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NUMBER THEORY

Reciprocal Stern Polynomials

by

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Summary. A partial answer is given to a problem of Ulas (2011), asking when the nth Stern polynomial is reciprocal.

Let $B_n(t)$ be defined by the formulae

$$B_1(t) = 1, \quad B_{2n}(t) = tB_n(t), \quad B_{2n+1}(t) = B_n(t) + B_{n+1}(t).$$

Klavžar, Milutinović and Petr [2] have called $B_n(t)$ the nth *Stern polynomial* and Ulas [4] asked when $B_n(t)$ is reciprocal, i.e.

$$B_n^*(t) = t^{\deg B_n} B_n(t^{-1}) = B_n(t). \tag{1}$$

As a partial answer we shall prove

THEOREM 1. *If n has binary expansion*

$$n = \overset{a_1}{1}\,\overset{a_2}{0} \ldots \overset{a_k}{1} \qquad (k \text{ odd}, a_i \geq 1 \text{ for all } 1 \leq i \leq k), \tag{2}$$

and $l_1, \ldots, l_j$ are the lengths of blocks of 1 occurring in the sequence $a_2, \ldots, a_k$, then (1) holds if and only if, identically in t,

$$\sum_{\mu=0}^{\lfloor k/2 \rfloor} \sum{}' \frac{T_{a_1} \cdots T_{a_k}}{T_{a_{i_1}} \cdots T_{a_{i_\mu}}} \left(\prod_{\lambda=1}^{\mu} \frac{t^{a_{i_\lambda}}}{T_{a_{i_\lambda+1}}} - t^d \prod_{\lambda=1}^{\mu} \frac{t^{a_{i_\lambda+1}-2}}{T_{a_{i_\lambda+1}}} \right) = 0, \tag{3}$$

where $\sum'$ is taken over all integer vectors $[i_1, \ldots, i_\mu]$ such that

$$1 \leq i_1 < \cdots < i_\mu < k, \quad i_{\lambda+1} \geq i_\lambda + 2 \ (1 \leq \lambda < \mu), \tag{4}$$

and where

$$T_a = \frac{t^a - 1}{t - 1}, \qquad d = \left\lfloor \frac{l_1 + 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_j + 1}{2} \right\rfloor.$$

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Corollary. *If n has binary expansion* (2) *and*

$$a_{i+1} = a_i + 2 \quad (1 \le i < k), \tag{5}$$

then (1) *holds.*

Theorem 2. *If n has binary expansion* (2) *and for all pairs $1 \le i < j \le k$,*

$$a_i + a_j > \max\{a_1, \ldots, a_k\} + 2, \tag{6}$$

then (1) *is equivalent to* (4).

The assumption (6) in Theorem 2 is not superfluous, as the two infinite sequences of odd n satisfying (1) discovered by M. Gawron [1] show, as also does the following

Theorem 3. *For $k \le 3$,* (1) *holds if and only if either $k = 1$, or $k = 3$ and* (4) *holds, or $a_1 = a - 1$, $a_2 = 2a$, $a_3 = a + 1$ (a an integer > 1), or $a_2 = 1$, $a_3 = 2$, or $a_1 = a_2 - 1$ ($a_2 > 1$), $a_3 = 1$.*

Proof of Theorem 1. It follows from [3, Theorem 1 and Lemma 5] that if (2) holds, then

$$B_n(t) = T_{a_1} \cdots T_{a_k}\Big(1 + \sum_{\mu=1}^{\lfloor k/2 \rfloor} \sum{}' \frac{1}{T_{a_{i_1}} \cdots T_{a_{i_\mu}}} \prod_{\lambda=1}^{\mu} \frac{t^{a_{i_\lambda}}}{T_{a_{i_\lambda + 1}}}\Big). \tag{7}$$

On the other hand, by [3, Theorem 2] and (2),

$$\deg B_n = a_1 + \cdots + a_k - k + d = \deg T_{a_1} \cdots T_{a_k} + d.$$

Also

$$T_a(t^{-1}) = t^{1-a} T_a(t),$$

hence by (1),

$$B_n^*(t) = t^d T_{a_1} \cdots T_{a_k}\Big(1 + \sum_{\mu=1}^{\lfloor k/2 \rfloor} \frac{1}{T_{a_{i_1}} \cdots T_{a_{i_\mu}}} \prod_{\lambda=1}^{\mu} \frac{t^{a_{i_{\lambda+1}} - 2}}{T_{a_{i_\lambda + 1}}}\Big). \tag{8}$$

Theorem 1 follows from (7) and (8). ∎

Proof of Corollary. If $a_{i+1} - a_i = 2$ $(1 \le i < k)$, then $d = 0$ and the Corollary follows from Theorem 1. ∎

For the proof of Theorem 2 we need two lemmas.

Lemma 1. *If $k \ge 2$, a_i $(1 \le i \le k)$ is a sequence of positive integers and*

$$\sum_{i=1}^{k-1} \Big(\frac{1}{t^{a_i}} - \frac{1}{t^{a_{i+1} - 2}}\Big) = 0, \tag{9}$$

identically in t*, then*

$$\{a_2, \ldots, a_{k-1}\} = \{a_1 + 2, \ldots, a_1 + 2(k-2)\}, \tag{10}$$

$$a_k = a_1 + 2(k-1). \tag{11}$$

Proof. Differentiating (9) and substituting afterwards $t = 1$ we obtain

$$\sum_{i=1}^{k-1} (a_i - a_{i+1} + 2) = 0,$$

thus (11) holds. Substituting in (9) we obtain

$$\begin{aligned}\sum_{i=1}^{k-1} \frac{t^2 - 1}{t^{a_{i+1}}} &= \sum_{i=1}^{k-1} \left(\frac{1}{t^{a_i}} - \frac{1}{t^{a_{i+1}}} \right) = \frac{1}{t^{a_1}} - \frac{1}{t^{a_k}} \\ &= \frac{t^{2(k-1)} - 1}{t^{a_1 + 2(k-1)}} = \frac{(t^2 - 1)(t^{2(k-2)} + t^{2(k-3)} + \cdots + 1)}{t^{a_1 + 2(k-1)}},\end{aligned}$$

and on dividing both sides by $t^2 - 1$,

$$\sum_{i=1}^{k-1} \frac{1}{t^{a_{i+1}}} = \sum_{i=1}^{k-1} \frac{1}{t^{a_1 + 2i}}\,.$$

Substituting $t = u^{-1}$ we obtain an identity for polynomials which implies (10). ∎

Lemma 2. *If* (6) *holds, then for any* $2 \le \mu < k$ *integers* i_λ $(1 \le \lambda \le \mu)$ *satisfying* (4) *we have*

$$a_{i_1} + \cdots + a_{i_\mu} \ge \frac{\mu}{2}\big(\max\{a_1, \ldots, a_k\} + 3\big). \tag{12}$$

Proof. By (6) for any positive integers $\lambda < \nu \le \mu$ we have

$$a_{i_\lambda} + a_{i_\nu} \ge \max\{a_1, \ldots, a_k\} + 3.$$

Summing over all pairs λ, ν in question we obtain

$$(\mu - 1)(a_{i_1} + \cdots + a_{i_\mu}) \ge \binom{\mu}{2}(\max\{a_1, \ldots, a_k\} + 3),$$

which implies (12). ∎

Proof of Theorem 2. Let us write the sum S occurring in (3) in the form

$$S = \sum_{\mu=0}^{\lfloor k/2 \rfloor} S_\mu.$$

If (6) holds, we have $a_i > 2$ for all $i \le k$, thus $d = 0$, $S_0 = 0$ and by

Theorem 1 and Lemma 2, for all $\mu \ge 2$,

$$\begin{aligned}\deg(t-1)^{k-2}S_\mu &\le a_1+\cdots+a_k-2+2\mu-\min{}^*\min\Big(-2\mu+\sum_{\lambda=1}^{\mu}a_{i_\lambda+1},\sum_{\lambda=1}^{\mu}a_{i_\lambda}\Big)\\ &\le \max\Big(a_1+\cdots+a_k-2+2\mu-\frac{\mu}{2}(\max\{a_1,\ldots,a_k\}-3),\\ &\qquad a_1+\cdots+a_k-2-\max\{a_1,\ldots,a_k\}-3\Big)\\ &< a_1+\cdots+a_k-\max\{a_1,\ldots,a_k\},\end{aligned}$$

where $\min^*$ is taken over all integer vectors $[i_1,\ldots,i_\mu]$ satisfying (4).

On the other hand, by (6), the sum of all terms of $(t-1)^{k-2}S_1$ of degree $\ge a_1+\cdots+a_k-\max\{a_1,\ldots,a_k\}$ equals

$$\sum_{i=1}^{k-1}t^{a_1+\cdots+a_k-a_{i+1}}-\sum_{\substack{i=1\\ a_i+2\le\max\{a_1,\ldots,a_k\}}}^{k-1}t^{a_1+\cdots+a_k-2-a_i}.$$

Substituting $t=1$ leads by Theorem 1 to the conclusion that for all $i<k$ we have $a_i+2\le\max\{a_1,\ldots,a_k\}$, and that

$$\sum_{i=1}^{k-1}\Big(\frac{1}{t^{a_{i+1}}}-\frac{1}{t^{a_i+2}}\Big)=0.$$

By Lemma 1 we obtain

$$\begin{gathered}a_k=a_1+2(k-1),\\ \{a_2,\ldots,a_{k-1}\}=\{a_1+2,\ldots,a_1+2(k-2)\},\\ s=\sum_{i=1}^{k}a_i=k(a_1+k-1).\end{gathered}$$

Therefore, we have

$$\begin{aligned}(t-1)^{k-2}S_1&=-\sum_{i=1}^{k-1}\sum_{\substack{h=1\\ h\ne i,i+1}}^{k}t^{s-a_{i+1}-a_h}\\ &\quad+\sum_{i=1}^{k-1}\sum_{\substack{h=1\\ h\ne i,i+1}}^{k}t^{s-2-a_i-a_h}+O(t^{s-3a_1-6}),\\ (t-1)^{k-2}S_2&=(t-1)^2\Big(\sum_{1<i+1<h<k}t^{s-a_{i+1}-a_{h+1}}\\ &\qquad-\sum_{1<i+1<h<k}t^{s-4-a_h-a_k}\Big)+O(t^{s-3a_1+4}),\\ (t-1)^{k-2}S_\mu&=O(t^{s-3a_1-8})\qquad(\mu\ge 3).\end{aligned}$$

We shall show by induction on $i < k$ that

(13) $$a_{i+1} = a_1 + 2i.$$

Suppose that
$$a_j = a_i + 2, \quad j > 2.$$
Then $(t-1)^{k-2}S_1$ contains the term $-t^{s-2a_1-2}$ (for $i=1$, $h=i-1$) which does not cancel with any other term of $(t-1)^{k-2}S_1$ since $2+a_i+a_h \le 2a_i+2$ is impossible for $i \ne h$. Thus
$$a_2 = a_1 + 2.$$
Assume now that $a_{i+1} = a_i + 2i$ for $i \le l \le k-3$, $l \ge 1$. Then
$$\begin{aligned}(t-1)^{k-2}S_1 = &-\sum_{i=l+1}^{k-1}\sum_{\substack{h=1\\ h\ne i,i+1}}^{k} t^{s-a_{i+1}-a_h}\\ &+\sum_{i=l+1}^{k-1}\sum_{\substack{h=1\\ h\ne i,i+1}}^{k} t^{s-2-a_i-a_h} + O(t^{s-3a_1-6}),\end{aligned}$$
$$\begin{aligned}(t-1)^{k-2}S_2 = (t-1)^2\Big(&\sum_{\substack{1<i+1<h<k\\ h>l}} t^{s-a_{i+1}-a_{h+1}}\\ &-\sum_{\substack{1<i+1<h<k\\ h>l}} t^{s-4-a_i-a_h}\Big) + O(t^{s-3a_1+4}).\end{aligned}$$
Suppose that
$$a_j = a_1 + 2l, \quad j > l+1.$$
Then $(t-1)^{k-2}S_2$ contains the term $-2t^{s-(2a_1+2l+1)}$ (for $i=1$, $h=j-1$), which does not cancel any other term of $(t-1)^{k-2}S$. Indeed, we have $2a_1+2l+1 < 3a_1+2$ and the terms of $(t-1)^{k-2}S_1$ of degree $\ge s-(3a_1+2)$ are of the form t^{s-2m}, m integer. So are also the terms of $(t-1)^{k-2}S_2$ of degree $\ge s-(3a_1+2)$ except the terms of
$$-2t\Big(\sum_{\substack{1<i+1<h<k\\ h>l}} t^{s-a_{i+1}-a_{h+1}} - \sum_{\substack{1<i+1<h<k\\ h>l}} t^{s-4-a_i-a_h}\Big).$$
However, for $h > l$ we have
$$3 + a_i + a_h > 2a_1 + 2l + 1.$$
This proves (13), and (4) follows. ∎

For the proof of Theorem 3 we need

LEMMA 3. *If $T_\alpha T_\beta = T_\gamma T_\delta$, where $\alpha,\beta,\gamma,\delta \in \mathbb{N}\setminus\{0\}$, $\alpha \le \beta$, $\gamma \le \delta$, then*
$$\alpha = \gamma, \quad \beta = \delta.$$

Proof. Assume that $\beta > \delta$. Then $T_\beta(\zeta_\beta) = 0$, where β is a positive βth root of unity, but $T_\gamma(\zeta_\beta) \neq 0$, $T_\delta(\zeta_\beta) \neq 0$, a contradiction. Thus $\beta \leq \delta$ and by symmetry $\beta = \delta$. Hence $T_\alpha = T_\gamma$ and $\alpha = \gamma$. ∎

Proof of Theorem 3. (1) holds obviously for $k = 1$. For $k = 3$ we shall consider successively the following cases:

A. $a_2 \geq 2$, $a_3 \geq 2$,
B. $a_2 = 1$, $a_3 \geq 2$,
C. $a_2 = 2$, $a_3 = 1$,
D. $a_2 = a_3 = 1$.

A. Here we have $d = 0$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_3}(t^{a_1} - t^{a_2-2}) + T_{a_1}(t^{a_2} - t^{a_3-2}) = 0,$$

thus to

$$\varepsilon t^{\min\{a_1,a_2-2\}} T_{a_3} T_{|a_1-a_2+2|} + \eta t^{\min\{a_2,a_3-2\}} T_{a_1} T_{|a_2-a_3+2|} = 0,$$

where

$$\varepsilon = \operatorname{sgn}(a_1 - a_2 + 2), \quad \eta = \operatorname{sgn}(a_2 - a_3 + 2). \tag{14}$$

Hence, there are the following possibilities: either

1. $\varepsilon = \eta = 0$, so $a_2 = a_1 + 2$, $a_3 = a_2 + 2$, and (4) holds; or
2. $\min\{a_1, a_2 - 2\} = \min\{a_2, a_3 - 2\}$,

$$\varepsilon = -\eta \neq 0, \tag{15}$$

and by Lemma 3 either

$$a_3 = a_1, \quad |a_1 - a_2 + 2| = |a_3 - a_2 + 2|, \tag{16}$$

or

$$a_3 = |a_2 - a_3 + 2|, \quad a_1 = |a_1 - a_2 + 2|. \tag{17}$$

The formulae (16) give $a_1 = a_2 = a_3$, contrary to (14) and (15). We cannot have $a_3 = a_3 - a_2 - 2$, thus from (17) we obtain $a_3 = a_2 - a_3 + 2$, from (14) $\eta = 1$, from (15) $\varepsilon = -1$ and from (17) $a_1 = a_2 - a_1 - 2$. Taking $a_2 = 2a$, we obtain $a_1 = a - 1$, $a_3 = a + 1$ (with integer $a > 1$).

B. Here we have $d = 1$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_1} T_{a_3}(1 - t) + T_{a_3}(t^{a_1} - 1) + T_{a_1}(t - t^{a_3-1}) = 0,$$

hence

$$T_{a_1}(t - t^{a_3-1}) = 0, \quad a_3 = 2.$$

C. Here we have $d = 1$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_1}T_{a_2}(1-t) + t^{a_1} - t^{a_2-1} + T_{a_1}(t^{a_2} - 1) = 0,$$

hence

$$t^{a_1} - t^{a_2-1} = 0, \qquad a_1 = a_2 - 1.$$

D. Here we have $d = 1$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_1}(1-t) + t^{a_1} - 1 + T_{a_1}(t-1) = 0,$$

hence

$$t^{a_1} = 1, \quad \text{impossible.} \ \blacksquare$$

References

[1] M. Gawron, *A note on the arithmetic properties of Stern polynomials*, Publ. Math. Debrecen 85 (2014), 453–465.

[2] S. Klavžar, U. Milutinović and C. Petr, *Stern polynomials*, Adv. Appl. Math. 39 (2007), 86–95.

[3] A. Schinzel, *Stern polynomials as numerators of continued fractions*, Bull. Polish Acad. Sci. Math. 62 (2014), 23–27.

[4] M. Ulas, *On certain arithmetic properties of Stern polynomials*, Publ. Math. Debrecen 79 (2011), 55–81.

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