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D sets and IP rich sets in $\mathbb{Z}$

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Abstract. We give combinatorial characterizations of IP rich sets (IP sets that remain IP upon removal of any set of zero upper Banach density) and D sets (members of idempotent ultrafilters, all of whose members have positive upper Banach density) in $\mathbb{Z}$. We then show that the family of IP rich sets strictly contains the family of D sets.

1. Introduction. In this paper we will be dealing with the space of ultrafilters on $\mathbb{Z}$, endowed with its usual algebraic structure and topology. A standard background reference is [HS].

A *filter* on $\mathbb{Z}$ is a non-empty set p of subsets of $\mathbb{Z}$ that is closed under finite intersections and supersets and does not contain $\emptyset$. An *ultrafilter* is a maximal filter, that is, a filter not properly contained in another filter. We denote the set of ultrafilters on $\mathbb{Z}$ by $\beta\mathbb{Z}$, and endow $\beta\mathbb{Z}$ with the topology generated by the (closed, as well as open) sets $\hat{A} = \{p \in \beta\mathbb{Z} : A \in p\}$. With this topology, $\beta\mathbb{Z}$ becomes a compact Hausdorff space.

Identifying $z \in \mathbb{Z}$ with the *principal* ultrafilter $e(z) = \{A \subset \mathbb{Z} : z \in A\}$, $\beta\mathbb{Z}$ becomes a representation of the Stone–Čech compactification of $\mathbb{Z}$. Now there is a unique associative extension to $\beta\mathbb{Z}$ of the operation $+$ on $\mathbb{Z}$ having the property that for every $q \in \beta\mathbb{Z}$ the function $p \mapsto p+q$ is continuous (thus making $(\beta\mathbb{Z}, +)$ a *compact right topological semigroup*). There are several ways to describe this extension; we will content ourselves with the classical one, i.e.

$$A \in p + q \iff \{x \in \mathbb{Z} : A - x \in q\} \in p.$$

According to a theorem of Ellis [E], any compact right topological semigroup has idempotents. It is easy to see that if $p \in \beta\mathbb{Z}$ is idempotent (that is, if $p = p+p$) then any member of p is an *IP set*, that is, a set that contains

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the set of finite sums of some sequence:

$$\mathrm{FS}(\langle x_i\rangle_{i=1}^{\infty}) = \{x_{i_1} + \cdots + x_{i_k} : i_1 < \cdots < i_k\}.$$

Conversely, any IP set is a member of some idempotent ultrafilter. A set $A \subset \mathbb{Z}$ is said to be *IP** if it belongs to *every* idempotent ultrafilter p. (Equivalently, A^c fails to be IP.) Note that as $\{0\}$ is an IP set, every IP* set contains $\{0\}$. Some authors require that IP sets be infinite. We shall call infinite IP sets *non-trivial*.

Recall that the *upper Banach density* of a set $A \subset \mathbb{Z}$ is defined as

$$d^*(A) = \limsup_{N-M\to\infty} \frac{|A \cap \{M, M+1, \ldots, N-1\}|}{N-M}.$$

We will be concerned here with two density-related strengthenings of the IP set notion. The first, that of *D set*, was introduced in [BD].

Definition 1.1. An ultrafilter $p \in \beta\mathbb{Z}$ having the property that $d^*(A) > 0$ for every $A \in p$ is said to be *essential*. If p is an essential idempotent and $A \in p$, we say that A is a *D set*. If B^c is not a D set (equivalently, if B belongs to every essential idempotent), we say that B is a *D^* set*.

The second notion, that of *IP rich set*, was recently developed by V. Bergelson and A. Leibman, who have proved (unpublished) that certain return times intersect all such sets.

Definition 1.2. A set $A \subset \mathbb{Z}$ is *IP r*ich, or an *AIP set*, if $A \setminus E$ is an IP set for every $E \subset \mathbb{Z}$ with $d^*(E) = 0$. If $B^c \subset \mathbb{Z}$ is not IP rich (equivalently, if $B \cup E$ is IP* for some E with $d^*(E) = 0$) we say that B is *AIP**.

As *AIP** is supposed to stand for *almost IP**, we prefer *IP rich* to *AIP*.

IP* sets are AIP* and D*. Since there are zero density IP sets, not every IP set is a D set, from which it follows that not every D* set is IP*. It is also clear that not every AIP* set is IP*, and routine that every AIP* set is D*. (If B is AIP* then $B \cup E$ is a member of every idempotent for some zero density E. As E does not belong to any essential idempotent, B must belong to all of them.)

V. Bergelson (personal communication) asked whether every D* set is AIP*. In this paper we give a negative answer to this question. That is, we prove

Theorem 1.3. *There are D^* subsets of $\mathbb{Z}$ that are not AIP*.*

This yields the following proper containments:

$$\mathrm{IP}^* \subsetneq \mathrm{AIP}^* \subsetneq \mathrm{D}^*, \quad \text{or} \quad \mathrm{IP} \supsetneq \mathrm{AIP} \supsetneq \mathrm{D}.$$

The proof, carried out in Section 3, proceeds via construction of an IP rich set that is not a D set. Workable characterizations of D sets and IP rich sets, which are of independent interest, are given in Section 2. Our equivalent condition for IP richness, which we call *FS tree richness*, already appears in the literature. In [HS, Theorem 20.17] it is shown to be a necessary property of D sets (making its non-sufficiency potentially interesting to a different crowd), while in [T] it is proved by elementary means to be a partition regular property. Our equivalent condition for D sets, meanwhile, is inspired by and comparable to a combinatorial characterization of *central sets* given in [HMS].

2. Tree structure characterizations of IP rich sets and D sets. In this section we give characterizations of D sets and IP rich sets. These are modeled on an elementary characterization of so-called *central sets* by Hindman, Maleki and Strauss [HMS]. We begin with several definitions.

DEFINITION 2.1. Let Ω be the set of finite sequences of integers, including the empty sequence.

Sometimes, we will want to include zero in our finite sums sets.

DEFINITION 2.2. $\mathrm{FS}_0(\langle x_i\rangle) = \mathrm{FS}(\langle x_i\rangle) \cup \{0\}$.

The following definition will be instrumental in the inductive process whereby we construct IP rich sets.

DEFINITION 2.3. If $A \subset \mathbb{Z}$ and $f = (x_1, \ldots, x_k) \in \Omega$, we say that A is *IP rich over* f if for every $E \subset \mathbb{Z} \setminus \mathrm{FS}(\langle x_1, \ldots, x_k\rangle)$ with $d^*(E) = 0$ there exist non-zero $x_{k+1}, x_{k+2}, \ldots \in \mathbb{Z}$ such that $\mathrm{FS}(\langle x_i\rangle_{i=1}^{\infty}) \subset A \setminus E$.

Here is a related notion. Recall that a non-trivial IP set is just an infinite IP set.

DEFINITION 2.4. Let $F \subset \mathbb{Z}$ be a finite set. An *IP*$_F$ *set* is a set $R + F$, where R is a non-trivial IP set. A set $J \subset \mathbb{Z}$ is *IP*$^*_F$ if J intersects every IP_F set non-trivially.

The following lemma is a generalization of the fact that for any IP^* set B and any $n \in \mathbb{Z}$, the set $n\mathbb{Z} \cap B$ is again IP^*.

LEMMA 2.5. *Let $F \subset \mathbb{Z}$ be a finite set. If $J \subset \mathbb{Z}$ is an IP$^*_F$ set and $n \in \mathbb{N}$ then $(n\mathbb{Z} + F) \cap J$ is IP$^*_F$ as well.*

Proof. For every non-trivial IP set R,

$$R + F \not\subset J^c \Rightarrow R \not\subset \bigcap_{f\in F} (J^c - f).$$

Therefore

$$\{0\} \cup \bigcup_{f\in F} (J - f) = \{0\} \cup \Big(\bigcap_{f\in F} (J^c - f)\Big)^c$$

is IP^*, which implies that

$$\{0\} \cup \Big(n\mathbb{Z} \cap \bigcup_{f\in F} (J - f)\Big)$$

is IP*. So, for every non-trivial IP set R there exist $r \in R$, $f \in F$, $z \in \mathbb{Z}$ and $j \in J$ such that $r = nz = j - f$, so that $r + f = nz + f = j$, whence $(R + F) \cap ((n\mathbb{Z} + F) \cap J) \neq \emptyset$. ∎

We now move to our characterization of IP rich sets.

DEFINITION 2.6. A set $A \subset \mathbb{Z}$ is *FS tree rich* if there is a subset $T \subset \Omega$ having the following properties:

I1. $() \in T$.

I2. If $f \in T$ then $d^*(B_f) > 0$, where

$$B_{(x_1,\ldots,x_k)} = \{x \in \mathbb{Z} : (x_1, \ldots, x_k, x) \in T\}.$$

I3. If $(x_1, \ldots, x_k) \in T$ then $\mathrm{FS}(\langle x_1, \ldots, x_k\rangle) \subset A$.

As mentioned in the introduction, Hindman and Strauss have shown (see [HS, Theorem 20.17]) that FS tree richness is necessary for D sets. We establish now that FS tree richness is necessary (and sufficient) for IP richness.

THEOREM 2.7. *Let $A \subset \mathbb{Z}$. Then A is IP rich if and only if it is FS tree rich.*

Proof. We start with

CLAIM. *If A is IP rich over $(x_1, \ldots, x_k)$ then*

$$B = \{x \in \mathbb{Z} \setminus \mathrm{FS}(\langle x_1, \ldots, x_k\rangle) : A \textit{ is IP rich over } (x_1, \ldots, x_k, x)\} \tag{2.1}$$

has positive upper Banach density.

Suppose the Claim is false. Pick recalcitrant $(x_1, \ldots, x_k)$ and let

$$F = \mathrm{FS}_0(\langle x_1, \ldots, x_k\rangle).$$

We will construct a set

$$E \subset \mathbb{Z} \setminus \mathrm{FS}(\langle x_1, \ldots, x_k\rangle)$$

with $d^*(E) = 0$ such that $A^c \cup E$ is IP^*_F, which will yield a contradiction.

Let

$$K = \{x \in \mathbb{Z} \setminus \mathrm{FS}(\langle x_1, \ldots, x_k\rangle) : \mathrm{FS}(\langle x_1, \ldots, x_k, x\rangle) \subset A\}.$$

Let $\prec$ be a well-order on $\mathbb{Z}$. We will construct sequences $(k_x)_{x\in K\setminus B}$ and $(E'_x)_{x\in K\setminus B}$ (of numbers tending to ∞ and sets, respectively) satisfying the following:

(a) For every $x \in K \setminus B$ and every interval I with $|I| \geq k_x$,

$$\Big| I \cap \bigcup_{y\in K\setminus B,\, y\prec x} E'_y \Big| \leq \frac{|I|}{|x| + 1}.$$

(b) For every $x \in K \setminus B$, $d^*(E'_x) = 0$.

(c) For every $x \in K \setminus B$, $E'_x \subset k_x\mathbb{Z}$.

(d) If $x, y \in K \setminus B$ with $y \prec x$ then $k_y \mid k_x$.

(e) For every $x \in K \setminus B$,
$$\bigcap_{y \in F_x} ((A \setminus E'_x) - y) \setminus \{0\}$$
is not IP, where $F_x = \mathrm{FS}_0(\langle x_1, \dots, x_k, x\rangle)$.

(f) For every $x \in K \setminus B$, $E'_x \subset \mathbb{Z} \setminus \mathrm{FS}(\langle x_1, \dots, x_k\rangle)$.

Supposing that this construction has been carried out, let
$$E = B \cup \bigcup_{x \in K \setminus B} E'_x.$$
By (2.1) and (f), $E \subset \mathbb{Z} \setminus \mathrm{FS}(\langle x_1, \dots, x_k\rangle)$. Also $d^*(E) = 0$. To see this, note that if I_x are intervals with $|I_x| = k_x$ then by (c) and (d) at most one member of $\bigcup_{y \in K \setminus B,\, y \not\prec x} E'_y$ can belong to I_x, whereas by (a) and the fact that $d^*(B) = 0$ one has
$$\Big| I_x \cap \bigcup_{y \in K \setminus B,\, y \prec x} E'_y \Big| + |I_x \cap B| = |I_x| o(1).$$
Moreover $A^c \cup E$ is IP^*_F as desired. For if its complement $A \setminus E$ contains $R + F$ for some (non-trivial) IP set R then picking $x \in R$ and an IP set R' not having 0 as a member such that $R' + \{0, x\} \subset R$ one will have
$$R' \subset \bigcap_{y \in F_x} ((A \setminus E) - y) \setminus \{0\}.$$
The latter set is therefore IP, but $x \in K \setminus B$ ($x \in K$ by definition and $x \notin E \supset B$) and $E'_x \subset E$, so by (e) it is *not* IP.

It remains to show that one can carry out the construction. Suppose $x \in K \setminus B$ and k_y, E'_y have been determined for all $y \in K \setminus B$ with $y \prec x$. Since $x \notin B$, A is not IP rich over $(x_1, \dots, x_k, x)$, so there is a set
$$E_x \subset \mathbb{Z} \setminus \mathrm{FS}(\langle x_1, \dots, x_k, x\rangle)$$
with $d^*(E_x) = 0$ such that $A \setminus E_x$ contains no set of the form $\mathrm{FS}(\langle x_i\rangle_{i=1}^{\infty})$ with $x_{k+1} = x$ and x_i non-zero for $i \geq k+2$. In particular,
$$\bigcap_{y \in F_x} ((A \setminus E_x) - y) \setminus \{0\}$$
is not IP.

It is clear that we may choose k_x in conformity with (a) and (d). Now set
$$E'_x = \Big(k_x \mathbb{Z} \cap \bigcup_{y \in F_x} (E_x - y) \Big) \setminus \{0\}.$$
Note that (b) and (c) are satisfied, and since $0 \notin E'_x$, (f) is as well provided k_x is large enough, which we may require. We now establish (e).

We know that $\bigcap_{y\in F_x}((A\setminus E_x)-y)\setminus\{0\}$ is not IP, so its complement

$$\{0\}\cup\bigcup_{y\in F_x}((A^c\cup E_x)-y)$$

is IP*. Thus

$$k_x\mathbb{Z}\cap\Big(\{0\}\cup\bigcup_{y\in F_x}((A^c-y)\cup(E_x-y))\Big)$$

is IP*, so that the potentially larger

$$\{0\}\cup\bigcup_{y\in F_x}(A^c-y)\cup\Big(k_x\mathbb{Z}\cap\bigcup_{y\in F_x}(E_x-y)\Big)=\{0\}\cup\bigcup_{y\in F_x}(A^c-y)\cup E'_x$$

is IP* as well. Since $0\in F_x$, this set is however contained in

$$\{0\}\cup\bigcup_{y\in F_x}((A^c\cup E'_x)-y),$$

which is therefore IP*, implying that its complement

$$\bigcap_{y\in F_x}((A\setminus E'_x)-y)\setminus\{0\}$$

is not IP, yielding (e) and establishing the Claim.

In light of the Claim, it is now easy to check that

$$T=\{f\in\Omega : A\text{ is IP rich over }f\}$$

satisfies I1–I3 above.

Conversely, suppose that T satisfies I1–I3 and let $E\subset\mathbb{Z}$ with $d^*(E)=0$. We must show that $A\setminus E$ contains an IP set. Since $()\in T$,

$$d^*(\{x\in\mathbb{Z}:(x)\in T\})>0,$$

and for all x in this set, $x\in A$. So we may choose x_1 such that $(x_1)\in T$ and $x_1\notin E$. Next we have

$$d^*(\{x\in\mathbb{Z}:(x_1,x)\in T\})>0,$$

and for every x in this set, $\{x,x+x_1\}\subset A$. Since

$$d^*(E\cup(E-x_1))=0,$$

we may choose x_2 such that $(x_1,x_2)\in T$ and $x_2\notin E\cup(E-x_1)$. Note now that

$$\mathrm{FS}(\{x_1,x_2\})\subset A\setminus E.$$

It is clear that this process can be continued and will yield a sequence $\langle x_i\rangle_{i=1}^\infty$ for which $\mathrm{FS}(\langle x_i\rangle_{i=1}^\infty)\subset A\setminus E$. ∎

We next move to our elementary characterization of D sets. One will immediately see that it is similar to the FS tree richness condition, but ostensibly stronger, in that the intersection of the successor sets of any finite family of nodes must have positive upper Banach density.

THEOREM 2.8. *Let $A \subset \mathbb{Z}$. Then A is a D set if and only if there is a subset $T \subset \Omega$ having the following properties:*

D1. $() \in T$.

D2. *If $f_1, \ldots, f_t \in T$ then $d^*(B_{f_1} \cap \cdots \cap B_{f_t}) > 0$, where*

$$B_{(x_1,\ldots,x_k)} = \{x \in \mathbb{Z} : (x_1, \ldots, x_k, x) \in T\}.$$

D3. *If $(x_1, \ldots, x_k) \in T$ then $\mathrm{FS}(\langle x_1, \ldots, x_k \rangle) \subset A$.*

Proof. We will be using the standard fact that if p is idempotent and $A \in p$ then $A \in p + p$, i.e. $\{m : A - m \in p\} \in p$. Let p be an essential idempotent with $A \in p$. Let

$$A_{()} = A \cap \{m : A - m \in p\}.$$

For $x \in A_{()}$, let

$$A_{(x)} = A \cap (A - x) \cap \{m : (A \cap (A - x)) - m \in p\} \in p.$$

Note that for any $x \in A_{()}$, one has $x \in A$. Now for $y \in A_{(x)}$, let

$$\begin{aligned} A_{(x,y)} = {} & A \cap (A - x) \cap (A - y) \cap (A - x - y) \\ & \cap \big\{m : \big(A \cap (A - x) \cap (A - y) \cap (A - x - y)\big) - m \in p\big\} \in p. \end{aligned}$$

Note that for any $x \in A_{()}$ and $y \in A_{(x)}$, $\mathrm{FS}(\langle x, y \rangle) \subset A$. Now for $z \in A_{(x,y)}$ one defines $A_{(x,y,z)} \in p$, etc. Continuing in this fashion, one defines p-sets $\{A_f : f \in T\}$ for some set $T \subset \Omega$. Letting

$$B_{(x_1,\ldots,x_k)} = \{x \in \mathbb{Z} : (x_1, \ldots, x_k, x) \in T\}$$

one gets $B_{(x_1,\ldots,x_k)} = A_{(x_1,\ldots,x_k)}$, and D1–D3 above are satisfied.

Conversely, suppose that T satisfies D1–D3. By expanding T if necessary, we can assume that T satisfies

D4. If $(x_1, \ldots, x_k) \in T$ and $L_1, \ldots, L_r$ are consecutive blocks of natural numbers whose union is $\{1, \ldots, k\}$ then, letting $y_i = \sum_{j \in L_i} x_j$, one has $(y_1, \ldots, y_r) \in T$.

To see this, note that once we include every such $(y_1, \ldots, y_r)$ for $(x_1, \ldots, x_k)$ originally in T, D4 will be already satisfied, and that after doing so

$$B_{(x_1,\ldots,x_k)} \subset B_{(y_1,\ldots,y_r)};$$

that is, every set of successors is a superset of an original set of successors, so D2 (and obviously D3) will still be satisfied.

Now let

$$S = \bigcap_{\substack{f \in T,\, E \subset \mathbb{Z} \\ d^*(E) = 0}} \overline{B_f \setminus E}.$$

As the sets $B_f \setminus E$ have the finite intersection property, S is non-empty and of course closed. Moreover, if $p \in S$ and $C \in p$ then $d^*(C) > 0$, as otherwise

$(B_{()} \setminus C) \in p$, a contradiction. Also, $A \in p$ for all $p \in S$. We claim that S is a semigroup and thus contains idempotents; such idempotents will be essential and will contain A, and this will complete the proof.

Let $p, q \in S$. We need to show that $p+q \in S$. Let $C \in p+q$ be arbitrary. It suffices to find $r \in S$ with $C \in r$. (If $p+q$ were not a member of the closed set S, one could find a basic neighborhood $\overline{C} = \{r : C \in r\}$ of $p+q$ disjoint from S.)

It suffices to show that $d^*(\bigcap_{i=1}^h B_{f_i} \cap C) > 0$ for every $f_1, \ldots, f_h \in T$, as then we can choose

$$r \in \bigcap_{\substack{f\in T,\, E\subset\mathbb{Z}\\ d^*(E)=0}} \overline{(B_f \cap C) \setminus E}.$$

One has $\{x \in \mathbb{Z} : C - x \in q\} \in p$, so since $p \in S$, for every $f_1, \ldots, f_h \in T$,

$$d^*\Big(\Big\{x \in \bigcap_{i=1}^h B_{f_i} : C - x \in q\Big\}\Big) > 0.$$

Fix $f_i = (x_1^{(i)}, \ldots, x_{k_i}^{(i)}) \in T$, $1 \le i \le h$. We may choose $x \in \bigcap_{i=1}^h B_{f_i}$ with $C - x \in q$. Since $q \in S$, we have

$$d^*\Big(\Big\{n \in \bigcap_{i=1}^h B_{(x_1^{(i)},\ldots,x_{k_i}^{(i)},x)} : n \in C - x\Big\}\Big) = d^*\Big(\bigcap_{i=1}^h B_{(x_1^{(i)},\ldots,x_{k_i}^{(i)},x)} \cap (C-x)\Big) > 0.$$

Put another way,

$$d^*\Big(\Big\{n \in \bigcap_{i=1}^h B_{(x_1^{(i)},\ldots,x_{k_i}^{(i)},x)} : n + x \in C\Big\}\Big) > 0.$$

Observe now that (by D4)

$$x + \Big\{n \in \bigcap_{i=1}^h B_{(x_1^{(i)},\ldots,x_{k_i}^{(i)},x)} : n + x \in C\Big\} \subset \bigcap_{i=1}^h B_{f_i} \cap C. \ \blacksquare$$

As mentioned in the introduction, Towsner [T] has shown by an elementary argument that for any finite partition of $\mathbb{Z}$, some cell is FS tree rich. In light of the example given in the next section, it is natural to pose the following

PROBLEM 2.9. *Give an elementary argument that for any finite partition of $\mathbb{Z}$, some cell A supports a tree T satisfying* D1–D3 *above, i.e. is a D set.*

3. An IP rich set that is not a D set. With characterizations in place, we move to our primary task.

THEOREM 3.1. *There exists a set $A \subset \mathbb{Z}$ such that A is IP rich and A is not a D set.*

Proof. Recall that a subset of $\mathbb{Z}$ is *thick* if it contains arbitrarily long intervals. It is an exercise that there exists a countable pairwise disjoint family $\{S_i : i \in \mathbb{N}\}$ of thick subsets of $\mathbb{N}$. We will be constructing countably many sets A_f of positive upper Banach density in this proof. Each of these will be assumed to be contained in a separate member of such a family. By an *n-spaced subset* of some S_i we mean a set $B \subset S_i \cap [n, \infty)$ having the property that if $x \in B$ and $0 < |x-y| < n$ then $y \in S_i \setminus B$.

Let $A_{()} \subset S_1$ be a set of odd numbers with $d^*(A_{()}) > 0$. Let x_1 be the least member of $A_{()}$. Choose m_1 with $2^{m_1} > x_1$ and let $A_{(x_1)}$ be a 2^{m_1+2}-spaced subset of S_2 consisting of numbers equal to $2^{m_1} \pmod{2^{m_1+1}}$ with $d^*(A_{(x_1)}) > 0$. Now pick the least member x_2 of $A_{()} \cup A_{(x_1)}$ not already used (i.e. not x_1). Suppose that x_2 comes from $A_{(x_1)}$. Choose $m_2 > m_1$ with $2^{m_2} > x_1 + x_2$ and let $A_{(x_1,x_2)}$ be a 2^{m_2+2}-spaced subset of S_3 consisting of numbers equal to $2^{m_2} \pmod{2^{m_2+1}}$ with $d^*(A_{(x_1,x_2)}) > 0$. Let x_3 be the least member of $A_{()} \cup A_{(x_1)} \cup A_{(x_1,x_2)}$ not already used. Say it comes from $A_{()}$. Choose $m_3 > m_2$ with $2^{m_3} > x_1 + x_2 + x_3$ and let $A_{(x_3)}$ be a 2^{m_3+2}-spaced subset of S_4 consisting of numbers equal to $2^{m_3} \pmod{2^{m_3+1}}$ with $d^*(A_{(x_3)}) > 0$.

Continue in this fashion; at the stage where we are ready to create a set within S_{k+1}, we let x_k be the least member of any of the sets constructed in the previous stages that was not already used. Say it comes from a set $A_{(a_1,\dots,a_t)}$. Choose $m_k > m_{k-1}$ with $2^{m_k} > x_1 + \dots + x_k$ and let $A_{(a_1,\dots,a_t,x_k)}$ be a 2^{m_k+2}-spaced subset of S_{k+1} consisting of numbers equal to $2^{m_k} \pmod{2^{m_k+1}}$ with $d^*(A_{(a_1,\dots,a_t,x_k)}) > 0$.

We note that:

A. $A_{(a_1,\dots,a_t,x_k)} + \mathrm{FS}_0(\langle a_1,\dots,a_t,x_k\rangle) + \{0,1,\dots,2^{m_k}\} \subset S_{k+1}$.
B. *Every member of $A_{(a_1,\dots,a_t,x_k)}$ is divisible by 2^{m_k}.*
C. *No member of $A_{(a_1,\dots,a_t,x_k)} + \mathrm{FS}_0(\langle a_1,\dots,a_t,x_k\rangle)$ is divisible by 2^{m_k+1}.*

Let T be the set of $(a_1,\dots,a_k) \in \Omega$ used as subscripts for sets $A_{(\cdot)}$ in this construction.

D. *Letting $B_{(a_1,\dots,a_k)} = \{x \in \mathbb{Z} : (a_1,\dots,a_k,x) \in T\}$, one has $B_{(a_1,\dots,a_k)} = A_{(a_1,\dots,a_k)}$.*

Next set

$$A = \bigcup_{(a_1,\dots,a_k)\in T} \big(A_{(a_1,\dots,a_k)} + \mathrm{FS}_0(\langle a_1,\dots,a_k\rangle)\big) = \bigcup_{(a_1,\dots,a_k)\in T} \mathrm{FS}(\langle a_1,\dots,a_k\rangle).$$

Then I1–I3 above are plainly satisfied, so A is IP rich. We now turn to showing that A is not a D set.

E. *If* $(a_1, \ldots, a_t) \in T$ *and* $m \in \mathbb{N}$ *then:*

(1) $4a_i \le a_{i+1}$, $1 \le i < t$.

(2) *If* $a_t \equiv 2^m \pmod{2^{m+1}}$ *then* $a_i \not\equiv 0 \pmod{2^m}$, $1 \le i < t$.

(3) *If for some* $1 \le i_1 < i_2 < \cdots < i_k \le t$ *one has* $(a_{i_1} + \cdots + a_{i_k}) \equiv 0 \pmod{2^m}$ *then* $a_{i_1} \equiv 0 \pmod{2^m}$. (*Hence* $a_{i_j} \equiv 0 \pmod{2^m}$, $1 \le j \le k$.)

Property (1) follows from the fact that $A_{(a_1,\ldots,a_j,x_k)}$ is $4(x_1 + \cdots + x_k)$-spaced. (Recall that $(a_1, \ldots, a_j)$ is a subsequence of $(x_1, \ldots, x_{k-1})$.)

For (2), note that for some $i_1 < \cdots < i_t$, $(a_1, \ldots, a_t) = (x_{i_1}, \ldots, x_{i_t})$. Since $x_{i_t} \in A_{(x_{i_1},\ldots,x_{i_{t-1}})}$, we have $x_{i_t} \equiv 2^{m_{i_{t-1}}} \pmod{2^{m_{i_{t-1}}+1}}$. This implies that $m = m_{i_{t-1}}$. Now use the fact that the sequence m_j increases with j.

For (3), assume the negation and choose a shortest (i.e. minimum k, but note $k \ge 2$) counterexample. Then obviously $a_{i_k} \not\equiv 0 \pmod{2^m}$. Choose r such that $a_{i_k} \equiv 2^r \pmod{2^{r+1}}$. Then

$$a_{i_1} + \cdots + a_{i_{k-1}} \equiv 0 \pmod{2^r}$$

but $a_{i_1} \not\equiv 0 \pmod{2^r}$ (again, since m_j increases with j). So this is a shorter counterexample, which is a contradiction.

F. *If* $\langle a_i \rangle_{i=1}^{\infty}$ *is a sequence having the property that* $(a_1, \ldots, a_t) \in T$ *for every* $t \in \mathbb{N}$ *then* $d^*(\mathrm{FS}(\langle a_i \rangle_{i=1}^{\infty})) = 0$.

This follows from **E**(1). For let $t \in \mathbb{N}$ and let I be any interval of length 4^t. Since $a_{t+1} \ge 4^t$, I contains at most one member of $x + \mathrm{FS}(\langle a_i \rangle_{i=t+1}^{\infty})$ for any $x \in \mathrm{FS}(\langle a_i \rangle_{i=1}^{t})$. Therefore, I contains no more than 2^t members of $\mathrm{FS}(\langle a_i \rangle_{i=1}^{\infty})$.

G. *For all* $x, y \in A$, *if there exists an IP set* $R \subset \mathbb{N}$ *such that*

$$R \cup (R + x) \cup (R + y) \subset A$$

then there exists some $(a_1, \ldots, a_k) \in T$ *with* $\{x, y\} \subset \mathrm{FS}(\langle a_1, \ldots, a_k \rangle)$.

To see this, pick m such that 2^m is greater than $\max\{x, y\}$. Under the hypothesis about R, A must contain a configuration of the form

$$\{h2^m, h2^m + x, h2^m + y\}.$$

By definition of A, $h2^m$ is a member of some set

$$A_{(a_1,\ldots,a_t,x_k)} + \mathrm{FS}_0(\langle a_1, \ldots, a_t, x_k \rangle).$$

By **C**, no member of that set is divisible by 2^{m_k+1}. This implies that $m \le m_k$, so that $\max\{x, y\} < 2^{m_k}$. Then by **A**,

$$\{h2^m, h2^m + x, h2^m + y\} \subset S_{k+1},$$

which implies that in fact

$$\{h2^m, h2^m + x, h2^m + y\} \subset A \cap S_{k+1} = A_{(a_1,\ldots,a_t,x_k)} + \mathrm{FS}_0(\langle a_1, \ldots, a_t, x_k \rangle).$$

But $A_{(a_1,\dots,a_t,x_k)}$ is 2^{m_k+2}-spaced, $2^{m_k} > x_1+\dots+x_k$ and $\max\{x,y\} < 2^{m_k}$, so for some $x_j \in A_{(a_1,\dots,a_t,x_k)}$ one actually has

$$\begin{aligned}\{h2^m, h2^m+x, h2^m+y\} &\subset \{x_j\} + \mathrm{FS}_0(\langle a_1,\dots,a_t,x_k\rangle)\\ &\subset \mathrm{FS}(\langle a_1,\dots,a_t,x_k,x_j\rangle).\end{aligned}$$

Write $(x_{i_1},\dots,x_{i_z}) = (a_1,\dots,a_t,x_k,x_j)$ and suppose that $x_{i_1},\dots,x_{i_q}$ are not divisible by 2^m while $x_{i_{q+1}},\dots,x_{i_z}$ are; this is possible by **E**(2). By **E**(3), no member of $\mathrm{FS}(\langle x_{i_1},\dots,x_{i_q}\rangle)$ is divisible by 2^m, so

$$h2^m \in \mathrm{FS}(\langle x_{i_{q+1}},\dots,x_{i_z}\rangle).$$

Now, every member of $\mathrm{FS}(\langle x_{i_{q+1}},\dots,x_{i_z}\rangle)$ is divisible by $2^{m_{i_q}}$. On the other hand, by **C**, $x_{i_{q+1}}$ is not divisible by $2^{m_{i_q}+1}$. Therefore $m_{i_q} \ge m$, whence $\max\{x,y\} < 2^{m_{i_q}}$. It is also the case (by stipulation; see the construction) that $x_{i_1}+\dots+x_{i_q} < 2^{m_{i_q}}$. Now since $M + x' = h2^m + x$ for some $M \in \mathrm{FS}(\langle x_{i_{q+1}},\dots,x_{i_z}\rangle)$ and $x' \in \mathrm{FS}(\langle x_{i_1},\dots,x_{i_q}\rangle)$, we have

$$2^{m_{i_q}} \mid (h2^m - M) = (x' - x).$$

So $x = x' \in \mathrm{FS}(\langle x_{i_1},\dots,x_{i_q}\rangle)$. As a similar argument applies to y, we have $\{x,y\} \subset \mathrm{FS}(\langle x_{i_1},\dots,x_{i_q}\rangle)$.

H. *Suppose that* $(x_{i_1},\dots,x_{i_k}),(x_{j_1},\dots,x_{j_t}) \in T$. *If*

$$\begin{aligned}&\big(\mathrm{FS}(\langle x_{i_1},\dots,x_{i_k}\rangle)\setminus \mathrm{FS}(\langle x_{i_1},\dots,x_{i_{k-1}}\rangle)\big)\\ &\qquad\cap \big(\mathrm{FS}(\langle x_{j_1},\dots,x_{j_t}\rangle)\setminus \mathrm{FS}(\langle x_{j_1},\dots,x_{j_{t-1}}\rangle)\big)\end{aligned}$$

is non-empty then $k = t$ *and* $i_s = j_s$, $1 \le s \le t$.

Note that, by construction, $(a_1,\dots,a_j) \in T$ is uniquely determined by a_j. (If $a_j \in S_{k+1}$ then $a_{j-1} = x_k$. Now use induction.) So by symmetry we may assume that if there is a counterexample to **H** then there is a counterexample with $x_{i_k} < x_{j_t}$. But since $x_{i_k} \in A_{(x_{i_1},\dots,x_{i_{k-1}})}$, it has distance at least $4(x_1+\dots+x_{i_{k-1}})$ from any other x_i. It follows that every member of $\mathrm{FS}(\langle x_{i_1},\dots,x_{i_k}\rangle)$ is less than x_{j_t}, a contradiction.

Suppose now that A is a D set. Then there is a tree $T' \subset \Omega$ which, together with its successor sets B'_f, satisfies D1–D3 above. In particular, for any $y,z \in B'_{()}$ there is some IP set $R \subset \mathbb{N}$ with

$$R \cup (R+y) \cup (R+z) \subset A.$$

By **G**, then, for every $y,z \in B'_{()}$ there exists some $(a_1,\dots,a_k) \in T$ such that $\{y,z\} \subset \mathrm{FS}(\langle a_1,\dots,a_k\rangle)$.

Consider now the map from $B'_{()}$ to T that sends $x \in B'_{()}$ to the unique (by **H**) $\pi(x) = (a_1,\dots a_k) \in T$ having the property that $x = a_k + y$ for some $y \in \mathrm{FS}_0(\langle a_1,\dots,a_{k-1}\rangle)$. What **G** tells us is that for every $y,z \in B'_{()}$, either $\pi(y)$ is an initial segment of $\pi(z)$ or vice-versa. Since for any fixed y

there can be only finitely many $z \in B'_{()}$ such that $\pi(z)$ is an initial segment of $\pi(y)$, the length of $\pi(y)$ as y ranges over the infinite set $B'_{()}$ is unbounded and there exists at least one infinite sequence $(a_1, a_2, \ldots)$ in the closure of $\pi(B'_{()})$ (topology of pointwise convergence). So $\pi(y)$ is an initial segment of $(a_1, a_2, \ldots)$ for every $y \in B'_{()}$ (otherwise we could find $z \in B'_{()}$ such that neither of $\pi(y), \pi(z)$ was an initial segment of the other). Therefore $B'_{()} \subset \mathrm{FS}(\langle a_i \rangle_{i=1}^{\infty})$, and since by $\mathbf{E}(1)$, $4a_i \leq a_{i+1}$ for every i, one has $d^*(\mathrm{FS}(\langle a_i \rangle_{i=1}^{\infty})) = 0$ by $\mathbf{F}$, which contradicts $d^*(B'_{()}) > 0$. ∎

Proof of Theorem 1.3. Let A be the set constructed in the previous theorem. Then $\mathbb{Z} \setminus A$ is D^* but not AIP^*. ∎

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