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Endpoint bounds of square functions associated with Hankel multipliers

by

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Abstract. We prove endpoint bounds for the square function associated with radial Fourier multipliers acting on L^p radial functions. This is a consequence of endpoint bounds for a corresponding square function for Hankel multipliers. We obtain a sharp Marcinkiewicz-type multiplier theorem for multivariate Hankel multipliers and L^p bounds of maximal operators generated by Hankel multipliers as corollaries. The proof is built on techniques developed by Garrigós and Seeger for characterizations of Hankel multipliers.

1. Introduction. Let S_t^λ be the Bochner–Riesz mean of order $\lambda > 0$ defined by

$$\mathcal{F}[S_t^\lambda f](\xi) = \left(1 - \frac{|\xi|^2}{t^2}\right)_+^\lambda \mathcal{F}f(\xi)$$

for $t > 0$, where $\mathcal{F}$ denotes the Fourier transform $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x)e^{ix\cdot\xi}\,dx$. One is interested in the convergence $S_t^\lambda f \to f$ as $t \to \infty$ in various senses. In this regard, L^p estimates of $S^\lambda := S_1^\lambda$ and the maximal operator $S_*^\lambda f(x) = \sup_{t>0} |S_t^\lambda f(x)|$ have been studied extensively. For λ below the critical index $(d-1)/2$, it is conjectured that S^λ is bounded on L^p if and only if

$$\frac{2d}{d+1+2\lambda} < p < \frac{2d}{d-1-2\lambda}$$

and that S_*^λ is bounded for the same p-range for $p \geq 2$.

Although the conjectures remain open in the full p-range for $d \geq 3$, they are indeed theorems for $d = 2$ by Carleson and Sjölin [6] and Carbery [2], respectively. In addition, the work by Carbery, Gasper, and Trebels [5] and Carbery [3] shows that the results for $d = 2$ are consequences of more general multiplier theorems which apply to all radial Fourier multipliers. This

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involves the square function G^α,

$$G^\alpha f(x) = \Big(\int_0^\infty |R_t^\alpha f(x)|^2 \,\frac{dt}{t}\Big)^{1/2},$$

where

$$\mathcal{F}[R_t^\alpha f](\xi) = \frac{|\xi|}{t}\Big(1-\frac{|\xi|}{t}\Big)_+^{\alpha-1}\mathcal{F}f(\xi).$$

The square function (with $S_t^\alpha f - S_t^{\alpha-1} f$ in place of $R_t^\alpha f$) was introduced by Stein [23] in order to prove the L^2 boundedness of S_*^λ for $\lambda > 0$.

Let m be a bounded function on $\mathbb{R}_+ := (0,\infty)$ and $\mathcal{T}_m$ be the operator defined by

$$\mathcal{F}[\mathcal{T}_m f](\xi) = m(|\xi|)\mathcal{F}f(\xi).$$

Then for $\alpha > 1/2$ and a fixed non-trivial smooth function ϕ supported on $[1,2]$, there is a pointwise estimate

$$g[\mathcal{T}_m f](x) \le C \sup_{t>0} \|m(t\,\cdot)\phi\|_{L^2_\alpha(\mathbb{R})} G^\alpha f(x), \tag{1.1}$$

where g is the standard Littlewood–Paley square function and $L^2_\alpha(\mathbb{R})$ is the L^2-Sobolev space (see [3]). Since $\|g(\mathcal{T}_m f)\|_{L^p(\mathbb{R}^d)}$ is comparable to $\|\mathcal{T}_m f\|_{L^p(\mathbb{R}^d)}$ for $1 < p < \infty$, (1.1) reduces L^p estimates of $\mathcal{T}_m$ to the study of G^α, which is independent of a specific multiplier m. Moreover, it was shown in [3] that G^α controls the maximal operator generated by $\mathcal{T}_m$ by a pointwise estimate, which gives effective L^p bounds for the maximal functions S_*^λ when $p \ge 2$. We refer the reader to [19] for an excellent overview of various square functions.

For $1 < p \le 2$, it is known that G^α is bounded on L^p if and only if $\alpha > d(1/p - 1/2) + 1/2$ (see [25]). On the other hand, in order for G^α to be bounded on $L^p(\mathbb{R}^d)$ for $p > 2$, the condition

$$\alpha > \max\Big(d\Big(\frac{1}{2}-\frac{1}{p}\Big), \frac{1}{2}\Big)$$

is necessary, and is conjectured to be sufficient. The conjecture for $d = 2$ was verified by Carbery [2, 4], yielding the L^4 bound for S_*^α as a corollary. For higher dimensions, the conjecture has been verified for $p > 2(d+2)/d$ in [18] (see also [7, 22]). Furthermore, $L^{p,2} \to L^p$ endpoint estimates for the critical index $\alpha = d(1/2 - 1/p)$ and a smaller p-range, $p > 2(d+1)/(d-1)$, were obtained in [19], where $L^{p,q}$ denotes the Lorentz space.

We show that, on the subspace of radial functions, the endpoint estimate is valid for the conjectured p-range.

THEOREM 1.1. *Let*

$$d \ge 2, \qquad \frac{2d}{d-1} < p < \infty \quad \textit{and} \quad \alpha = d\Big(\frac{1}{2}-\frac{1}{p}\Big) > \frac{1}{2}.$$

Then

$$\|G^\alpha f\|_{L^p_{\mathrm{rad}}(\mathbb{R}^d)} \le C\|f\|_{L^{p,2}_{\mathrm{rad}}(\mathbb{R}^d)}.$$

This implies a radial version of the conjecture for G^α by real interpolation. As a consequence, one may obtain a new proof of the sharp estimate for radial Fourier multipliers acting on radial functions in terms of Sobolev spaces (see [14]). A much stronger result is known. Garrigós and Seeger [12] obtained a necessary and sufficient condition for $L^p_{\mathrm{rad}}(\mathbb{R}^d)$ boundedness of $\mathcal{T}_m$ for $1 < p < 2d/(d+1)$. We note that our proof of Theorem 1.1 is based on [12].

Let $M_m f := \sup_{t>0} |\mathcal{T}_{m(t\cdot)} f|$, where we additionally assume that m is compactly supported in $(0,\infty)$. For the range $1 < p < 2d/(d+1)$, a necessary and sufficient condition for $L^p_{\mathrm{rad}}(\mathbb{R}^d)$ boundedness of M_m is known (see [13]). By Theorem 1.1, we may obtain a sharp sufficient condition for L^p_{rad} boundedness of M_m in terms of Sobolev spaces for $2 \le p < \infty$ (see Corollary 2.4).

Our primary motivation for Theorem 1.1 comes from a more general situation when multipliers and functions are assumed to be multi-radial. Let $n \in \mathbb{N}$ and $\vec{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$. We say that f is *$\vec{d}$-radial* if there is a function f_0 on $(0,\infty)^n$ such that $f(x_1, \dots, x_n) = f_0(|x_1|, \dots, |x_n|)$, where $x_j \in \mathbb{R}^{d_j}$. In this case, we say that f is the *$\vec{d}$-radial extension* of f_0.

In this paper, we are interested in the Fourier multiplier transformation given by a $\vec{d}$-radial multiplier m acting on $\vec{d}$-radial functions. A typical m would be a tensor product of radial multipliers. In that case, one may easily obtain L^p bounds by iteration. Unfortunately, this argument fails for general m. Nevertheless, it is easy to iterate Theorem 1.1 to obtain estimates for product square functions. As a consequence, we obtain sharp Marcinkiewicz type multiplier theorems for the $\vec{d}$-radial case. This will be carried out in the multivariate Hankel multiplier setting, which improves a result of Wróbel [26] (see Theorem 3.1).

Here we state a special case of Theorem 3.1. Let us denote by $\mathbb{R}^{\vec{d}}$ and $L^p_{\mathrm{rad}}(\mathbb{R}^{\vec{d}})$ the product space $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$ and the subspace of $\vec{d}$-radial functions in $L^p(\mathbb{R}^{\vec{d}})$, respectively. Let ϕ be a tensor product of n non-trivial smooth functions supported on the interval $[1,2]$. It would be convenient to define the subspace $L^2_{\mathrm{loc},\vec{\alpha}}(\mathbb{R}^n)$ of $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ equipped with the norm

$$\|m_0\|^2_{L^2_{\mathrm{loc},\vec{\alpha}}(\mathbb{R}^n)} := \sup_{\vec{t}\in(0,\infty)^n} \int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^n}[m_0(\vec{t}\,\cdot)\phi](\xi)|^2 \prod_{j=1}^n (1+|\xi_j|)^{2\alpha_j}\, d\xi,$$

where $(\vec{t}\,\cdot)$ denotes the n-parameter dilation $(t_1\cdot, \dots, t_n\cdot)$.

THEOREM 1.2. *Let $1 < p < \infty$ and $\vec{d} = (d_1, \dots, d_n)$ with $d_j \ge 2$ for all j. Assume that m is the $\vec{d}$-radial extension of a bounded function m_0 in*

$L^2_{\mathrm{loc},\vec{\alpha}}(\mathbb{R}^n)$ *for some* $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ *such that*

$$\alpha_j > \max\left(\frac{1}{2}, d_j\left|\frac{1}{p} - \frac{1}{2}\right|\right) \quad \textit{for } 1 \le j \le n.$$

Then

$$\|\mathcal{F}^{-1}[m\mathcal{F}f]\|_{L^p(\mathbb{R}^{\vec{d}})} \le C\|m_0\|_{L^2_{\mathrm{loc},\vec{\alpha}}}\|f\|_{L^p_{\mathrm{rad}}(\mathbb{R}^{\vec{d}})}.$$

The paper is organized as follows. In Section 2, we formulate Theorem 1.1 in a slightly more general context in terms of a square function associated with Hankel multipliers. In Section 3, we extend the results in Section 2 to multivariate Hankel multipliers. We include an application to Bochner–Riesz type multipliers. In Section 4, product square functions are discussed. Pointwise estimates for multivariate Hankel multiplier transformations in terms of the product square functions are obtained, which leads to multiplier theorems. The rest of the paper is devoted to the proof of Theorem 2.2, which is slightly more general than Theorem 1.1. In Appendix, we give a proof of L^p bounds of a Littlewood–Paley square function considered in this paper.

In what follows, we frequently write $A \lesssim B$ if $A \le CB$ for some universal implicit constant C which may depend on parameters including n, p, $\vec{d}$, and $\vec{\alpha}$. Throughout the paper, we assume that $\alpha > 1/2$ unless otherwise stated.

2. Hankel multipliers: Single variable case. Consider a radial function F on $\mathbb{R}^d$ such that $F(x) = f(|x|)$ for a function f on $\mathbb{R}_+ := (0, \infty)$. It is well-known that the Fourier transform of F can be expressed by an integral transform of f which involves the Bessel function. Indeed, $\mathcal{F}_{\mathbb{R}^d}[F](\xi) = (2\pi)^d \mathcal{H}_d f(|\xi|)$. Here, $\mathcal{H}_d$ is the modified Hankel transform defined by

$$\mathcal{H}_d f(s) = \int_0^\infty B_d(sr) f(r)\, d\mu_d(r),$$

where $B_d(x) = x^{-(d-2)/2} J_{(d-2)/2}(x)$, J_α denotes the standard Bessel function of order α, and μ_d is the measure on $\mathbb{R}_+$ given by $d\mu_d(r) = r^{d-1}\, dr$ (see [24]). In what follows, we let d be a real number greater than 1.

The operator $\mathcal{H}_d$ enjoys many properties analogous to those of the Fourier transform including the inversion formula and Plancherel's theorem. Let $\mathcal{S}(\mathbb{R}_+)$ be the space of (restrictions to $\mathbb{R}_+$ of) even Schwartz functions on $\mathbb{R}$. Then $\mathcal{H}_d$ is an isomorphism on $\mathcal{S}(\mathbb{R}_+)$ and an isometry of $L^2(\mu_d)$ with $\mathcal{H}_d^{-1} = \mathcal{H}_d$.

We are now ready to define a variant of the square function G^α relevant to Hankel multipliers. We shall work with H-valued functions f on $\mathbb{R}_+$, where H is a separable Hilbert space, for an iteration argument to be used later in

Section 4.1. We define the square function $\mathcal{G}^\alpha$ by

$$\mathcal{G}^\alpha f(r) = \bigg(\int_0^\infty |R_t^\alpha f(r)|_H^2 \, \frac{dt}{t}\bigg)^{1/2},$$

where

$$\mathcal{H}_d[R_t^\alpha f](\rho) = \frac{\rho}{t}\bigg(1 - \frac{\rho}{t}\bigg)_+^{\alpha-1} \mathcal{H}_d f(\rho) \quad \text{for } \alpha > 1/2.$$

For $1 < p \le 2$, $\mathcal{G}^\alpha$ is bounded on $L^p(\mu_d)$ if and only if $\alpha > d(1/p-1/2) + 1/2$. The proof is essentially contained in the proof of $L^p(\mathbb{R}^d)$ bounds of G^α. For $2 \le p < \infty$, one may verify that $\mathcal{G}^\alpha$ is bounded on $L^p(\mu_d)$ only if $\alpha > \alpha(d,p)$, where

$$\alpha(d,p) := \max\bigg(\frac{1}{2}, d\bigg|\frac{1}{p} - \frac{1}{2}\bigg|\bigg).$$

This can be done, for instance, by examining its consequences (see e.g. Corollary 2.3). We show that the condition is also sufficient.

Theorem 2.1. *Let $d > 1$ and $2 \le p < \infty$. Then*

$$\|\mathcal{G}^\alpha f\|_{L^p(\mu_d)} \le C\|f\|_{L^p(\mu_d,H)} \quad \textit{if and only if} \quad \alpha > \alpha(d,p).$$

This result is obtained by real interpolation between the $L^2(\mu_d)$ bound for $\alpha > 1/2$ and the following endpoint bounds.

Theorem 2.2. *Let*

$$d > 1, \qquad \frac{2d}{d-1} < p < \infty \quad \textit{and} \quad \alpha = d\bigg(\frac{1}{2} - \frac{1}{p}\bigg) > \frac{1}{2}.$$

Then

$$\|\mathcal{G}^\alpha f\|_{L^p(\mu_d)} \le C\|f\|_{L^{p,2}(\mu_d,H)}.$$

Theorem 1.1 is an immediate consequence of Theorem 2.2. Indeed, observe that $G^\alpha F(x) = \mathcal{G}^\alpha f(|x|)$ if $F(x) = f(|x|)$ and that we may identify $L^{p,q}_{\mathrm{rad}}(\mathbb{R}^d)$ with $L^{p,q}(\mu_d)$. In fact, all results to be discussed in this paper on Hankel multipliers $m(\rho)$ with $L^{p,q}(\mu_d)$ norm can be similarly translated into statements on radial Fourier multipliers $m(|\cdot|)$ with $L^{p,q}_{\mathrm{rad}}$ norm.

Remark. The Lorentz space $L^{p,2}$ in Theorem 2.2 cannot be replaced by $L^{p,q}$ for $q > 2$ (see [19]). We do not know if $\mathcal{G}^\alpha$ is actually bounded from $L^{p,2}(\mu_d, H)$ to $L^{p,q}(\mu_d)$ for some $q < p$, in particular for $q = 2$.

Next, we shall state multiplier theorems which follow from the square function estimates. Let m be a bounded function on $\mathbb{R}_+$ and T_m be the operator defined by

$$\mathcal{H}_d[T_m f](\rho) = m(\rho)\mathcal{H}_d f(\rho).$$

Let $\Phi \in \mathcal{S}(\mathbb{R}_+)$ be such that $\Phi(0) = 0$, and Φ_t be a Hankel multiplier transformation defined by $\mathcal{H}_d[\Phi_t f](\rho) = \Phi(\rho/t)\mathcal{H}_d f(\rho)$. We define a Littlewood–Paley function g_Φ by

$$g_\Phi f(r) = \Big(\int_0^\infty |\Phi_t f(r)|_H^2 \frac{dt}{t}\Big)^{1/2}.$$

Then $\|g_\Phi(f)\|_{L^p(\mu_d)}$ is comparable to $\|f\|_{L^p(\mu_d)}$ for $1 < p < \infty$ (see Appendix).

We shall use a specific Φ given by $\Phi(\rho) = \rho\phi(\rho)$, where ϕ is a non-trivial smooth function supported on the interval $[1, 2]$. Then for $\alpha > 1/2$, there is a pointwise estimate similar to (1.1),

$$g_\Phi[T_m f](r) \le C \sup_{t>0} \|m(t\,\cdot)\phi\|_{L^2_\alpha(\mathbb{R})} \mathcal{G}^\alpha f(r) \tag{2.1}$$

(see Section 4.2). Thus, we obtain the following sharp multiplier theorem in terms of localized L^2 Sobolev spaces.

Corollary 2.3. *Let $d > 1$, $1 < p < \infty$, and ϕ be a non-trivial smooth function supported on* $[1, 2]$. *Suppose that* $\sup_{t>0} \|m(t\,\cdot)\phi\|_{L^2_\alpha(\mathbb{R})} < \infty$ *for some $\alpha > \alpha(d, p)$. Then the operator T_m is bounded on $L^p(\mu_d)$.*

As was discussed in Introduction, Corollary 2.3 is not new. See also [15, 8] for multiplier theorems on L^1 and Hardy spaces.

Next, we turn to the maximal operators $M_m f := \sup_{t>0} |T_{m(t\,\cdot)} f|$ for a multiplier m supported in $[1/2, 2]$. From the square function estimate, we have L^p bounds for the maximal operators M_m for the range $p \ge 2$.

Corollary 2.4. *Let $d > 1$ and $2 \le p < \infty$. Suppose that m is supported in $[1/2, 2]$ and $m \in L^2_\alpha(\mathbb{R})$ for $\alpha > \alpha(d, p)$. Then*

$$\|M_m f\|_{L^p(\mu_d)} \le C\|m\|_{L^2_\alpha(\mathbb{R})}\|f\|_{L^p(\mu_d)}.$$

This is a consequence of the pointwise estimate

$$M_m f(r) \le C\|m\|_{L^2_\alpha(\mathbb{R})} \mathcal{G}^\alpha f(r) \tag{2.2}$$

(see Section 4.2) for $\alpha > 1/2$.

Corollary 2.4 is sharp in the sense that the required number of derivatives, $\alpha(d, p)$ cannot be decreased. This can be seen by considering the truncated Bochner–Riesz multiplier $m(\rho) = (1-\rho^2)_+^\lambda \chi(\rho)$ where χ is a smooth function supported near $\rho = 1$, discarding a harmless part near the origin. Corollary 2.4 also implies L^p_{rad} bounds of S^λ_* for $2 \le p < 2d/(d - 1 - 2\lambda)$, which was previously obtained by Kanjin [16] ([1]).

([1]) In fact, the optimal p-range $\frac{2d}{d+1+2\lambda} < p < \frac{2d}{d-1-2\lambda}$ was obtained in [16].

While it is sharp, the L^2-Sobolev condition is too restrictive to yield endpoint bounds. However, we recently proved that

$$\|M_m\|_{L^{p,q}(\mu_d)\to L^p(\mu_d)} \approx \|\mathcal{H}_d m\|_{L^{p',q'}(\mu_d)}$$

for $2d/(d-1) < p < \infty$ and $1 \le q \le p$ (see [17]), which covers endpoint bounds. See [19] for $L^{p,1}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ bounds of S_*^λ for a smaller p-range.

3. Hankel multipliers: Multivariate case. The goal of this section is to extend the results of the previous section to the multivariate setting. Fix $n \in \mathbb{N}$ and $\vec{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n$ such that $d_j \ge 1$. The Hankel transform $\mathcal{H}_{\vec{d}}$ acting on functions on $(\mathbb{R}_+)^n$ is defined by $\mathcal{H}_{\vec{d}} f(s) := \mathcal{H}_{d_n} \cdots \mathcal{H}_{d_1} f(s)$, where $\mathcal{H}_{d_k}$ acts only on the kth variable.

For $\vec{d} \in \mathbb{N}^n$, $\mathcal{H}_{\vec{d}}$ generalizes the Fourier transform of $\vec{d}$-radial functions. Suppose that $\tilde{m}$ is the $\vec{d}$-radial extension of a bounded function m on $(\mathbb{R}_+)^n$. Then the study of $\mathcal{T}_{\tilde{m}}$ acting on $\vec{d}$-radial functions can be reduced to the study of T_m defined by $\mathcal{H}_{\vec{d}}[T_m f] = m\mathcal{H}_{\vec{d}} f$.

The operators T_m have been studied only recently (see e.g. [1, 26, 9]). In particular, Wróbel [26] proved a Marcinkiewicz type multiplier theorem, where a smoothness condition was given in terms of a variant of L^2 Sobolev space. We introduce some notation in order to simplify the presentation.

Notation. Let $\mu_{\vec{d}}$ be the measure on $(\mathbb{R}_+)^n$ given by

$$d\mu_{\vec{d}}(s) = \prod_{k=1}^n d\mu_{d_k}(s_k).$$

For $\vec{t}, \vec{s} \in \mathbb{R}^n$, we define the vectors $\vec{t}\vec{s}$ and $\vec{t}/\vec{s}$ to be given by component-wise product and division, respectively. We write $\vec{t} > \vec{s}$ if $t_k > s_k$ for all $1 \le k \le n$. If $s \in \mathbb{R}$, we write $\vec{t} > s$ if $t_k > s$ for all k. For $1 \le p < \infty$, let $\vec{\alpha}(\vec{d}, p) \in (\mathbb{R}_+)^n$ be the vector whose kth component is

$$\max\left(\frac{1}{2}, d_k\left|\frac{1}{p} - \frac{1}{2}\right|\right).$$

For a given $\vec{\alpha} \in (\mathbb{R}_+)^n$, we shall denote by $L^2_{\vec{\alpha}}(\mathbb{R}^n)$ the Sobolev space equipped with the norm

$$\|f\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)} = \|w_{\vec{\alpha}} \mathcal{F}_{\mathbb{R}^n}[f]\|_{L^2(\mathbb{R}^n)}, \quad \text{where} \quad w_{\vec{\alpha}}(\xi) = \prod_{k=1}^n (1 + |\xi_k|)^{\alpha_k}.$$

Let $\{\phi_k\}_{1\le k\le n}$ be a collection of non-trivial smooth functions supported on $[1, 2]$, and let $\phi(r) = \prod_{k=1}^n \phi_k(r_k)$. It was shown in [26] that T_m is bounded on $L^p(\mu_{\vec{d}})$ for $1 < p < \infty$ if

$$\sup_{\vec{t}>0} \|m(\vec{t}\,\cdot)\phi\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)} < \infty \tag{3.1}$$

for some $\vec{\alpha} > \vec{\alpha}(\vec{d}, 1)$. Here we have used the notation $(\vec{t}\,\cdot) = (t_1\,\cdot, \ldots, t_n\,\cdot)$ for the n-parameter dilation.

By using a product version of Theorem 2.1 (see Theorem 4.1), we may improve the above result as follows.

THEOREM 3.1. *Let $\vec{d} > 1$ and $1 < p < \infty$. Suppose that* (3.1) *holds for some $\vec{\alpha} > \vec{\alpha}(\vec{d}, p)$. Then the operator T_m is bounded on $L^p(\mu_{\vec{d}})$.*

This is sharp in the sense that $\vec{\alpha}(\vec{d}, p)$ cannot be decreased. One may verify this from the sharpness of Corollary 2.3 by considering product type multipliers. Theorem 3.1 implies the following.

COROLLARY 3.2. *Let $1 < p < \infty$, $\vec{\alpha} \in \mathbb{Z}^n$, and $\vec{\alpha} > \vec{\alpha}(\vec{d}, p)$. Suppose that*

$$\sup_{\vec{t}>0} \int_{t_n}^{2t_n} \cdots \int_{t_1}^{2t_1} |\rho^{\vec{\beta}} D^{\vec{\beta}} m(\rho)|^2 \frac{d\rho_1}{t_1} \cdots \frac{d\rho_n}{t_n} < \infty \tag{3.2}$$

for all $\vec{\beta} \in \mathbb{Z}^n$ with $0 \le \vec{\beta} \le \vec{\alpha}$. Then T_m is bounded on $L^p(\mu_{\vec{d}})$. In particular, (3.2) *holds if $|D^{\vec{\beta}} m(\rho)| \le C\rho^{-\vec{\beta}}$ for all $0 \le \vec{\beta} \le \vec{\alpha}$.*

We may also extend Corollary 2.4 to the n-parameter maximal operator $M_m f := \sup_{\vec{t}>0} |T_{m(\vec{t}\,\cdot)} f|$.

THEOREM 3.3. *Let $\vec{d} > 1$ and $2 \le p < \infty$. Suppose that m is supported in $[1/2, 2]^n$ and $m \in L^2_{\vec{\alpha}}(\mathbb{R}^n)$ for some $\vec{\alpha} > \vec{\alpha}(\vec{d}, p)$. Then*

$$\|M_m f\|_{L^p(\mu_{\vec{d}})} \le C\|m\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)} \|f\|_{L^p(\mu_{\vec{d}})}.$$

See Section 4 for the proof of Theorems 3.1 and 3.3.

3.1. Application to Bochner–Riesz type multipliers. Let $m^\lambda(\rho) = (1 - |\rho|^2)_+^\lambda$, where $|\rho|^2 = \rho_1^2 + \cdots + \rho_n^2$, and T^λ be the operator defined by $\mathcal{H}_{\vec{d}}[T^\lambda f] = m^\lambda \mathcal{H}_{\vec{d}} f$. Let us temporarily assume that $\vec{d} \in \mathbb{N}^n$. Then the study of the usual Bochner–Riesz means for the Fourier transform acting on $\vec{d}$-radial functions reduces to the study of T^λ, since

$$S^\lambda F(x_1, \ldots, x_n) = T^\lambda f(|x_1|, \ldots, |x_n|)$$

if F is the $\vec{d}$-radial extension of f. Note that T^λ cannot be bounded on $L^p(\mu_{\vec{d}})$ unless $|1/p - 1/2| < (1/\|d\|)(\lambda + 1/2)$, where $\|d\| = \sum_{k=1}^n d_k$. As a corollary of Theorem 3.1, we have the following.

COROLLARY 3.4. *Let $\vec{d} > 1$ and*

$$\lambda > \max_{1 \le k \le n} \frac{\|d\|}{2d_k} - \frac{1}{2}.$$

Then T^λ is bounded on $L^p(\mu_{\vec{d}})$ if

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{\|d\|}\left(\lambda + \frac{1}{2}\right). \tag{3.3}$$

Unfortunately, Corollary 3.4 does not seem to give any improvement over the known results for the boundedness of the Bochner–Riesz means even under the additional assumption on the function side. However, we expect that one may improve the range of λ by taking advantage of m^λ being radial. We hope to return to this issue in future work.

To prove Corollary 3.4, we need the following.

LEMMA 3.5. *Let ϕ be a tensor product of n smooth functions supported on $[1,2]$. If $0 \le \beta < \lambda + 1/2$, then*

$$\sup_{\vec{t}>0} \|m^\lambda(\vec{t}\,\cdot)\phi\|_{L^2_\beta(\mathbb{R}^n)} < \infty.$$

Let us show how Corollary 3.4 follows from Lemma 3.5. Assume (3.3). Then there is $\epsilon(p) > 0$ such that if we define

$$\alpha_k = \frac{d_k}{\|d\|}\left(\lambda + \frac{1}{2}\right) - \epsilon(p)$$

for $1 \le k \le n$, then $\vec{\alpha} > \vec{\alpha}(\vec{d}, p)$ and $\|\vec{\alpha}\| := \sum_{j=1}^n \alpha_j < \lambda + 1/2$. By Theorem 3.1 together with Lemma 3.5 and the trivial embedding $L^2_{\|\vec{\alpha}\|} \hookrightarrow L^2_{\vec{\alpha}}$, we have Corollary 3.4.

Proof of Lemma 3.5. Although the proof seems to be standard, we include it for completeness. We apply the standard dyadic decomposition for the Bochner–Riesz multipliers. Take a smooth function χ supported on $1/2 \le x \le 2$ such that $\sum_{l=0}^\infty \chi(2^l x) = 1$ if $0 < x \le 1$. Then one may write $m^\lambda(\rho) = \sum_{l=0}^\infty 2^{-l\lambda} m_l^\lambda(\rho)$, where

$$m_l^\lambda(\rho) = 2^{l\lambda}(1 - |\rho|^2)^\lambda \chi(2^l(1 - |\rho|^2)).$$

Then we have

$$\sup_{\vec{t}>0} \|m^\lambda(\vec{t}\,\cdot)\phi\|_{L^2_\beta(\mathbb{R}^n)} \le \sum_{l=0}^\infty 2^{-l\lambda} \sup_{\vec{t}>0} \|m_l^\lambda(\vec{t}\,\cdot)\phi\|_{L^2_\beta(\mathbb{R}^n)}.$$

Since $m_l^\lambda(\vec{t}\,\cdot)\phi \equiv 0$ if $|\vec{t}| > 1$, we may assume that the supremum is taken over $|\vec{t}| \le 1$. For $l \le 2$, it is easy to show that

$$\sup_{\vec{t}>0} \|m_l^\lambda(\vec{t}\,\cdot)\phi\|_{L^2_\beta(\mathbb{R}^n)} < \infty$$

for any $\beta \ge 0$.

For $l > 2$, we need to show that

$$\sup_{\vec{t}>0} \|m_l^\lambda(\vec{t}\,\cdot)\phi\|_{L^2_\beta(\mathbb{R}^n)} \le C 2^{l(\beta - 1/2)}. \tag{3.4}$$

Here, we may further assume that $1/4 \le |\vec{t}| \le 1$ since m_l^λ is compactly supported away from the origin. Moreover, by interpolation, it is enough to show (3.4) for integer $\beta \ge 0$.

For $\|\vec{\gamma}\| \le \beta$ and $|\vec{t}| \le 1$, we have

$$|D^{\vec{\gamma}}[m_l^\lambda(\vec{t}\cdot)\phi]|(\rho) \lesssim 2^{l\beta}\tilde{\chi}(2^l(1-|\vec{t}\cdot\rho|^2))\tilde{\phi}(\rho)$$

by the Leibniz rule, where $\tilde{\chi}$ and $\tilde{\phi}$ are finite sums of absolute values of derivatives of χ and ϕ, respectively.

Thus, (3.4) follows from

$$\sup_{1/4\le|\vec{t}|\le 1} \int |\tilde{\chi}(2^l(1-|\vec{t}\cdot\rho|^2))\tilde{\phi}(\rho)|^2\,d\rho \lesssim 2^{-l}. \tag{3.5}$$

The integral on the left hand side of (3.5) is bounded by

$$\int_{t_n}^{2t_n}\cdots\int_{t_1}^{2t_1} |\tilde{\chi}(2^l(1-|\rho|^2))|^2\,\frac{d\rho_1}{t_1}\cdots\frac{d\rho_n}{t_n}.$$

Thus, we are led to consider the volume of the intersection between the box $\prod_{j=1}^n[t_j,2t_j]$ and an annulus of radius and width comparable to 1 and 2^{-l}, respectively. Assume, without loss of generality, that $t_1\ge\cdots\ge t_n$. Then $t_1\ge c_n$ for some $c_n>0$ since $|\vec{t}|>1/4$.

We claim that

$$\int_{t_1}^{2t_1} |\tilde{\chi}(2^l(1-|\rho|^2))|^2\,\frac{d\rho_1}{t_1}\lesssim 2^{-l}$$

provided that $t_1\ge c_n$, which would imply (3.5). When evaluating the integral, we may assume that $|\rho'|^2\le 1-c_n^2<1$ since $\rho_1\ge c_n$, where $\rho=(\rho_1,\rho')$. The claim follows from the fact that the ρ_1 support of $\chi(2^l(1-|\rho|^2))$ for each fixed ρ' with $|\rho'|^2\le 1-c_n^2$, is contained in an interval of size $O(2^{-l})$. This proves the claim, and thus (3.5). ∎

4. Product variants

4.1. Product square functions. Let $\Phi^{(k)}(\rho_k)=\rho_k\phi_k(\rho_k)$ for ϕ_k as in Section 3,

$$\Phi_{\vec{t}}f=\Phi_{t_n}^{(n)}\cdots\Phi_{t_1}^{(1)}f \quad\text{and}\quad R_{\vec{t}}^{\vec{\alpha}}f=R_{t_n}^{\alpha_n}\cdots R_{t_1}^{\alpha_1}f,$$

where $\Phi_{t_k}^{(k)}$ and $R_{t_k}^{\alpha_k}$ act only on the kth variable. For H-valued functions f, we define $\mathcal{G}^{\vec{\alpha}}$ by

$$\mathcal{G}^{\vec{\alpha}}f(r)=\bigg(\int_{(\mathbb{R}_+)^n}|R_{\vec{t}}^{\vec{\alpha}}f(r)|_H^2\,\frac{d\vec{t}}{\vec{t}}\bigg)^{1/2}, \tag{4.1}$$

where $d\vec{t}/\vec{t}$ is the measure $\prod_{k=1}^n dt_k/t_k$ on $(\mathbb{R}_+)^n$; and g_Φ is defined similarly to (4.1), with $\Phi_{\vec{t}}$ in place of $R_{\vec{t}}^{\vec{\alpha}}$.

We first note that for $1<p<\infty$, there is a constant $C>0$ such that

$$C^{-1}\|f\|_{L^p(\mu_{\vec{d}},H)}\le\|g_\Phi f\|_{L^p(\mu_{\vec{d}})}\le C\|f\|_{L^p(\mu_{\vec{d}},H)}.$$

The second inequality follows from the case $n = 1$ (see Appendix) by an iteration argument (see [11, Section 2]). The first inequality follows from the second by the polarization identity and $\|g_\Phi f\|_{L^2(\mu_{\vec{d}})} = C\|f\|_{L^2(\mu_{\vec{d}},H)}$.

There are product versions of the pointwise estimates (2.1) and (2.2). For $\vec{\alpha} > 1/2$, we have

$$g[T_m f](r) \le C \sup_{\vec{t}>0} \|m(\vec{t}\,\cdot)\phi\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)} \mathcal{G}^{\vec{\alpha}} f(r), \tag{4.2}$$

$$M_m f(r) \le C\|m\|_{L^2_{\vec{\alpha}}(\mathbb{R})} \mathcal{G}^{\vec{\alpha}} f(r), \tag{4.3}$$

where we additionally assume that m is supported in $[1/2, 2]^n$ in (4.3). We defer the proof of the estimates to the following section.

Given the pointwise estimates, Theorems 3.1 and 3.3 are consequences of the following theorem.

THEOREM 4.1. *Let $\vec{d} > 1$ and $2 \le p < \infty$. Then*

$$\|\mathcal{G}^{\vec{\alpha}} f\|_{L^p(\mu_{\vec{d}})} \le C\|f\|_{L^p(\mu_{\vec{d}},H)} \quad \textit{if and only if} \quad \vec{\alpha} > \vec{\alpha}(\vec{d}, p).$$

Proof. For the necessity, it is enough to consider a function $f(r) = \prod_{1\le k\le n} f_k(r_k)$ such that $f_k \in L^p(\mu_{d_k}, H)$. Then $\mathcal{G}^{\vec{\alpha}} f(r) = \prod_{k=1}^n \mathcal{G}^{\alpha_k} f_k(r_k)$, thus the necessity follows from Theorem 2.1.

An iteration argument in [11] gives the sufficiency, but we shall include the argument for the convenience of the reader. We use induction on n, with the case $n = 1$ given by Theorem 2.1. Suppose that the assertion is true for the dimension $n - 1$. Let $\vec{\alpha} = (\vec{\alpha}', \alpha_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be such that $\vec{\alpha} > \vec{\alpha}(\vec{d}, p)$. Set $F(r', r_n) = R^{\alpha_n}_{t_n}[f(r', \cdot)](r_n)$. We regard F as an $\tilde{H}$-valued function, where $\tilde{H} = L^2(\mathbb{R}_+, dt_n/t_n, H)$ is a Hilbert space.

We have $|F(r', r_n)|_{\tilde{H}} = \mathcal{G}^{\alpha_n}[f(r', \cdot)](r_n)$ and $\mathcal{G}^{\vec{\alpha}} f(r) = \mathcal{G}^{\vec{\alpha}'}[F(\cdot, r_n)](r')$. Thus, $\|\mathcal{G}^{\vec{\alpha}} f\|^p_{L^p(\mu_{\vec{d}})}$ is equal to

$$\begin{aligned}
&\int_0^\infty \int_{(\mathbb{R}_+)^{n-1}} |\mathcal{G}^{\vec{\alpha}'}[F(\cdot, r_n)](r')|^p \, d\mu_{\vec{d}'}(r') \, d\mu_{d_n}(r_n) \\
&\quad \lesssim \int_0^\infty \int_{(\mathbb{R}_+)^{n-1}} |F(r', r_n)|^p_{\tilde{H}} \, d\mu_{\vec{d}'}(r') \, d\mu_{d_n}(r_n) \\
&\quad = \int_{(\mathbb{R}_+)^{n-1}} \int_0^\infty |\mathcal{G}^{\alpha_n}[f(r', \cdot)](r_n)|^p \, d\mu_{d_n}(r_n) \, d\mu_{\vec{d}'}(r') \\
&\quad \lesssim \int_{(\mathbb{R}_+)^{n-1}} \int_0^\infty |f(r', r_n)|^p_H \, d\mu_{d_n}(r_n) \, d\mu_{\vec{d}'}(r') = C\|f\|^p_{L^p(\mu_{\vec{d}},H)},
\end{aligned}$$

where the first inequality follows from the induction hypothesis. ∎

4.2. Pointwise estimates. In this section, we prove (4.2) and (4.3). The Riemann–Liouville lemma on fractional differentiation played an important role for the pointwise estimate (1.1) given in [3]. We shall need a product version of that lemma.

Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\vec{\alpha} \geq 0$. We define the fractional differentiation $D^{\vec{\alpha}}$ for $f \in L^2(\mathbb{R}^n)$ by

$$\mathcal{F}[D^{\vec{\alpha}} f](\xi) = \prod_{k=1}^{n} (-i\xi_k)^{\alpha_k} \mathcal{F} f(\xi),$$

which coincides with the usual differentiation up to a constant when $\vec{\alpha} \in \mathbb{Z}^n$. In addition, let $I^{\vec{\alpha}}$ be the fractional integral defined by

$$I^{\vec{\alpha}} f(x) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} \frac{(u_k - x_k)_+^{\alpha_k - 1}}{\Gamma(\alpha_k)} f(u)\, du.$$

We shall work with the Sobolev space $L^2_{\vec{\alpha}}(\mathbb{R}^n)$ defined in Section 3.

LEMMA 4.2 (Riemann–Liouville). *Let $\vec{\alpha} > 1/2$. Suppose that $f \in L^2_{\vec{\alpha}}(\mathbb{R}^n)$ and* $\operatorname{supp} f \subset \prod_{k=1}^{n} (-\infty, a_k]$. *Then* $\operatorname{supp} D^{\vec{\alpha}} f \subset \prod_{k=1}^{n} (-\infty, a_k]$ *and*

$$f(x) = I^{\vec{\alpha}}[D^{\vec{\alpha}} f](x) \qquad a.e.$$

Proof. The proof for $n = 1$ was given in [3]. It still works for $n > 1$ with a few minor changes. For the convenience of the reader, we include the proof for $n = 1$ and indicate the changes for $n > 1$.

First consider the operators D^{α}_{ϵ} associated with the Fourier multipliers $(\epsilon - i\xi)^{\alpha}$ for $\epsilon > 0$. Then $D^{\alpha}_{\epsilon} f \to D^{\alpha} f$ in L^2 as $\epsilon \to 0$ by the dominated convergence theorem.

With the aid of Cauchy's theorem, one may verify that

$$g^{\alpha}_{\epsilon}(x) = \mathcal{F}^{-1}_{\mathbb{R}}[(\epsilon - i\xi)^{-\alpha}](x) = \frac{1}{\Gamma(\alpha)} (-x)_+^{\alpha - 1} e^{\epsilon x} \in L^1.$$

Thus, the convolution operator $I^{\alpha}_{\epsilon} f = f * g^{\alpha}_{\epsilon}$ is the inverse of D^{α}_{ϵ} and $I^{\alpha}_{\epsilon} f \in L^2$.

For the support condition, observe that

$$D^{\alpha}_{\epsilon} f(x) = D^{[\alpha]+1}_{\epsilon} h(x),$$

where $h = I^{[\alpha]+1-\alpha}_{\epsilon} f$ and $[\alpha]$ is the least integer such that $[\alpha] \geq \alpha$. Then $\operatorname{supp} h \subset (-\infty, a]$. Moreover, $D^{[\alpha]+1}_{\epsilon} h$ is a linear combination of (non-fractional) derivatives of h, preserving the support of h.

Next, for a fixed $x \in (-\infty, a)$, one estimates $|f(x) - I^{\alpha}(D^{\alpha} f)(x)|$ by

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_x^a (u - x)^{\alpha - 1} |e^{-\epsilon(u - x)} D^{\alpha}_{\epsilon} f(t) - D^{\alpha} f(u)|\, du \\
&\qquad \leq \frac{1}{\Gamma(\alpha)} \|(\cdot - x)^{\alpha - 1}\|_{L^2(I_x)} \|e^{-\epsilon(\cdot - x)} D^{\alpha}_{\epsilon} f - D^{\alpha} f\|_{L^2(I_x)},
\end{aligned}$$

where $I_x = (x, a)$. The first norm is finite and the second norm tends to 0 as $\epsilon \to 0$.

The proof easily extends to the case $n > 1$. Indeed, one may extend D_ϵ^α by $D_\epsilon^{\vec{\alpha}}$ which is associated with the Fourier multipliers $\prod_{k=1}^n (\epsilon - i\xi_k)^{\alpha_k}$ and make similar multi-parameter extensions. ∎

With Lemma 4.2, product versions of the pointwise estimates given in [3] can be obtained without much additional work. We conclude this section with the proofs of (4.2) and (4.3).

Proof of (4.2). We may assume that $\|m(\vec{t}\,\cdot)\phi\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)} < \infty$ for each $\vec{t}$. Fix $\vec{t} \in (\mathbb{R}_+)^n$, and let $h(\rho) = m(\vec{t}\rho)\phi(\rho)$. Then by the Riemann–Liouville lemma, the support of $D^{\vec{\alpha}}h$ is contained in $(-\infty, 2]^n$. Moreover,

$$\begin{aligned}
\mathcal{H}_{\vec{d}}[\Phi_{\vec{t}}[T_m f]](\rho) &= \prod_{k=1}^n (\rho_k/t_k)\phi_k(\rho_k/t_k) m(\rho)\mathcal{H}_{\vec{d}} f(\rho) \\
&= C_{\vec{\alpha}} \int_{\mathbb{R}^n} \prod_{k=1}^n \frac{\rho_k}{t_k}\left(u_k - \frac{\rho_k}{t_k}\right)_+^{\alpha_k - 1} \mathcal{H}_{\vec{d}} f(\rho) D^{\vec{\alpha}} h(u)\, du \\
&= C_{\vec{\alpha}} \int_{[0,2]^n} \prod_{k=1}^n u_k^{\alpha_k} \frac{\rho_k}{u_k t_k}\left(1 - \frac{\rho_k}{u_k t_k}\right)_+^{\alpha_k - 1} \mathcal{H}_{\vec{d}} f(\rho) D^{\vec{\alpha}} h(u)\, du
\end{aligned}$$

for a.e. $\rho \in (\mathbb{R}_+)^n$. Applying $\mathcal{H}_{\vec{d}}$ to both sides of the above equality yields

$$\begin{aligned}
|\Phi_{\vec{t}}[T_m f](r)|_H &= C_{\vec{\alpha}} \Big| \int_{[0,2]^n} \prod_{k=1}^n u_k^{\alpha_k} R_{\vec{t}u}^{\vec{\alpha}} f(r) D^{\vec{\alpha}} h(u)\, du \Big|_H \\
&\lesssim \sup_{\vec{t}>0} \|m(\vec{t}\cdot)\phi\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)} \Big(\int_{[0,2]^n} |R_{\vec{t}u}^{\vec{\alpha}} f(r)|_H^2\, du \Big)^{1/2}.
\end{aligned}$$

The proof is completed by taking the $L^2((\mathbb{R}_+)^n, d\vec{t}/\vec{t})$ norm and making use of Fubini's theorem and a change of variable. ∎

Proof of (4.3). First observe that one may write

$$\frac{m(\rho)}{\rho_1 \cdots \rho_n} = m(\rho)\chi(\rho)$$

for a smooth function χ supported in $[1/4, 4]^n$ if m supported in $[1/2, 2]^n$.

We apply the product version of the Riemann–Liouville lemma to the function $m\chi$ without the use of the square function g. Arguing as in the proof of (4.2), we obtain

$$|T_{m(\cdot/\vec{t})}f(r)| = C_{\vec{\alpha}}\Big|\int_{(\mathbb{R}_+)^n} R^{\vec{\alpha}}_{\vec{t}u}f(r)u^{\vec{\alpha}+\vec{1}}D^{\vec{\alpha}}[m\chi](u)\,\frac{du}{u}\Big|$$
$$\lesssim \mathcal{G}^{\vec{\alpha}}f(r)\Big(\int_{(\mathbb{R}_+)^n}|u^{\vec{\alpha}+\vec{1}}D^{\vec{\alpha}}[m\chi](u)|^2\,\frac{du}{u}\Big)^{1/2}$$
$$\lesssim \mathcal{G}^{\vec{\alpha}}f(r)\|m\chi\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)},$$

where $\vec{1}=(1,\dots,1)$, $du/u=\prod_{k=1}^n du_k/u_k$, and we have used the fact that $\operatorname{supp}(D^{\vec{\alpha}}[m\chi])\subset(-\infty,2]^n$. Finally, $\|m\chi\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)}\lesssim\|m\|_{L^2_{\vec{\alpha}}(\mathbb{R}^n)}$ since $\hat{\chi}$ is a Schwartz function. ∎

5. Reductions toward Theorem 2.2

5.1. Littlewood–Paley theory. Split the multiplier of R^α as

$$\rho(1-\rho)_+^{\alpha-1}=\rho\chi_0(\rho)+(1-\rho)_+^{\alpha-1}\chi(\rho),$$

where $\chi_0,\chi\in\mathcal{S}(\mathbb{R}_+)$ and χ_0 and χ are supported on $[0,3/4]$ and $[1/2,2]$, respectively. Then g_Φ with $\Phi(\rho)=\rho\chi_0(\rho)$ is a standard Littlewood–Paley function, which is bounded on $L^p(\mu_d)$ for $1<p<\infty$. Thus, we may assume that

$$\mathcal{H}_d[R^\alpha_t f](\rho)=\Big(1-\frac{\rho}{t}\Big)_+^{\alpha-1}\chi\Big(\frac{\rho}{t}\Big)\mathcal{H}_d f(\rho).$$

Using this reduction and the Littlewood–Paley theory, we may localize the t-integral on the interval $[1,2]$. Choose a cut-off function $\eta\in C_0^\infty(\mathbb{R}^+)$ supported on $(1/8,8)$ such that $\eta(\rho)=1$ on $[1/4,4]$ and define the Littlewood–Paley projection L_j by $\mathcal{H}_d[L_jf](\rho)=\eta(2^{-j}\rho)\mathcal{H}_df(\rho)$. We have

$$[\mathcal{G}^\alpha f(r)]^2=\int_0^\infty|R^\alpha_t f(r)|_H^2\,\frac{dt}{t}=\sum_j\int_{2^j}^{2^{j+1}}|R^\alpha_t f(r)|_H^2\,\frac{dt}{t}$$
$$=\sum_j\int_1^2|R^\alpha_{2^jt}f(r)|_H^2\,\frac{dt}{t}=\sum_j\int_1^2|R^\alpha_{2^jt}[L_jf](r)|_H^2\,\frac{dt}{t},$$

where the last equality follows from $\eta(\rho)\chi(\rho/t)=\chi(\rho/t)$ for $t\in[1,2]$. Thus by the Littlewood–Paley inequality (see (7.15)), Theorem 2.2 follows from

$$\Big\|\Big(\sum_j\int_1^2|R^\alpha_{2^jt}f_j|_H^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mu_d)}\lesssim\Big\|\Big(\sum_j|f_j|_H^2\Big)^{1/2}\Big\|_{L^{p,2}(\mu_d)}\tag{5.1}$$

for $p=2d/(d-2\alpha)$ and $1/2<\alpha<d/2$.

5.2. Dualization. Let us denote by L_t^2 the Hilbert space $L^2([1,2],dt/t)$. If g_j is a function which takes values in $L_t^2(H^*)$, namely that $(g_j)_t(r) := [g_j(r)](t) \in H^*$, then

$$\int_0^\infty \int_1^2 \langle R^\alpha_{2^j t} f_j(r), (g_j)_t(r)\rangle \,\frac{dt}{t}\, d\mu_d(r) = \int_0^\infty \langle f_j(s), \mathcal{R}^\alpha_j g_j(s)\rangle\, d\mu_d(s),$$

where

$$\mathcal{R}^\alpha_j g(s) = \int_1^2 R^\alpha_{2^j t} g_t(s)\,\frac{dt}{t}$$

for $L_t^2(H^*)$-valued functions g. Thus by duality, (5.1) is equivalent to

$$\Big\|\Big(\sum_j |\mathcal{R}^\alpha_j g_j|_H^2\Big)^{1/2}\Big\|_{L^{p,2}(\mu_d)} \lesssim \Big\|\Big(\sum_j |g_j|^2_{L_t^2(H)}\Big)^{1/2}\Big\|_{L^p(\mu_d)} \tag{5.2}$$

for $p = 2d/(d+2\alpha)$ and $1/2 < \alpha < d/2$.

5.3. Decomposition. We make a dyadic decomposition following [12]. For $m \in \mathbb{Z}$, we let $I_m = [2^m, 2^{m+1})$, $I_m^* = [2^{m-1}, 2^{m+2})$, $I_m^{**} = [2^{m-2}, 2^{m+3})$, $L_m = (0, 2^m)$, and $R_m = [2^m, \infty)$. Then we may write

$$\begin{aligned}\mathcal{R}^\alpha_j f &= \sum_m [E^\alpha_{j,m} f + S^\alpha_{j,m} f + V^\alpha_{j,m} f]\\ &= \sum_m E^\alpha_{j,m-j} f + \sum_m S^\alpha_{j,m} f + \sum_m V^\alpha_{j,m-j} f,\end{aligned}$$

where $E^\alpha_{j,m} f$, $S^\alpha_{j,m} f$ and $V^\alpha_{j,m} f$ are defined by $\mathcal{R}^\alpha_j(f\chi_{I_m})$ times the characteristic functions $\chi_{L_{m-2}}$, $\chi_{I_m^{**}}$ and $\chi_{R_{m+3}}$, respectively.

We shall prove the following propositions in Section 7.

Proposition 5.1. *Let $1/2 < \alpha < d/2$ and $p = 2d/(d+2\alpha)$. Then there is a constant $\delta(p) > 0$ such that*

$$\|V^\alpha_{j,m-j} f\|_{L^{p,2}(\mu_d,H)} \le C 2^{-|m|\delta(p)} \|f\chi_{I_{m-j}}\|_{L^{p,\infty}(\mu_d, L_t^2(H))},$$

where the constant C does not depend on j or m.

Proposition 5.2. *Let $\alpha > 1/2$ and $1 \le p \le 2$. Then there is a constant $\delta > 0$ such that*

$$\|E^\alpha_{j,m-j} f\|_{L^p(\mu_d,H)} \le C 2^{-|m|\delta} \|f\chi_{I_{m-j}}\|_{L^p(\mu_d, L_t^2(H))},$$

where the constant C does not depend on j or m. In fact, one may take $\delta = \min(\alpha - 1/2, d)$.

We shall use the vector notation $\vec{f} = \{f_j\}_j$ and $\vec{S}^\alpha_m \vec{f} = \{S^\alpha_{j,m} f_j\}_j$.

Proposition 5.3. *Let $\alpha > 1/2$. For each $1 < p \le 2$,*

$$\|\vec{S}^\alpha_m \vec{f}\|_{L^p(\mu_d, l^2(H))} \le C \|\vec{f}\chi_{I_m}\|_{L^p(\mu_d, l^2(L_t(H)))}$$

with a constant C which does not depend on m.

Proof of (5.2) given Propositions 5.1–5.3. The proof is just a minor modification of the proof given in [12], but we include it for the convenience of the reader. We show (5.2) for $\sum_m V^\alpha_{j,m-j}$ first:

$$\begin{aligned}\Big\|\Big(\sum_j\Big|\sum_m V^\alpha_{j,m-j}f_j\Big|_H^2\Big)^{1/2}\Big\|_{L^{p,2}(\mu_d)}&\\ &\le\sum_m\Big(\sum_j\|V^\alpha_{j,m-j}f_j\|^p_{L^{p,2}(\mu_d,H)}\Big)^{1/p}\\ &\lesssim\sum_m 2^{-|m|\delta(p)}\Big(\int_0^\infty\sum_j\chi_{I_{m-j}}(r)|f_j(r)|^p_{L^2_t(H)}\,d\mu_d(r)\Big)^{1/p},\end{aligned}$$

where we have used Minkowski's inequality to pull out the sum over m, and the inclusion $l^p(\mathbb{Z})\subset l^2(\mathbb{Z})$. To be more precise, [12, Lemma 2.1] was used in order to deal with the Lorentz space $L^{p,2}$. Next, the trivial bound

$$|f_j(r)|^p_{L^2_t(H)}\le\Big(\sum_j|f_j(r)|^2_{L^2_t(H)}\Big)^{p/2}$$

will finish the proof by making use of the disjointness of $\chi_{I_{m-j}}$ and the summability of $2^{-|m|\delta(p)}$. The proof is similar for $\sum_m E^\alpha_{j,m-j}$ except that we may show the stronger $L^p(\mu_d)$ bounds.

For the case $\sum_m S^\alpha_{j,m}$, we shall use the fact that I^{**}_m overlap only finitely many times to get

$$\begin{aligned}\Big(\sum_j\Big|\sum_m\chi_{I^{**}_m}S^\alpha_{j,m}f_j\Big|_H^2\Big)^{1/2}&\lesssim\Big(\sum_m\chi_{I^{**}_m}\sum_j|S^\alpha_{j,m}f_j|^2_H\Big)^{1/2}\\ &\lesssim\sum_m\chi_{I^{**}_m}\Big(\sum_j|S^\alpha_{j,m}f_j|^2_H\Big)^{1/2}.\end{aligned}$$

Thus, by Proposition 5.3,

$$\begin{aligned}\Big\|\Big(\sum_j\Big|\sum_m S^\alpha_{j,m}f_j\Big|_H^2\Big)^{1/2}\Big\|_{L^p(\mu_d)}&\lesssim\Big(\sum_m\Big\|\Big(\sum_j|S^\alpha_{j,m}f_j|^2_H\Big)^{1/2}\Big\|^p_{L^p(\mu_d)}\Big)^{1/p}\\ &\lesssim\Big(\sum_m\Big\|\chi_{I_m}\Big(\sum_j|f_j|^2_{L^2_t(H)}\Big)^{1/2}\Big\|^p_{L^p(\mu_d)}\Big)^{1/p}\\ &=\Big\|\Big(\sum_j|f_j|^2_{L^2_t(H)}\Big)^{1/2}\Big\|_{L^p(\mu_d)}.\ \blacksquare\end{aligned}$$

6. Kernel estimate. Let $\alpha>1/2$ be fixed. In what follows, we shall often suppress the dependence on α.

6.1. Estimate I. The goal of this section is to obtain an estimate for the kernel of the operator $\mathcal{R}_j^\alpha$. Note that $\mathcal{R}_j^\alpha f(r)$ can be written as

$$\int_0^\infty 2^{jd} \int_0^\infty \int_1^2 \left(1-\frac{\rho}{t}\right)_+^{\alpha-1} \chi\left(\frac{\rho}{t}\right) f_t(s)\, \frac{dt}{t}\, B_d(2^j r\rho) B_d(2^j s\rho)\, d\mu_d(\rho)\, d\mu_d(s)$$
$$:= \int_0^\infty 2^{jd} K(2^j r, 2^j s)[f(s)]\, d\mu_d(s),$$

where $K(r,s)$ is a bounded linear operator from $L_t^2(H) := L^2([1,2], dt/t, H)$ to H. For $f \in L_t^2(H)$, define the operator $\mathcal{K}$ by

$$\mathcal{K}[f](u) = \int_1^2 t\kappa(tu) f(t)\, \frac{dt}{t},$$

where $\mathcal{F}_{\mathbb{R}}\kappa(\rho) = (1-\rho)_+^{\alpha-1}\chi(\rho)$. Then $K(r,s)[f]$ can be written as

$$K(r,s)[f] = \int_0^\infty \int_1^2 \left(1-\frac{\rho}{t}\right)_+^{\alpha-1} \chi\left(\frac{\rho}{t}\right) f(t)\, \frac{dt}{t}\, B_d(r\rho) B_d(s\rho)\, d\mu_d(\rho)$$
$$= \int_0^\infty \mathcal{F}_{\mathbb{R}}[\mathcal{K}[f]](\rho) B_d(r\rho) B_d(s\rho)\, d\mu_d(\rho).$$

We shall borrow the kernel estimate from [12] for the characterization of Hankel multipliers to obtain a rather precise estimate for the kernel $K(r,s)$. We quote here a special case of [12, Proposition 3.1]. Here and in what follows, we set $\omega_N(u) = (1+|u|)^{-N}$.

Proposition A. *Let $d \geq 1$, $N > 1$, and $m \in L^2$ be compactly supported in $(0,\infty)$. Then for $\beta, \gamma = 0, 1, 2, \ldots$,*

$$\left|\partial_r^\beta \partial_s^\gamma \int_0^\infty m(\rho) B_d(\rho r) B_d(\rho s)\, d\mu_d(\rho)\right| \leq C_{N,\beta,\gamma} \sum_{\pm,\pm} \frac{|\mathcal{F}_{\mathbb{R}}^{-1}[m]| * \omega_N(\pm r \pm s)}{[(1+r)(1+s)]^{(d-1)/2}}, \tag{6.1}$$

where $C_{N,\beta,\gamma}$ does not depend on m, r, s.

It is a routine matter to verify that (6.1) continues to work for functions m taking values in a Hilbert space. Applying (6.1) with $\mathcal{F}_{\mathbb{R}}^{-1}[m] = \mathcal{K}[f]$ we immediately obtain the following.

Proposition 6.1. *Let $f \in L_t^2(H)$. Then for $N \geq 1$,*

$$|K(r,s)[f]|_H \leq C_{N,\alpha} \sum_{\pm,\pm} \frac{W[f](\pm r \pm s)}{[(1+r)(1+s)]^{(d-1)/2}}, \tag{6.2}$$

where $W[f](x) = |\mathcal{K}[f]|_H * \omega_{N+\alpha}(x)$.

6.2. Estimate II. In what follows, we shall write $f^\alpha(t) = t^{-\alpha} f(t)$. The main result of this section is the following.

PROPOSITION 6.2. *Let $f \in L^2_t(H)$. Then for all sufficiently large N,*

$$W[f](x) \lesssim \frac{1}{(1+|x|)^\alpha} |\mathcal{F}_{\mathbb{R}}^{-1}[f^\alpha]|_H * \omega_N(x) + \frac{1}{(1+|x|)^{\alpha+1}} |f|_{L^2_t(H)}. \tag{6.3}$$

In addition, there is a uniform estimate

$$W[f](x) \lesssim |f|_{L^2_t(H)}. \tag{6.4}$$

Proof. Since $W[f]$ is a convolution of $|\mathcal{K}[f]|_H$ and $\omega_{N+\alpha}$, it suffices to prove

$$|\mathcal{K}[f](u)|_H \lesssim \frac{1}{(1+|u|)^\alpha} |\mathcal{F}_{\mathbb{R}}^{-1}[f^\alpha](u)|_H + \frac{1}{(1+|u|)^{\alpha+1}} |f|_{L^2_t(H)}, \tag{6.5}$$

$$|\mathcal{K}[f](u)|_H \lesssim |f|_{L^2_t(H)}. \tag{6.6}$$

Indeed, we may obtain (6.3) from (6.5) via the inequality

$$\left| \frac{g}{(1+|\cdot|)^\alpha} \right| * \omega_{N+\alpha}(x) \lesssim \frac{|g| * \omega_N(x)}{(1+|x|)^\alpha},$$

which follows from $(1+|x-u|)^{-1} \le (1+|u|)(1+|x|)^{-1}$.

(6.6) follows from the Cauchy–Schwarz inequality since $\kappa \in L^\infty(\mathbb{R})$.

For (6.5), we need a standard asymptotic formula for κ (see [10, p. 48]):

$$\kappa(u) = \frac{\Gamma(\alpha)}{2\pi} e^{iu} (iu)^{-\alpha} + O(|u|^{-(\alpha+1)}) \quad \text{as } u \to \pm\infty. \tag{6.7}$$

Let us assume $|u| \ge C$ for some large constant C. By (6.7),

$$\begin{aligned} |\mathcal{K}[f](u)|_H &\lesssim \left| \int_1^2 e^{itu} (itu)^{-\alpha} f(t)\, dt \right|_H + \left| \int_1^2 O(|tu|^{-(\alpha+1)}) f(t)\, dt \right|_H \\ &\lesssim |u|^{-\alpha} |\mathcal{F}_{\mathbb{R}}^{-1}[f^\alpha](u)|_H + |u|^{-(\alpha+1)} |f|_{L^2_t(H)}. \end{aligned}$$

This estimate combined with (6.6) implies (6.5). ∎

7. Proof of the main propositions.

7. Proof of the main propositions. Throughout the section, we shall omit the summation notation $\sum_{\pm,\pm}$ in the kernel estimate (6.2).

7.1. Proof of Proposition 5.1. By a scaling argument, we may assume that $j = 0$. Indeed, we have $V^\alpha_{j,m-j} f(r) = V^\alpha_{0,m}[f(2^{-j}\cdot)](2^j r)$. By (6.2),

$$|V^\alpha_{0,m} f(r)|_H \lesssim \int_{I_m} \frac{\chi_{R_{m+3}}(r) W[f(s)](\pm r \pm s)}{(1+r)^{(d-1)/2}} \frac{d\mu_d(s)}{(1+s)^{(d-1)/2}}.$$

Then by the Minkowski integral inequality (for Banach function spaces), $\|V_{0,m}^{\alpha} f\|_{L^{p,2}(\mu_d,H)}$ is bounded by

$$C \int_{I_m} \left\| \frac{\chi_{R_{m+3}} W[f(s)](\pm \cdot \pm s)}{(1+\cdot)^{(d-1)/2}} \right\|_{L^{p,2}(\mu_d)} \frac{d\mu_d(s)}{(1+s)^{(d-1)/2}}. \tag{7.1}$$

The norm inside of the integral is bounded by

$$\left\| \frac{\chi_{R_{m+2}} W[f(s)](\pm \cdot)}{(1+\cdot)^{(d-1)/2}} \right\|_{L^{p,2}(\mu_d)} \lesssim \left\| \frac{W[f(s)](\cdot)}{(1+|\cdot|)^{(d-1)/2}} \right\|_{L^{p,2}(\nu_d)} \tag{7.2}$$

by a change of variable $r \to r \pm s$ and $(r \pm s) \sim r$, where ν_d is the measure on $\mathbb{R}$ defined by $d\nu_d(x) = (1+|x|)^{d-1}\, dx$.

We claim that (7.2) is bounded by $C|f(s)|_{L_t^2(H)}$, which would imply

$$\begin{aligned} \|V_{0,m}^{\alpha} f\|_{L^{p,2}(\mu_d,H)} &\lesssim \int_{I_m} |f(s)|_{L_t^2(H)} \frac{d\mu_d(s)}{(1+s)^{(d-1)/2}} \\ &\lesssim \|f\chi_{I_m}\|_{L^{p,\infty}(\mu_d, L_t^2(H))} \|\chi_{I_m}(1+\cdot)^{-(d-1)/2}\|_{L^{p',1}(\mu_d)} \end{aligned}$$

by a variant of Hölder's inequality in Lorentz spaces (see [20]). The proof is complete if we observe that

$$\|\chi_{I_m}(1+\cdot)^{-(d-1)/2}\|_{L^{p',1}(\mu_d)} \le \min(2^{md/p'}, 2^{-m(d(1/p-1/2)-1/2)}).$$

Here we have used the assumption that $1 < p < 2d/(d+1)$.

We turn to the proof of the claim. We separately estimate the main term and the error term given by Proposition 6.2. For the error term, we control the $L^{p,2}$ norm by the L^p norm to obtain

$$\Big(\int_{\mathbb{R}} (1+|x|)^{-p[\alpha+1-(d-1)(1/p-1/2)]}\, dx \Big)^{1/p} |f(s)|_{L_t^2(H)} \lesssim |f(s)|_{L_t^2(H)}.$$

For the main term, we apply Hölder's inequality and Plancherel's identity:

$$\begin{aligned} &\left\| \frac{|\mathcal{F}_{\mathbb{R}}^{-1}[f^{\alpha}(s)]|_H * \omega_N (1+|\cdot|)^{-(d-1)/2}}{(1+|\cdot|)^{\alpha}} \right\|_{L^{p,2}(\nu_d)} \\ &\qquad \lesssim \big\| |\mathcal{F}_{\mathbb{R}}^{-1}[f^{\alpha}(s)]|_H * \omega_N \big\|_{L^2(\mathbb{R})} \|(1+|\cdot|)^{-\alpha}\|_{L^{(1/p-1/2)^{-1},\infty}(\nu_d)} \\ &\qquad \lesssim \|\omega_N\|_{L^1(\mathbb{R})} \Big(\int_1^2 |f_t(s)|_H^2 t^{-2\alpha}\, dt \Big)^{1/2} \\ &\qquad \lesssim |f(s)|_{L_t^2(H)}. \end{aligned}$$

For the second inequality, $\alpha = d(1/p-1/2)$ was required.

7.2. Proof of Proposition 5.2. By scaling, we may assume that $j = 0$.

7.2.1. *The case* $m \le 0$. By (6.2) and the change of variable $s \to s \pm r$,

$$\begin{aligned}|E^\alpha_{0,m} f(r)|_H &\lesssim 2^{m(d-1)} \chi_{L_{m-2}}(r) \int_{I_m} W[f\chi_{I_m}(s)](\pm r \pm s)\, ds \\ &\lesssim 2^{m(d-1)} \chi_{L_{m-2}}(r) \int_{I_m^*} W[f\chi_{I_m}(s \pm r)](\pm s)\, ds.\end{aligned}$$

By Minkowski's inequality, $\|E^\alpha_{0,m} f\|_{L^p(\mu_d, H)}$ is bounded by

$$\begin{aligned}&C 2^{m(d-1)} 2^{m(d-1)/p} \int_{I_m^*} \Big(\int_{L_{m-2}} W[f\chi_{I_m}(s \pm r)](\pm s)^p\, dr \Big)^{1/p} ds \\ &\qquad \lesssim 2^{m(d-1)} \int_{I_m^*} \Big(\int_{I_m^{**}} W[f\chi_{I_m}(r)](\pm s)^p r^{d-1}\, dr \Big)^{1/p} ds,\end{aligned}$$

since $I_m^* \pm L_{m-2} \subset I_m^{**}$. Applying the uniform estimate (6.4), we get

$$\|E^\alpha_{0,m} f\|_{L^p(\mu_d, H)} \lesssim 2^{md} \|f\chi_{I_m}\|_{L^p(\mu_d, L^2_t(H))}.$$

7.2.2. *The case* $m \ge 0$. By (6.2) and the change of variable $s \to s \pm r$,

$$\begin{aligned}&|E^\alpha_{0,m} f(r)|_H \\ &\qquad \lesssim 2^{m(d-1)/2} \chi_{L_{m-2(r)}} (1+r)^{-(d-1)/2} \int_{I_m} W[f\chi_{I_m}(s \pm r)](\pm s)\, ds.\end{aligned}$$

Then we take the $L^p(\mu_d)$ norm, and next apply Minkowski's inequality. With the use of $r \le 2^m$ and (6.3), $\|E^\alpha_{0,m} f\|_{L^p(\mu_d, H)}$ is bounded by

$$\begin{aligned}&C 2^{m(d-1)/p} \int_{I_m^*} \Big(\int_{L_{m-2}} W[f\chi_{I_m}(s \pm r)](\pm s)^p\, dr \Big)^{1/p} ds \\ &\qquad \lesssim \int_{I_m^*} \Big(\int_{I_m^{**}} W[f\chi_{I_m}(r)](\pm s)^p r^{d-1}\, dr \Big)^{1/p} ds \lesssim \mathrm{I} + \mathrm{II},\end{aligned} \tag{7.3}$$

where

$$\mathrm{I} = \int_{I_m^*} \Big(\int_{I_m^{**}} [|\mathcal{F}^{-1}_{\mathbb{R}}[[f\chi_{I_m}(r)]^\alpha]|_H * \omega_N(\pm s)]^p r^{d-1}\, dr \Big)^{1/p} \frac{ds}{(1+|s|)^\alpha}$$

$$\mathrm{II} = \int_{I_m^*} \Big(\int_{I_m^{**}} |f\chi_{I_m}(r)|^p_{L^2_t(H)} r^{d-1}\, dr \Big)^{1/p} \frac{ds}{(1+|s|)^{\alpha+1}},$$

II is the error term. Observe that

$$\mathrm{II} \lesssim 2^{-m\alpha} \|f\chi_{I_m}\|_{L^p(\mu_d, L^2_t(H))}$$

which has the desired decay term $2^{-m\alpha}$.

The estimate for the main term I uses the assumption $\alpha > 1/2$. We first apply the Cauchy–Schwarz inequality for the s-integral. Then I is bounded by a constant times

$$(7.4)\quad 2^{-m(\alpha-1/2)}\Big(\int_{I_m^*}\Big(\int_{I_m^{**}}[|\mathcal{F}_{\mathbb{R}}^{-1}[[f\chi_{I_m}(r)]^\alpha]|_H * \omega_N(\pm s)]^p r^{d-1}\,dr\Big)^{2/p} ds\Big)^{1/2}.$$

By Minkowski's inequality, Young's inequality, and Plancherel's identity, (7.4) is bounded by

$$2^{-m(\alpha-1/2)}\Big(\int_0^\infty \|\mathcal{F}_{\mathbb{R}}^{-1}[[f\chi_{I_m}(r)]^\alpha]\|_{L^2(\mathbb{R},H)}^p r^{d-1}\,dr\Big)^{1/p} \lesssim 2^{-m(\alpha-1/2)}\|f\chi_{I_m}\|_{L^p(\mu_d,L_t^2(H))}.$$

7.3. Proof of Proposition 5.3. We may assume that the function $\vec{f}$ is supported on I_m. To show that the estimate holds for $p=2$, it is enough to show that

$$\|T_{j,m}^\alpha f_j\|_{L^2(\mu_d,H)} \lesssim \|f_j\|_{L^2(\mu_d,L_t(H))},$$

uniformly in j. But this easily follows from Plancherel's identity and the Cauchy–Schwarz inequality since $\|\mathcal{H}_d[R_{2^jt}^\alpha]\|_{L^\infty} \le 1$.

Thus it suffices to prove a weak type inequality for $p=1$, namely

$$(7.5)\qquad \mu_d(\{r \in I_m^{**} : |\vec{S}_m^\alpha \vec{f}|_{l^2(H)} > \lambda\}) \le \frac{C}{\lambda}\|\vec{f}\|_{L^1(\mu_d,B)}$$

for $\lambda > 0$ and $B = l^2(L_t(H))$, by a vector-valued version of the Marcinkiewicz interpolation theorem.

We follow the usual strategy for proving weak type inequalities. We apply the Calderón–Zygmund decomposition, Proposition 7.1, to get $\vec{f} = \vec{g} + \vec{b} = \vec{g} + \sum_\nu \vec{b}_\nu$, where $\vec{b}_\nu = \{b_{\nu,j}\}_j$ is supported on $J_\nu \subset I_m$ and has cancellation. Let us denote by J_ν^* the interval with the same center as J_ν and twice its length and by Ω the union of the J_ν^*.

Then (7.5) for $\vec{g}$ can be shown as usual by applying the $L^2(\mu_d, B)$ boundedness of $\vec{S}_m^\alpha$. In addition, (7.5) for $\vec{b}$ reduces to

$$(7.6)\qquad \mu_d(\{r \in I_m^{**} \setminus \Omega : |\vec{S}_m^\alpha \vec{b}|_{l^2(H)} > \lambda\}) \le \frac{C}{\lambda}\|\vec{f}\|_{L^1(\mu_d,B)}.$$

The left hand side of (7.6) is bounded by

$$(7.7)\quad \lambda^{-1}\int_{I_m^{**}\setminus\Omega} |\vec{S}_m^\alpha \vec{b}(r)|_{l^1(H)}\,d\mu_d(r) \le \lambda^{-1}\sum_\nu\sum_j \int_{I_m^{**}\setminus J_\nu^*} |S_{j,m}^\alpha b_{\nu,j}(r)|_H\,d\mu_d(r).$$

Let us denote the integral on the right hand side of (7.7) by $\mathcal{I}_{j,\nu}$. We claim that there is $\epsilon > 0$ such that

$$\mathcal{I}_{j,\nu} \lesssim \min(2^j|J_\nu|, [2^j|J_\nu|]^{-\epsilon})\|b_{\nu,j}\|_{L^1(\mu_d, L^2_t(H))}. \tag{7.8}$$

Then (7.8) implies

$$\begin{aligned}\sum_j \mathcal{I}_{j,\nu} &\lesssim \sum_j \min(2^j|J_\nu|, [2^j|J_\nu|]^{-\epsilon})\|\vec{b}_\nu\|_{L^1(\mu_d,B)} \\ &\lesssim \|\vec{b}_\nu\|_{L^1(\mu_d,B)}.\end{aligned}$$

Then (7.7) is bounded by

$$\begin{aligned}C\lambda^{-1}\sum_\nu \|\vec{b}_\nu\|_{L^1(\mu_d,B)} &\lesssim \sum_\nu \mu_d(J_\nu) \\ &\lesssim \lambda^{-1}\|\vec{f}\|_{L^1(\mu_d,B)},\end{aligned}$$

as desired by Proposition 7.1.

Proof of the claim (7.8). By the kernel estimate (6.2),

$$\begin{aligned}|S^\alpha_{j,m} b_{\nu,j}(r)|_H &\lesssim \chi_{I^{**}_m}(r)\int_{J_\nu} \frac{2^{jd} W[b_{\nu,j}(s)](2^j(\pm r \pm s))\, d\mu_d(s)}{[(1+2^j r)(1+2^j s)]^{(d-1)/2}} \\ &\lesssim \frac{\chi_{I^{**}_m}(r)}{2^{m(d-1)}}\int_{J_\nu} 2^j W[b_{\nu,j}(s)](2^j(\pm r \pm s))\, d\mu(s)\end{aligned}$$

using $r \sim s \sim 2^m$. Then

$$\begin{aligned}\mathcal{I}_{j,\nu} &\lesssim \int_{J_\nu}\int_{I^{**}_m\setminus J^*_\nu} 2^j W[b_{\nu,j}(s)](2^j(\pm r \pm s))\, dr\, d\mu_d(s) \\ &\lesssim \int_{J_\nu}\int_{|x|\ge |J_\nu|/2} 2^j W[b_{\nu,j}(s)](2^j x)\, dx\, d\mu_d(s) \\ &\lesssim \int_{J_\nu}\int_{|x|\ge 2^{j-1}|J_\nu|} W[b_{\nu,j}(s)](x)\, dx\, d\mu_d(s) \\ &\lesssim \int_{J_\nu}\int_{|x|\ge 2^{j-1}|J_\nu|} \left(\frac{|\mathcal{F}^{-1}_{\mathbb{R}}[b^\alpha_{\nu,j}(s)]|_H * \omega_N(x)}{(1+|x|)^\alpha} + \frac{|b_{\nu,j}(s)|_{L^2_t(H)}}{(1+|x|)^{\alpha+1}}\right) dx\, d\mu_d(s).\end{aligned} \tag{7.9}$$

For the second inequality, we have used the fact that

$$|\pm r \pm s| \ge |r - s| \ge |J_\nu|/2$$

whenever $r \in I^{**}_m \setminus J^*_\nu$ and $s \in J_\nu$. For the last inequality, we have used Proposition 6.2.

Choose $\epsilon = (\alpha - 1/2)/2 > 0$. Then the main term of (7.9) is bounded by

$$
\begin{aligned}
(7.10)\qquad & C(2^j|J_\nu|)^{-\epsilon} \int_{J_\nu}\int_{\mathbb{R}} \frac{|\mathcal{F}_{\mathbb{R}}^{-1}[b^\alpha_{\nu,j}(s)]|_H * \omega_N(x)}{(1+|x|)^{\alpha-\epsilon}}\,dx\,d\mu_d(s)\\
&\lesssim (2^j|J_\nu|)^{-\epsilon} \int_{J_\nu} \big\||\mathcal{F}_{\mathbb{R}}^{-1}[b^\alpha_{\nu,j}(s)]|_H * \omega_N\big\|_{L^2(\mathbb{R})}\,d\mu_d(s)\\
&\lesssim (2^j|J_\nu|)^{-\epsilon} \int_{J_\nu} \|b_{\nu,j}(s)\|_{L^2_t(H)}\,d\mu_d(s)\\
&= (2^j|J_\nu|)^{-\epsilon}\|b_{\nu,j}\|_{L^1(\mu_d,L^2_t(H))},
\end{aligned}
$$

where we have applied the Cauchy–Schwarz inequality, Young's inequality, and Plancherel's identity. The estimation of the error term of (7.9) is straightforward.

Next we seek an inequality which is good when $2^j|J_\nu|$ is small. Let s_ν be the center of the interval J_ν. Using the cancellation of $b_{\nu,j}$, we may write

$$
S^\alpha_{j,m} b_{\nu,j}(r) = \chi_{I^{**}_m}(r)\int_{J_\nu} 2^{jd}[K(2^jr,2^js)-K(2^jr,2^js_\nu)][b_{\nu,j}(s)]\,d\mu_d(s).
$$

Let f be an $L^2_t(H)$-valued function. Then $[K(2^jr,2^js)-K(2^jr,2^js_\nu)][f]$ can be written as

$$
\begin{aligned}
&\int \mathcal{F}_{\mathbb{R}}[\mathcal{K}[f]](\rho)B_d(2^jr\rho)[B_d(2^js\rho)-B_d(2^js_\nu\rho)]\,d\mu_d(\rho)\\
&\qquad = 2^j(s-s_\nu)\int_0^1\Big[\int \mathcal{F}_{\mathbb{R}}[\mathcal{K}[f]](\rho)B_d(2^jr\rho)\rho B_d'(2^js(\tau)\rho)\,d\mu_d(\rho)\Big]\,d\tau,
\end{aligned}
$$

where $s(\tau) = s_\nu + \tau(s-s_\nu)$.

We apply (6.1) (with $\beta = 0$ and $\gamma = 1$) to the inner integral above. Since $|s-s_\nu| \le |J_\nu|$ and $r \sim s(\tau) \sim 2^m$ for $\tau \in [0,1]$, we may bound $2^{jd}|[K(2^jr,2^js)-K(2^jr,2^js_\nu)][f]|_H$ by a constant times

$$
\begin{aligned}
&2^j|J_\nu|\int_0^1 \frac{2^{jd}W[f](2^j(\pm r\pm s(\tau)))}{[(1+2^jr)(1+2^js(\tau))]^{(d-1)/2}}\,d\tau\\
&\qquad\qquad \lesssim 2^j|J_\nu|\int_0^1 \frac{2^{jd}W[f](2^j(\pm r\pm s(\tau)))}{2^{(m+j)(d-1)}}\,d\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(7.11)\qquad \mathcal{I}_{j,\nu} &= \int_{I^{**}_m\setminus J^*_\nu} |S^\alpha_{j,m} b_{\nu,j}(r)|_H\,d\mu_d(r)\\
&\lesssim 2^j|J_\nu|\int_0^1\int_{J_\nu}\int_{I^{**}_m\setminus J^*_\nu} 2^jW[b_{\nu,j}(s)](2^j(\pm r\pm s(\tau)))\,dr\,d\mu_d(s)\,d\tau
\end{aligned}
$$

$$\lesssim 2^j |J_\nu| \int_{J_\nu} \int_{\mathbb{R}} 2^j W[b_{\nu,j}(s)](2^j x)\, dx\, d\mu_d(s)$$
$$\lesssim 2^j |J_\nu|\, \|b_{\nu,j}\|_{L^1(\mu_d, L_t^2(H))},$$

where we have used the fact that $2^{-m(d-1)} r^{d-1} \lesssim 1$ if $r \in I_m^{**}$ in the second line. The last inequality follows by the change of variable $x \to 2^{-j}x$ and arguing as in (7.10) except that we do not need the decay $(2^j|J_\nu|)^{-\epsilon}$. ∎

Appendix: A note on g-function. In this section, we shall give a proof of L^p bounds of the square function g_Φ by Calderón–Zygmund theory in the vector-valued setting. We shall obtain an analogue of the gradient condition for the Hankel convolution operator $T^K f := K *_d f$. The material to be discussed is quite standard and well-known, but Lemma 7.3 does not seem to appear in the literature.

Calderón–Zygmund decomposition. Let B be a Banach space, and let $L^p(\mu_d, B)$ be the Bochner space, i.e.

$$\|f\|_{L^p(\mu_d,B)}^p = \int_0^\infty |f(r)|_B^p\, d\mu_d(r)$$

for strongly measurable functions $f : \mathbb{R}_+ \to B$. Then there is a Calderón–Zygmund decomposition for functions $f \in L^1(\mu_d, B)$:

PROPOSITION 7.1. *Let $f \in L^1(\mu_d, B)$ and $\lambda > 0$. Then there are dyadic intervals J_ν with disjoint interiors and a decomposition*

$$f = g + b = g + \sum_\nu b_\nu$$

such that:

(i) $|g(s)|_B \le C\lambda$ *s-a.e. and* $\|g\|_{L^1(\mu_d,B)} \le \|f\|_{L^1(\mu_d,B)}$.
(ii) b_ν *is supported on* J_ν *and* $\int b_\nu(s)\, d\mu_d(s) = 0$.
(iii) $\int |b_\nu(s)|_B\, d\mu_d(s) \lesssim \lambda \mu_d(J_\nu)$.
(iv) $\sum_\nu \mu_d(J_\nu) \lesssim \lambda^{-1} \|f\|_{L^1(\mu_d,B)}$.

A proof may be found in [12], but we shall give a sketch. Split $f = \sum_j f_j$ where $f_j = f\chi_{I_j}$ and $I_j = [2^j, 2^{j+1})$. Define $F_j(r)$ by the equation

$$2^{j(d-1)} F_j(r) = f_j(r) r^{d-1} \tag{7.12}$$

and perform the usual Calderón–Zygmund decomposition $F_j = G_j + B_j$ for the B-valued function F_j. Then we obtain $f_j = g_j + b_j$, where g_j and b_j are given by equations similar to (7.12), and then sum in j.

The square function g_Φ. Consider the Hankel convolution operator

$$T^K f(r) = K *_d f(r) = \int_0^\infty \tau^s K(r) f(s)\, d\mu_d(s),$$

where τ^s is the generalized translation given by $\mathcal{H}_d[\tau^s f](\rho) = B_d(s\rho)\mathcal{H}_d f(\rho)$. See (7.16) and (7.17) for an explicit formula for τ^s.

As in the Euclidean case (see e.g. [21]), one may extend this to the vector-valued setting. Let A and B Banach spaces, and $K : \mathbb{R}_+ \to \mathcal{L}(A,B)$ be an operator-valued kernel, where $\mathcal{L}(A,B)$ is the space of bounded linear operators from A to B.

Lemma 7.2 (Calderón–Zygmund). *Suppose that T^K is a bounded operator from $L^r(\mu_d, A)$ to $L^r(\mu_d, B)$ for some r, $1 \le r \le \infty$. In addition, suppose that*

$$\int\limits_{|r-s|>2|s-\bar{s}|} |\tau^s K(r) - \tau^{\bar{s}} K(r)|_{\mathcal{L}(A,B)}\, d\mu_d(r) \le C. \tag{7.13}$$

Then T^K is bounded from $L^p(\mu_d, A)$ to $L^p(\mu_d, B)$ for $1 < p < \infty$, and there is a weak-type inequality

$$\mu_d(\{r \in \mathbb{R}_+ : |T^K f(r)|_B > \lambda\}) \le \frac{C}{\lambda}\|f\|_{L^1(\mu_d, A)}.$$

This can be shown by using Proposition 7.1, or by the general theory of spaces of homogeneous type. We provide here a condition on K which implies (7.13) and is easier to verify.

Lemma 7.3. *The condition*

$$|K'(r)|_{\mathcal{L}(A,B)} \le C r^{-(d+1)} \tag{7.14}$$

implies (7.13).

Before we turn to the proof of this fact, we give an application for Littlewood–Paley square functions. Note that the g-function defined in Section 2 can be regarded as a vector-valued convolution operator

$$g_\Phi f(r) = \mathcal{H}_d[\Phi(\cdot/t)] *_d f(r),$$

where we regard $\mathcal{H}_d[\Phi(\cdot/t)]$ as an operator-valued kernel taking values in $\mathcal{L}(H, \tilde{H})$ for H-valued functions f, where $\tilde{H} = L^2(\mathbb{R}_+, dt/t, H)$.

Theorem 7.4. *Let $\Phi \in \mathcal{S}(\mathbb{R}_+)$ with $\Phi(0) = 0$, and $\tilde{H}$ and g_Φ be as described above. Then for $1 < p < \infty$,*

$$C_p^{-1}\|f\|_{L^p(\mu_d, H)} \le \|g_\Phi f\|_{L^p(\mu_d, \tilde{H})} \le C_p\|f\|_{L^p(\mu_d, H)}.$$

Proof. We prove the second inequality. The first follows from the second via the polarization identity. We may assume that $H = l^2$ and $f = \{f_j\}_j$, since, for instance, we may write an H-valued function f as the sum $f(x) = \sum_j f_j(x) e_j$ for an orthonormal basis $\{e_j\}_j$, then using Parseval's identity.

First we consider the case $p = 2$. By Plancherel's identity,

$$\|g(f)\|^2_{L^2(\mu_d,\tilde{H})} = \int_0^\infty \sum_j \|\mathcal{H}_d[\Phi(\cdot/t)] *_d f_j\|^2_{L^2(\mu_d)} \frac{dt}{t}$$
$$= \Big(\int_0^\infty |\Phi(t)|^2 \frac{dt}{t}\Big) \sum_j \|f_j\|^2_{L^2(\mu_d)} = C\|f\|^2_{L^2(\mu_d,H)}.$$

Next we verify (7.14). Let $K = \mathcal{H}_d[\Phi(\cdot/t)] = t^d\mathcal{H}_d[\Phi](t\,\cdot)$. As $\Phi \in \mathcal{S}(\mathbb{R}_+)$, we have

$$|K'(r)|_{\mathcal{L}(H,\tilde{H})} = |K'(r)|_{L^2(\mathbb{R}_+,dt/t)}$$
$$= \Big(\int_0^\infty |t^{d+1}\mathcal{H}_d[\Phi]'(tr)|^2 \frac{dt}{t}\Big)^{1/2} = \frac{C}{r^{d+1}}. \blacksquare$$

Remark. Choose a cut-off function $\eta \in C_0^\infty(\mathbb{R}^+)$ supported on $(1/8, 8)$ such that $\eta(\rho) = 1$ on $[1/4, 4]$ and define the Littlewood–Paley projection L_j by $\mathcal{H}_d[L_jf](\rho) = \eta(2^{-j}\rho)\mathcal{H}_df(\rho)$. Consider the $l^2(H)$-valued operator $g(f) = \{L_jf\}$. Then by using the above argument, one can verify that

$$\|g(f)\|_{L^p(\mu_d,l^2(H))} \le C_p\|f\|_{L^p(\mu_d,H)} \tag{7.15}$$

for $1 < p < \infty$. Moreover, by real interpolation, one can replace L^p by the Lorentz spaces $L^{p,q}$ for $1 \le q \le \infty$.

For the proof of Lemma 7.3, we need explicit formulae for the generalized translation. In what follows, we shall ignore multiplicative constants, and write $A = B$ if $A = CB$ for a constant C depending only on d. One has

$$\tau^s f(r) = \int_0^\pi f((r,s)_\theta)\, d\nu(\theta), \tag{7.16}$$

where $(r,s)_\theta = (r^2 + s^2 - 2rs\cos\theta)^{1/2}$ and $d\nu(\theta)$ is a probability measure on $[0,\pi]$. One may also write

$$\tau^s f(r) = \int_{|r-s|}^{r+s} f(t) dW_{r,s}(t), \tag{7.17}$$

where $dW_{r,s}(t)$ is a probability measure on $[|r-s|, r+s]$ (see [15]).

Proof of Lemma 7.3. This observation is a combination of estimates from [15], where an analogue of the Hörmander–Mikhlin multiplier theorem for Hankel multipliers is proved. We shall denote by $|\cdot|$ the operator norm $|\cdot|_{\mathcal{L}(A,B)}$. By (7.16),

$$\begin{aligned}\int\limits_{|r-s|\ge 2|s-\bar{s}|} &|\tau^s K(r) - \tau^{\bar{s}} K(r)|\, d\mu_d(r)\\ &= \int\limits_{|r-s|\ge 2|s-\bar{s}|} \Big|\int\limits_0^\pi K((r,s)_\theta) - K((r,\bar{s})_\theta)\, d\nu(\theta)\Big|\, d\mu_d(r)\\ &= \int\limits_{|r-s|\ge 2|s-\bar{s}|} \Big|\int\limits_0^\pi\int\limits_0^1 \frac{d}{dt}[K((r, ts+(1-t)\bar{s})_\theta)]\, dt\, d\nu(\theta)\Big|\, d\mu_d(r).\end{aligned}$$

Let $\Psi(t) = (r, ts + (1-t)\bar{s})_\theta$. Then $|\Psi'(t)| \le |s-\bar{s}|$ (see [15, eq. (2.9)]). Therefore, the last integral is bounded by

$$\begin{aligned}|s-\bar{s}|\int\limits_0^1 \int\limits_{|r-s|\ge 2|s-\bar{s}|} \int\limits_0^\pi |K'|((r, ts+(1-t)\bar{s})_\theta)\, d\nu(\theta)\, d\mu_d(r)\, dt&\\ \le |s-\bar{s}|\int\limits_0^1\int\limits_0^\infty \tau^{ts+(1-t)\bar{s}}|K'|(r)\chi(r)\, d\mu_d(r)\, dt&,\end{aligned}\tag{7.18}$$

where $\chi(r)$ is the characteristic function of the set $\{r : |r-s| \ge 2|s-\bar{s}|\}$.

Next, we use the identity $\int \tau^s f(r) g(r)\, d\mu_d(r) = \int f(r)\tau^s g(r)\, d\mu_d(r)$ and then analyse $\tau^{ts+(1-t)\bar{s}}\chi(r)$. It follows by considering (7.17) that

$$\tau^{ts+(1-t)\bar{s}}\chi(r) \le \chi_{[|s-\bar{s}|,\infty)}(r)$$

for any $t \in [0,1]$, as was observed in [15, eq. (3.9) and (3.10)]. Thus, (7.14) implies that (7.18) is bounded by

$$|s-\bar{s}| \int\limits_{|s-\bar{s}|}^\infty |K'(r)|\, d\mu_d(r) \lesssim |s-\bar{s}| \int\limits_{|s-\bar{s}|}^\infty r^{-(d+1)} r^{d-1}\, dr \le C. \ \blacksquare$$

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