_m) = g \circ \psi(w_m), \qquad m = 1, 3, 5, \dots.$$

This defines a semicharacter of $H_1$ and thus an embedding of $K$ into $H_1^\wedge$. Suppose first that $\gamma$ belongs to $R_*(H, H_1)$ so there is a semicharacter $\phi$ extending $\gamma$ to $H$, and moreover $|\phi| > 0$ everywhere. For each $v$ in $E_1$, $v$ divides $\psi(v)$ in $H$, whence $|\phi \circ \psi(v)| \le |\phi(v)| = |g \circ \psi(v)|$. Since the range of $\psi$ is all of $H_2$, we obtain $|\phi| \le g$ and $g$ is a majorant. To prove the reverse implication, suppose that $g$ is a majorant, that is, $0 < |\phi| \le g$, where $\phi$ is a semicharacter of $H_2$. To define an extension $\gamma$ of $\phi$ to all of $H$, we keep the same values on $H_2$, define $\gamma$ on $H_1$ by the formula $\gamma(v) = \phi \circ \psi(v)$, and then define $\gamma$ on the last part of $S$ by algebra. Then $0 < |\gamma| \le 1$ on all of $S$, and so on $H$. We observe that when $g \circ \psi$ is everywhere positive, then the semicharacter shares this property.

The next step is to focus on a certain closed subset $K_0$ of $K$, namely the set of submultiplicative maps, that is,

$$g(x + y) \le g(x)g(y) \quad \text{for all } x, y \in H_2.$$

When $F$ is a subset of $K_0$, then $\sup F$ is the pointwise supremum of $F$; of course $\sup F$ belongs to $K_0$. In the next result, we use only the fact that $H_2$ is countable and commutative.

LEMMA 2.3. *Let $F$ be a closed subset of $K_0$, and suppose that* $\sup F$ *is a majorant. Then some element of $F$ is everywhere positive on $H_2$.*

*Proof.* Taking absolute values of semicharacters, we may assume that they take values in $[0,1]$. Moreover, applying log allows us to switch to subadditive and additive maps taking values in $[-\infty, 0]$. Let $(h_m)$ be an enumeration of $H_2$, $(a_m)$ a sequence of positive numbers such that $\sum a_m^{-1} \le 1$, and $\lambda$ an additive map of $H_2$ into $(-\infty, 0]$ such that $\lambda \le \sup F$. We now show that $F$ contains an element $g$ such that $a_m\lambda(h_m) \le g(h_m)$ for each $m$. Let $r$ be a natural number; we will find an approximate solution over the elements $h_1, \ldots, h_r$. The set $V$ of elements at which $\lambda$ takes the value 0 requires special care.

Let $N$ be a positive integer, and define integers $n_1, \ldots, n_r$ by the formula

$$n_m = \begin{cases} [-Na_m^{-1}\lambda(h_m)^{-1}] & \text{if } \lambda(h_m) < 0, \\ N^2 & \text{if } h_m \in V. \end{cases}$$

We can take $N$ so large that all $n_m \ge 1$. Let $y = n_1h_1 + \cdots + n_rh_r$, so that $\lambda(y) > -N$. Then $g(y) > -N$ for some $g$ in $F$. Since $g$ is subadditive, for each $m$ we have

$$n_m g(h_m) \ge g(n_m h_m) \ge g(y) > -N,$$

whence $g(h_m) > -N/n_m$. Making $N \to \infty$, we obtain an element $\tilde{g}$ of $F$ such that $\tilde{g}(h_m) \ge a_m\lambda(h_m)$ for $m = 1, \ldots, r$. As $r \to \infty$, we obtain the element sought. ∎

REMARK 2.4. With the aid of a theorem of Hahn–Banach type for commutative semigroups [Ka] this can be strengthened: some element of $F$ is already a majorant. We observe that this lemma is closely related to Beppo Levi's theorem in real analysis; the countability of $H$ seems to be necessary.

*Proof of Theorem 2.2.* The set $A$ is the projection of a $G_\delta$ set $V$ in $X \times I$, and $V$ in turn is the intersection of a decreasing sequence $V_n$ of open sets in $X \times I$. (Here $I = [0,1]$, but the interval could be replaced by any uncountable compact metric space.) For each positive integer $n$, we define a continuous map $u_n$ on $X \times I$, taking values in $[-\infty, 0]$, with finite values on $V_n$ and $-\infty$ on its complement. We define now a continuous map of $X \times I$ to the set of additive maps on $H_2$ by the formula

$$\lambda(x, t; h) = \sum e_n u_n(x, t),$$

where $h = \sum e_n w_{2n}$, $x \in X$, $t \in I$. Here we specify that the sum extends over coefficients $e_n > 0$. Taking a little care with the last point, we see that this defines a continuous map of $X \times I$ into the set of additive maps. Then we define

$$\theta(x;h) = \sup\{\lambda(x,t;h) \colon t \in I\}.$$

As the interval $I$ is compact, this is continuous on $X$, and for each fixed $x$, there is a closed set $F$ of $K_0$, as in Lemma 2.3. When $x \in A$, there is some $t$ such that $u_n(x,t) > -\infty$ for each $n$, whence $\theta(x)$ is a majorant. Conversely, if $\theta(x)$ is a majorant, then for some $t$, each function $u_n$ must be finite at $(x,t)$, i.e., $x \in A$. Since the supremum $\theta(x)$ depends continuously on $x$, this shows that the set of majorants is a complete analytic set.

A slight change in the definition of the sets $V_n$ yields an interesting improvement of Theorem 2.1. Adding the set $X \times (0, 1/n)$ to $V_n$, we do not change the intersection $V$. But now the map $\theta$ takes its values in semicharacters of $H_1$ that are never 0. ∎

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Robert Kaufman
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green St.
Urbana, IL 61801, U.S.A.
E-mail: rpkaufma@math.uiuc.edu

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