

## Hölder's inequality for roots of symmetric operator spaces

by

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**Abstract.** We prove a version of Hölder's inequality with a constant for  $p$ th roots of symmetric operator spaces of operators affiliated to a semifinite von Neumann algebra factor, and with constant equal to 1 for strongly symmetric operator spaces.

**1. Introduction.** Let  $\mathcal{B}$  be an infinite-dimensional,  $\sigma$ -finite von Neumann algebra factor equipped with a normal, faithful, semifinite (or finite) trace  $\tau$  and acting on a Hilbert space. A closed, densely defined, unbounded operator  $T$  on the Hilbert space is *affiliated with  $\mathcal{B}$*  if the partial isometry from its polar decomposition and all spectral projections of its absolute value lie in  $\mathcal{B}$ , and it is said to be  $\tau$ -*measurable* if for every  $\epsilon > 0$ , there is a projection  $p \in \mathcal{B}$  such that  $\tau(p) < \epsilon$  and  $T(1-p)$  is bounded.  $\mathcal{B}$  together with the set of  $\tau$ -measurable operators  $T$  as described above is a  $*$ -algebra under natural operations (performing addition and multiplication with appropriate domains, and taking closures; see, for example, [8] for details.) We let  $S(\mathcal{B}, \tau)$  denote this  $*$ -algebra. Note that when  $\mathcal{B}$  is a type I von Neumann algebra, then  $S(\mathcal{B}, \tau) = \mathcal{B}$ .

We will consider subspaces  $\mathcal{I} \subseteq S(\mathcal{B}, \tau)$  that are  $\mathcal{B}$ -bimodules and such that there is a complete symmetric norm  $\|\cdot\|_{\mathcal{I}}$  on  $\mathcal{I}$  (see Definition 2.1). Such a pair  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  is called a *symmetric operator space* [8]. Note that if  $\mathcal{I} \subseteq \mathcal{B}$ , then  $\mathcal{I}$  is an ideal of  $\mathcal{B}$ . Among the symmetric operator spaces are the fully symmetric Schatten-von Neumann ideals  $S^p$ , noncommutative  $L^p$  spaces,  $1 \leq p < \infty$ , and the Marcinkiewicz (or Lorentz) operator spaces

$$(1.1) \quad \mathcal{I}_{\psi} := \left\{ A \in S(\mathcal{B}, \tau) : \|A\|_{\mathcal{I}_{\psi}} := \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu_s(A) ds < \infty \right\},$$

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where  $\psi$  is a concave function satisfying

$$\lim_{t \rightarrow 0^+} \psi(t) = 0, \quad \lim_{t \rightarrow \infty} \psi(t) = \infty$$

and  $\mu(A)$  is the generalized singular value function of  $A$  (see (2.1)).

For  $1 < p < \infty$ , define the set

$$(1.2) \quad \mathcal{I}^{1/p} = \{A \in S(\mathcal{B}, \tau) : |A|^p \in \mathcal{I}\}$$

and the function

$$(1.3) \quad \|A\|_{\mathcal{I}^{1/p}} := \||A|^p\|_{\mathcal{I}}^{1/p} \quad \text{for } A \in \mathcal{I}^{1/p}.$$

The space  $\mathcal{I}^{1/p}$  endowed with the function  $\|\cdot\|_{\mathcal{I}^{1/p}}$  is also a symmetric operator space (see Theorem 2.3). When  $\mathcal{I}$  is the noncommutative  $L^1$  space  $\{A \in S(\mathcal{B}, \tau) : \tau(|A|) < \infty\}$ , then  $\mathcal{I}^{1/p}$  is the noncommutative  $L^p$  space, and the noncommutative Hölder inequality

$$(1.4) \quad \|AB\|_{\mathcal{I}} \leq \|A\|_{\mathcal{I}^{1/p}} \|B\|_{\mathcal{I}^{1/q}} \\ (A \in \mathcal{I}^{1/p}, B \in \mathcal{I}^{1/q}, 1 < p < \infty, 1/p + 1/q = 1),$$

holds [3, Theorem 4.2]. By [2, Proposition 2.5], the Hölder inequality (1.4) holds when  $\mathcal{I}$  is a Marcinkiewicz operator space of a  $\sigma$ -finite semifinite von Neumann algebra factor  $\mathcal{B}$ , with norm  $\|A\|_{\mathcal{I}_\psi}$  as indicated in (1.1) (and also—see the erratum to [2]—for  $\mathcal{I}_\psi \cap \mathcal{B}$  with the symmetric norm  $\max\{\|\cdot\|, \|\cdot\|_{\mathcal{I}_\psi}\}$ ).

In this note (see Theorem 2.10), we show that a weaker Hölder-type inequality

$$(1.5) \quad \|AB\|_{\mathcal{I}} \leq 4\|A\|_{\mathcal{I}^{1/p}} \|B\|_{\mathcal{I}^{1/q}}$$

holds in an arbitrary symmetric operator space  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ . We also show that the Hölder inequality (1.4) holds in an arbitrary strongly symmetric operator space (see Definition 3.1). The class of strongly symmetric operator spaces includes the fully symmetric operator space, and the first example of a symmetric operator space that is not fully symmetric is due to [10]. An example of a symmetric (Banach) function space that is not strongly symmetric can be found in [11], and examples of symmetric operator spaces that are not strongly symmetric can be constructed based on Theorem 2.3.

Observe that if we take the equivalent norms  $\|\cdot\|_{1/p} = 2\|\cdot\|_{\mathcal{I}^{1/p}}$  and  $\|\cdot\|_{1/q} = 2\|\cdot\|_{\mathcal{I}^{1/q}}$  on  $\mathcal{I}^{1/p}$  and  $\mathcal{I}^{1/q}$ , respectively, then from (1.5) we get the inequality

$$(1.6) \quad \|AB\|_{\mathcal{I}} \leq \|A\|_{1/p} \|B\|_{1/q}.$$

Our interest in these versions of Hölder's inequality for symmetric operator spaces is motivated in part by the second order trace formulas in [2, 12], which are proved in the case of a symmetric operator ideal  $\mathcal{I}$  for perturbations in the square root ideal  $\mathcal{I}^{1/2}$  that possesses a norm  $\|\cdot\|_{1/2}$  satisfying (1.6) with  $p = q = 2$ .

**2. A Hölder-type inequality in symmetric operator spaces.** The generalized singular value function  $\mu(A)$  of  $A \in S(\mathcal{B}, \tau)$  is defined by

$$(2.1) \quad \mu_t(A) := \inf\{s \geq 0 : \tau(\chi_{(s, \infty)}(|A|)) \leq t\}, \quad t > 0$$

(see [3]). It has properties analogous to those of the decreasing rearrangement and to usual singular values of operators (see [3, Lemma 2.5]).

DEFINITION 2.1. A norm on a  $\mathcal{B}$ -bimodule  $\mathcal{I}$  of  $S(\mathcal{B}, \tau)$  is called *symmetric* if  $A \in S(\mathcal{B}, \tau)$ ,  $B \in \mathcal{I}$ ,  $\mu(A) \leq \mu(B)$  implies  $A \in \mathcal{I}$  and  $\|A\|_{\mathcal{I}} \leq \|B\|_{\mathcal{I}}$ .

We note that the properties of an operator ideal that were important in [2, 12] naturally hold for an ideal in a symmetric operator space. If a  $\mathcal{B}$ -bimodule  $\mathcal{I} \subseteq S(\mathcal{B}, \tau)$  is symmetric, then  $A \in \mathcal{B}$ ,  $B \in \mathcal{I}$ ,  $0 \leq A \leq B$  implies  $\|A\|_{\mathcal{I}} \leq \|B\|_{\mathcal{I}}$ . It is proved in [8, Theorem 2.5.2] that for a Banach bimodule  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  of  $S(\mathcal{B}, \tau)$ , the symmetry of the norm  $\|\cdot\|_{\mathcal{I}}$  is equivalent to the property

$$\|ABC\|_{\mathcal{I}} \leq \|A\| \|B\|_{\mathcal{I}} \|C\|, \quad (A, C \in \mathcal{B}, B \in \mathcal{I}).$$

It was also assumed in [2, 12] that  $\mathcal{I}$  lies in  $\mathcal{B}$  and the norm has the property  $\|\cdot\| \leq K\|\cdot\|_{\mathcal{I}}$  for some constant  $K > 0$ , where  $\|\cdot\|$  is the operator norm. If  $\mathcal{J}$  is a symmetric operator space in  $S(\mathcal{B}, \tau)$ , then  $\mathcal{J} \cap \mathcal{B}$  endowed with the norm  $\max\{\|\cdot\|, \|\cdot\|_{\mathcal{J}}\}$  is a symmetric operator ideal satisfying these properties.

There is a correspondence between symmetric operator spaces and Banach function and sequence spaces that is provided by the generalized singular values of operators. In the case of a type  $I_{\infty}$  factor, this is the well known Calkin correspondence. We will recall some salient aspects here; see, for example, [8] for a thorough exposition. Let

$$J = \begin{cases} \mathbb{N}, & \mathcal{B} \text{ a type } I_{\infty} \text{ factor,} \\ [0, 1], & \mathcal{B} \text{ a type } II_1 \text{ factor,} \\ [0, \infty), & \mathcal{B} \text{ a type } II_{\infty} \text{ factor,} \end{cases}$$

equipped with counting measure if  $J = \mathbb{N}$  and Lebesgue measure otherwise; let  $D(J)$  denote the vector space of all measurable, real-valued functions  $f$  on  $J$  such that for every  $\epsilon > 0$ , the measure of  $\{t \in J : |f(t)| > \epsilon\}$  is finite (and where elements of  $D(J)$  are regarded as being the same if they differ only on a set of measure zero). Let  $x^*$  denote the decreasing rearrangement of  $|x|$  when  $x \in D(J)$ . Properties of the decreasing rearrangement can be found, for example, in [1, Proposition 1.7].

DEFINITION 2.2. A *symmetric function space* (also called, when  $J = \mathbb{N}$ , a *symmetric sequence space*) is a subspace  $E$  of  $D(J)$  equipped with a complete norm  $\|\cdot\|_E$  such that  $x \in D(J)$ ,  $y \in E$ ,  $x^* \leq y^*$  implies  $x \in E$  and  $\|x\|_E \leq \|y\|_E$ .

Henceforth, we will refer to these as symmetric function spaces, including the possibility that they are symmetric sequence spaces.

By [4], every  $\mathcal{B}$ -bimodule  $\mathcal{I} \subseteq S(\mathcal{B}, \tau)$  can be uniquely described by its characteristic set

$$\mu(\mathcal{I}) := \{\mu(A) : A \in \mathcal{I}\}.$$

Alternatively, the correspondence can be seen at the level of function spaces and this goes further to characterize also symmetric norms. In particular, given a symmetric operator space  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  in  $S(\mathcal{B}, \tau)$  consider the subspace

$$E_{\mathcal{I}} := \{x \in D(J) : x^* \in \mu(\mathcal{I})\}$$

and for  $x \in E_{\mathcal{I}}$  set  $\|x\|_{E_{\mathcal{I}}} := \|A\|_{\mathcal{I}}$  for  $A \in \mathcal{I}$  such that  $\mu(A) = x^*$ . Conversely, given a symmetric function space  $(E, \|\cdot\|_E)$ , define

$$\mathcal{I}_E := \{A \in S(\mathcal{B}, \tau) : \mu(A) \in E\}, \quad \|A\|_{\mathcal{I}_E} := \|\mu(A)\|_E.$$

We have the following correspondence between symmetric operator spaces and symmetric function spaces.

**THEOREM 2.3** ([8, Theorem 3.1.1]; see also [5, Theorem 8.11]). *The map*

$$(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \mapsto (E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$$

*is a bijection from the set of all symmetric operator spaces in  $S(\mathcal{B}, \tau)$  onto the set of all symmetric function spaces in  $D(J)$ , whose inverse is the map*

$$(E, \|\cdot\|_E) \mapsto (\mathcal{I}_E, \|\cdot\|_{\mathcal{I}_E}).$$

It is proved in [6, Theorem II.4.1] that every symmetric function subspace  $E$  of  $D(J)$  satisfies

$$L^1 \cap L^\infty \subseteq E \subseteq L^1 + L^\infty.$$

Due to Theorem 2.3, every symmetric operator space  $\mathcal{I}$  in  $S(\mathcal{B}, \tau)$  satisfies

$$L^1(\mathcal{B}, \tau) \cap L^\infty(\mathcal{B}, \tau) \subseteq \mathcal{I} \subseteq L^1(\mathcal{B}, \tau) + L^\infty(\mathcal{B}, \tau).$$

We will use the Hölder inequality in Banach lattices to obtain the inequality (1.5) in symmetric operator spaces.

**DEFINITION 2.4** ([7, Definition 1.a.1]). A partially ordered Banach space  $X$  over the field  $\mathbb{R}$  is called a *Banach lattice* provided

- (i)  $x \leq y$  implies  $x + z \leq y + z$ , for every  $x, y, z \in X$ ,
- (ii)  $ax \geq 0$  for every  $0 \leq x \in X$  and every  $a \in (0, \infty)$ ,
- (iii) for all  $x, y \in X$  there exists a least upper bound  $x \vee y \in X$  and a greatest lower bound  $x \wedge y \in X$ ,
- (iv)  $\|x\|_X \leq \|y\|_X$  whenever  $|x| \leq |y|$ , for  $x, y \in X$ , where  $|x| := x \vee (-x)$ .

Every symmetric function space  $(E, \|\cdot\|_E)$  is a Banach lattice with the partial order defined by pointwise inequality (almost everywhere), with  $\vee$  and  $\wedge$  then taken pointwise. Indeed, the properties (i) and (ii) hold trivially and the property (iv) is an immediate consequence of monotonicity of the

decreasing rearrangement and the symmetric property of  $E$ . Let  $x, y \in E$ . Since  $0 \leq |x \vee y| \leq |x| \vee |y| \leq |x| + |y|$ , by monotonicity of the decreasing rearrangement and the symmetric property of  $E$ , we get  $x \vee y \in E$ . Since  $x \wedge y = x + y - x \vee y$ , we also have  $x \wedge y \in E$ . Thus, the property (iii) also holds.

In a Banach lattice  $X$ , there is a functional calculus

$$X^n \ni (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n) \in X$$

for functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  that are homogeneous of degree 1 (see [7, Theorem 1.d.1]). In the case of a symmetric function space, this functional calculus coincides with defining  $f(x_1, \dots, x_n)$  pointwise.

PROPOSITION 2.5 ([7, Proposition 1.d.2(i)]). *Let  $X$  be a Banach lattice. For every  $0 < \theta < 1$  and every  $x, y \in X$ ,*

$$\| |x|^\theta \cdot |y|^{1-\theta} \|_X \leq \|x\|_X^\theta \cdot \|y\|_X^{1-\theta},$$

where  $|x|^\theta |y|^{1-\theta} \in X$  is given by the functional calculus described above.

DEFINITION 2.6. Let  $(E, \|\cdot\|_E)$  be a symmetric function space in  $D(J)$ . For  $1 < p < \infty$ , let

$$E^{1/p} := \{x \in D(J) : |x|^p \in E\},$$

and for  $x \in E^{1/p}$ , let

$$(2.2) \quad \|x\|_{E^{1/p}} := \| |x|^p \|_E^{1/p}.$$

PROPOSITION 2.7 ([9, Proposition 2.23(i)]). *Let  $E$  be a symmetric function space. Then, for  $1 < p < \infty$ ,  $(E^{1/p}, \|\cdot\|_{E^{1/p}})$  is a symmetric function space.*

Proposition 2.7, asserts, in particular, the completeness of the norm (2.2) on  $E^{1/p}$ . As an immediate consequence of Proposition 2.5, we obtain the following.

PROPOSITION 2.8. *Let  $E$  be a symmetric function space. For every  $1 < p, q < \infty$  with  $1/p + 1/q = 1$ , and every  $x \in E^{1/p}$  and  $y \in E^{1/q}$ , we have  $xy \in E$  and*

$$\|xy\|_E \leq \| |x|^p \|_E^{1/p} \| |y|^q \|_E^{1/q}.$$

From Proposition 2.7 and the correspondence described in Theorem 2.3, we immediately obtain the analogue of Proposition 2.7 for symmetric operator spaces.

THEOREM 2.9. *Let  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  be a symmetric operator space. Then  $(\mathcal{I}^{1/p}, \|\cdot\|_{\mathcal{I}^{1/p}})$  defined in (1.2) and (1.3) is a symmetric operator space.*

Here is the main result of this note.

**THEOREM 2.10.** *Let  $\mathcal{I}$  be a symmetric operator space in  $S(\mathcal{B}, \tau)$ . Then for every  $1 < p, q < \infty$  with  $1/p + 1/q = 1$ , and every  $A \in \mathcal{I}^{1/p}$  and  $B \in \mathcal{I}^{1/q}$ , we have  $AB \in \mathcal{I}$  and*

$$(2.3) \quad \|AB\|_{\mathcal{I}} \leq 4\|A\|_{\mathcal{I}^{1/p}}\|B\|_{\mathcal{I}^{1/q}}.$$

*Proof.* Let  $(E, \|\cdot\|_E) = (E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$  be as in the correspondence from Theorem 2.3. Suppose firstly that  $\mathcal{B}$  is semifinite but not finite (namely, that  $\mathcal{B}$  is type  $\text{II}_{\infty}$  or  $\text{I}_{\infty}$ ). We can choose isometries  $V_1, V_2 \in \mathcal{B}$  so that  $V_1V_1^* + V_2V_2^* = 1$  and then  $\tau(V_1DV_1^*) = \tau(D)$  for every  $D \in S(\mathcal{B}, \tau)$  satisfying  $D \geq 0$ . If  $C, D \in S(\mathcal{B}, \tau)$ , then by  $C \oplus D \in S(\mathcal{B}, \tau)$  we will mean the element  $V_1CV_1^* + V_2DV_2^*$ .

By the properties of the generalized singular values (see [3, Proposition 2.5]), we have

$$(2.4) \quad \mu_t(AB) \leq \mu_{t/2}(A)\mu_{t/2}(B) = \mu_t(A \oplus A)\mu_t(B \oplus B),$$

with  $A \oplus A \in \mathcal{I}^{1/p}$  and  $B \oplus B \in \mathcal{I}^{1/q}$ . Applying Proposition 2.8 gives  $\mu(A \oplus A)\mu(B \oplus B) \in E$  and

$$\begin{aligned} \|\mu(A \oplus A)\mu(B \oplus B)\|_E &\leq \|(\mu(A \oplus A))^p\|_E^{1/p} \|(\mu(B \oplus B))^q\|_E^{1/q} \\ &= \|\mu(A \oplus A)\|_{E^{1/p}} \|\mu(B \oplus B)\|_{E^{1/q}}. \end{aligned}$$

From (2.4) and the symmetry condition, we have  $\mu(AB) \in E$  and

$$\|\mu(AB)\|_E \leq \|\mu(A \oplus A)\|_{E^{1/p}} \|\mu(B \oplus B)\|_{E^{1/q}}.$$

Thus,  $AB \in \mathcal{I}$ . The equality of norms  $\|C\|_{\mathcal{I}} = \|\mu(C)\|_E$  for  $C \in \mathcal{I}$  (and similarly for  $C$  in  $\mathcal{I}^{1/p}$  and  $\mathcal{I}^{1/q}$ ) immediately implies

$$(2.5) \quad \|AB\|_{\mathcal{I}} \leq \|A \oplus A\|_{\mathcal{I}^{1/p}} \|B \oplus B\|_{\mathcal{I}^{1/q}}.$$

Since we have

$$(2.6) \quad \|A \oplus A\|_{\mathcal{I}^{1/p}} \leq \|A \oplus 0\|_{\mathcal{I}^{1/p}} + \|0 \oplus A\|_{\mathcal{I}^{1/p}} = 2\|\mu(A)\|_{E^{1/p}} = 2\|A\|_{\mathcal{I}^{1/p}},$$

and a similar inequality for  $B$ , from (2.5) we get (2.3). This completes the proof when  $\mathcal{B}$  is not of type  $\text{II}_1$ .

When  $\mathcal{B}$  has type  $\text{II}_1$ , essentially the same proof works. However, instead of  $A \oplus A$  we use  $W_1|A|W_1^* + W_2|A|W_2^*$  where  $W_1$  and  $W_2$  are partial isometries satisfying  $W_i^*W_i = P$  for  $i \in \{1, 2\}$ , and  $W_1W_1^* + W_2W_2^* = 1$  for a projection  $P$  of trace  $1/2$  in  $\mathcal{B}$  that commutes with  $|A|$  and so that

$$\mu_r(|A|P) = \begin{cases} \mu_r(|A|), & 0 < r < 1/2, \\ 0, & 1/2 \leq r \leq 1. \end{cases}$$

A similar procedure is performed for  $B$ , but with a projection  $Q$  and partial isometries  $U_1$  and  $U_2$ . Then arguing as above, instead of (2.5) we get

$$\|AB\|_{\mathcal{I}} \leq \|W_1|A|W_1^* + W_2|A|W_2^*\|_{\mathcal{I}^{1/p}} \|U_1|B|U_1^* + U_2|B|U_2^*\|_{\mathcal{I}^{1/q}},$$

and instead of (2.6) we get

$$\|W_1|A|W_1^* + W_2|A|W_2^*\|_{\mathcal{I}^{1/p}} \leq 2\|\mu(|A|P)\|_{E^{1/p}} = 2\||A|P\|_{\mathcal{I}^{1/p}} \leq 2\|A\|_{\mathcal{I}^{1/p}},$$

and a similar inequality for  $|B|Q$ . Thus, also in this case we get (2.3). ■

QUESTION 2.11. What is the best constant in (2.3)? In particular, can 4 be replaced by 1?

In the next section, we see that the constant is 1 in strongly symmetric operator spaces.

### 3. The Hölder inequality in strongly symmetric operator spaces.

For a symmetric Banach function space  $E$ , recall that we have  $E \subseteq L^1(J) + L^\infty(J)$ . We have the notion of the Hardy–Littlewood submajorization: for  $x, y \in E$ , we write  $x \prec\prec y$  to mean

$$\int_0^t x^*(s) ds \leq \int_0^t y^*(s) ds, \quad t > 0.$$

Similarly, for  $A, B$  in a symmetric operator space  $\mathcal{I} \subseteq L^1(\mathcal{B}, \tau) + L^\infty(\mathcal{B}, \tau)$ , we say that  $A$  is *submajorized* by  $B$  and write  $A \prec\prec B$  if

$$\int_0^t \mu_s(A) ds \leq \int_0^t \mu_s(B) ds, \quad t > 0.$$

The following definition is standard.

DEFINITION 3.1.

- (i) A symmetric function space  $E \subseteq L^1(J) + L^\infty(J)$  is said to be *strongly symmetric* if  $x, y \in E$ ,  $x \prec\prec y$  implies  $\|x\|_E \leq \|y\|_E$ .
- (ii) A symmetric operator space  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  is said to be *strongly symmetric* if the corresponding symmetric function space  $(E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$  is strongly symmetric. Equivalently, a symmetric operator space  $\mathcal{I} \subseteq L^1(\mathcal{B}, \tau) + L^\infty(\mathcal{B}, \tau)$  is strongly symmetric if  $A, B \in \mathcal{I}$ ,  $A \prec\prec B$  implies that  $\|A\|_{\mathcal{I}} \leq \|B\|_{\mathcal{I}}$ .

Now we see that the Hölder inequality holds (with constant 1) in strongly symmetric operator spaces.

THEOREM 3.2. *Let  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  be a strongly symmetric operator space in  $S(\mathcal{B}, \tau)$ . Then for every  $1 < p, q < \infty$  with  $1/p + 1/q = 1$ , and every  $A \in \mathcal{I}^{1/p}$ ,  $B \in \mathcal{I}^{1/q}$ , we have  $AB \in \mathcal{I}$  and*

$$(3.1) \quad \|AB\|_{\mathcal{I}} \leq \|A\|_{\mathcal{I}^{1/p}} \|B\|_{\mathcal{I}^{1/q}}.$$

*Proof.* Let  $(E, \|\cdot\|_E) = (E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$ . Making use of Proposition 2.8 and the functional calculus described above it, we infer  $\mu(A)\mu(B) \in E$  and

$$(3.2) \quad \|\mu(A)\mu(B)\|_E \leq \|\mu(A)\|_{E^{1/p}} \|\mu(B)\|_{E^{1/q}}.$$

By Theorem 2.10, we also have  $\mu(AB) \in E$ , and [3, Theorem 4.2] (with  $f(x) = x$ ) yields  $\mu(AB) \prec \mu(A)\mu(B)$ . As  $E$  is strongly symmetric, we have

$$\|\mu(AB)\|_E \leq \|\mu(A)\mu(B)\|_E.$$

Now (3.1) follows from this and (3.2). ■

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