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LOCAL CONVERGENCE FOR A FAMILY OF ITERATIVE METHODS BASED ON DECOMPOSITION TECHNIQUES

Abstract. We present a local convergence analysis for a family of iterative methods obtained by using decomposition techniques. The convergence of these methods was shown before using hypotheses on up to the seventh derivative although only the first derivative appears in these methods. In the present study we expand the applicability of these methods by showing convergence using only the first derivative. Moreover we present a radius of convergence and computable error bounds based only on Lipschitz constants. Numerical examples are also provided.

1. Introduction. In this paper the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0 \tag{1.1}$$

is analysed. Here $F : D \subseteq X \to Y$ is a Fréchet-differentiable operator, X, Y are Banach spaces and D is a convex subset of X. Newton-like methods are widely used for finding solutions of (1.1), and their convergence is usually studied using semi-local and local convergence analysis. The semi-local convergence analysis is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local analysis is, based on the information around a solution, to find estimates of the radii of convergence balls [3, 4, 20–22, 24, 25].

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Third order methods such as Euler's, Halley's, super Halley's and Chebyshev's methods [1]–[33] require the evaluation of the second derivative F'' at each step, which in general is very expensive. That is why many authors have used higher order multi-point methods [1]–[33]. In this paper, we introduce the three-step iterative method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1} F(x_n), \\ z_n &= y_n - \beta A_n^{-1} F(y_n), \\ x_{n+1} &= z_n - \gamma B_n^{-1} F(z_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, $\alpha, \beta, \gamma \in \mathbb{R}$, $A_n = \sum_{i=1}^{p} w_i F'(x_n + \theta_i(y_n - x_n))$, $B_n = \sum_{i=1}^{p} w_i F'(x_n + \theta_i(z_n - x_n))$, $\theta_i \in [0, 1]$ for $i = 1, \dots, p$, p is a positive integer and the weights $w_i \in S$ satisfy $\sum_{i=1}^{p} \|w_i\| = 1$. Method (1.2) reduces to earlier methods resulting from the Adomian decomposition [2], other decompositions [1–3, 23–27], quadrature formulae and other methods in the special case when $X = Y = \mathbb{R}$.

Let us mention some special cases. Notice that $x_n, y_n, w_i, \theta_i, i = 1, \dots, p$, and F' define A_n and B_n.

- Noor et al. [25–27] fourth order method: $\alpha = \beta = \gamma = 1$, and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ z_n &= y_n - A_n^{-1} F(y_n), \\ x_{n+1} &= z_n - B_n^{-1} F(z_n). \end{aligned} \tag{1.3}$$

- Noor et al. [25–27] third order method: $\alpha = \beta = 1$ and $\gamma = 0$, and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= y_n - A_n^{-1} F(y_n). \end{aligned} \tag{1.4}$$

- Newton's second order method: $\alpha = 1$ and $\beta = \gamma = 0$, and

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \tag{1.5}$$

with efficiency index 1.4142.

- Two step Newton's third order method considered by Traub [5, 6]: $\alpha = \gamma = p = w_1 = \theta_1 = 1$ and $\beta = 0$, and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= y_n - F'(x_n)^{-1} F(y_n), \end{aligned} \tag{1.6}$$

with efficiency index 1.4422.

- Midpoint two step Newton's third order method [5, 6, 30, 32]: $\alpha = \beta = p = w_1 = 1, \theta = 1/2$ and $\gamma = 0$, and

$$
(1.7)\qquad \begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ x_{n+1} &= y_n - F'\Big(\frac{x_n+y_n}{2}\Big)^{-1}F(y_n), \end{aligned}
$$

with efficiency index 1.3161.

• Third order method: $\alpha=\beta=1$, $\gamma=0$, $p=2$, $w_1=1/4$, $w_2=3/4$, $\theta_1=0$ and $\theta_2=2/3$, and

$$
(1.8)\qquad \begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ x_{n+1} &= y_n - \frac{1}{4}\Big(F'(x_n)+3F'\Big(\frac{x_n+2y_n}{3}\Big)\Big)^{-1}F(y_n), \end{aligned}
$$

with efficiency index 1.3161.

• Fourth order method: $\alpha=\beta=\gamma=p=1$ and $\theta_1=0$, and

$$
(1.9)\qquad \begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ z_n &= y_n - F'(x_n)^{-1}F(y_n),\\ x_{n+1} &= z_n - F'(x_n)^{-1}F(z_n), \end{aligned}
$$

with efficiency index 1.4142.

Notice that methods (1.4)–(1.9) are special cases of method (1.3). Many other choices are also possible [5, 6, 29]. Therefore it is important to study these methods in a unified way. A problem with these methods is that they require the existence of the fourth derivative of F. This limits the applicability of these methods. As a motivational example, let us define a function f on $D=[-1/2,5/2]$ by

$$
f(x)=\begin{cases} x^3\ln x^2+x^5-x^4, & x\neq 0,\\ 0, & x=0. \end{cases}
$$

Choose $x^*=1$. We have

$$
\begin{aligned} f'(x) &= 3x^2\ln x^2+5x^4-4x^3+2x^2, \qquad f'(1)=3,\\ f''(x) &= 6x\ln x^2+20x^3-12x^2+10x,\\ f'''(x) &= 6\ln x^2+60x^2-24x+22. \end{aligned}
$$

Then, obviously, f''' is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative. This way we expand the applicability of method (1.2).

The rest of the paper is organized as follows. The local convergence of method (1.2) is analysed in Section 2, whereas the numerical examples are given in the concluding Section 3.

2. Local convergence analysis. In this section we present the local convergence analysis of method (1.2). Let $L_0, L>0$, $M\geq 1$ and $\alpha,\beta,\gamma\in\mathbb{R}$ be given parameters. It is convenient for the local convergence analysis of method (1.2) to define some scalar functions and parameters. Define functions g_1, f_2, h_{f_2} on the interval $[0, 1/L_0)$ by

$$g_1(t) = \frac{1}{2(1 - L_0 t)}(Lt + 2M|1 - \alpha|),$$
$$f_2(t) = L_0 \sum_{i=1}^{p} \|w_i\|((1 - \theta_i) + \theta_i g_1(t))t,$$
$$h_{f_2}(t) = f_2(t) - 1,$$

and parameters

$$r_1 = \frac{2(1 - M|1 - \alpha|)}{2L_0 + L}, \qquad r_A = \frac{2}{2L_0 + L}.$$

Suppose that $M|1 - \alpha| < 1$. Then $g_1(r_1) = 1$ and $0 \le g_1(t) < 1$ for each $t \in [0, r_1)$ and $r_1 < r_A < 1/L_0$. We have $h_{f_2}(0) = -1 < 0$ and $h_{f_2}(t) \to \infty$ as $t \to 1/L_0{}^-$. It follows from the Intermediate Value Theorem that h_{f_2} has zeros in $(0, 1/L_0)$. Denote by r_{f_2} the smallest such zero. Define functions g_2 and h_2 on $[0, r_{f_2})$ by

$$g_2(t) = \left(1 + \frac{|\beta|M}{1 - f_2(t)}\right) g_1(t), \qquad h_2(t) = g_2(t) - 1.$$

Suppose that $(1+M|\beta|)M|1-\alpha| < 1$. Then $h_2(0) = (1+M|\beta|)M|1-\alpha|-1 < 0$ and $h_2(t) \to \infty$ as $t \to r_{f_2}^-$. Denote by r_2 the smallest zero of h_2 in $(0, r_{f_2})$. Moreover, define functions f_3, h_3 on $[0, r_2)$ by

$$f_3(t) = L_0 \sum_{i=1}^{p} \|w_i\|((1 - \theta_i) + \theta_i g_2(t))t, \qquad g_3(t) = \left(1 + \frac{M|\beta|}{1 - f_3(t)}\right) g_2(t)$$

and $h_3(t) = g_3(t) - 1$. Then again as above we have $f_3(0) - 1 = -1 < 0$ and $f_3(t) \to \infty$ as $t \to 1/L_0{}^-$. Hence $f_3(t) - 1$ has a smallest zero denoted by r_3 in $(0, 1/L_0)$.

Suppose that

$$(1 + |\beta|M)(1 + |\gamma|M|)M|1 - \alpha| < 1. \tag{2.1}$$

Then

$$h_3(0) = (1 + M|\beta|)(1 + M|\gamma|)M|1 - \alpha| - 1 < 0$$

and

$$h_3(r_2) = \frac{M|\gamma|}{1 - f_3(r_2)} > 0,$$

since $f_3(r_2) < 1$. Denote by $\overline{r}_3$ the smallest zero of h_3 in $(0, r_2)$. Define

$r = \min\{r_1, r_2, \bar{r}_3\}$. Then for each $t \in [0, r)$,

$$0 \leq g_1(t) < 1, \tag{2.2}$$

$$0 \leq g_2(t) < 1, \quad 0 \leq f_2(t) < 1, \tag{2.3}$$

$$0 \leq g_3(t) < 1, \quad 0 \leq f_3(t) < 1. \tag{2.4}$$

Denote by $U(v, \rho)$, $\bar{U}(v, \rho)$ the open and closed balls in S with center $v \in S$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notation.

THEOREM 2.1. *Let $F : D \subset X \to Y$ be a Fréchet differentiable operator. Suppose that there exist $x^* \in D$, $L_0, L > 0$, $M \geq 1$ and $\alpha, \beta, \gamma \in S$ such that for each $x, y \in D$,* (2.1) *holds and*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X), \tag{2.5}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|, \tag{2.6}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L \|x - y\|, \tag{2.7}$$

$$\|F'(x^*)^{-1} F'(x)\| \leq M, \tag{2.8}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.9}$$

where r is defined before Theorem 2.1. *Then the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method* (1.2) *is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \dots$ and converges to x^*. Moreover,*

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.10}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.11}$$

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.12}$$

where the "g" functions are defined before Theorem 2.1. *Furthermore, for $T \in [r, 2/L_0)$, x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.*

Proof. We shall show (2.10) and (2.12) by induction. Using (2.6), the definition of r and the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, we obtain

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0 \|x_0 - x^*\| < L_0 r < 1. \tag{2.13}$$

It follows from (2.13) and the Banach Lemma on invertible operators [3, 4, 26, 28] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1} F'(x^*)\| \leq \frac{1}{1 - L_0 \|x_0 - x^*\|}. \tag{2.14}$$

Hence y_0 is well defined by the first substep of method (1.2) for $n = 0$. By

(2.5) we can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)\, d\theta. \tag{2.15}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| \le \|x_0 - x^*\| < r$. Hence, $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Then, by (2.8) and (2.15), we get

$$\begin{aligned} \|F'(x^*)^{-1}F(x_0)\| &= \left\|\int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)\, d\theta\right\| \\ &\le M\|x_0 - x^*\|. \end{aligned} \tag{2.16}$$

Then it follows from method (1.2) for $n = 0$, (2.2), (2.5), (2.7), (2.15), (2.16) and the definition of r that

$$\begin{aligned} \|y_0 - x^*\| &\le \|x_0 - x^* - F'(x_0)^{-1}F(x_0\| + |1-\alpha|\, \|F'(x^*)^{-1}F(x_0)\| \\ &\le \|F'(x_0)^{-1}F'(x^*)\| \left\|\int_0^1 F'(x^*)^{-1}(F'(x_0 + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)\, d\theta\right\| \\ &\quad + |1-\alpha|\, \|F'(x_0)^{-1}F'(x^*)\|\, \|F'(x^*)^{-1}F(x_0)\| \\ &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{M\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.17}$$

which shows (2.10) for $n = 0$ and $y_0 \in U(x^*, r)$. We have

$$\begin{aligned} \|x_0 + \theta_i(y_0 - x_0) - x^*\| &\le (1-\theta_i)\|x_0 - x^*\| + \theta_i\|y_0 - x^*\| \\ &< (1-\theta_i)r + \theta_i r = r. \end{aligned}$$

Hence, $x_0 + \theta_i(y_0 - x^*) \in U(x^*, r)$. We shall show that $A_0^{-1} \in L(Y, X)$. Using (2.2), (2.16), (2.17) and the definition of r, we obtain

$$\begin{aligned} &\|F'(x^*)^{-1}(A_0 - F'(x^*))\| \\ &\quad \le \sum_{i=1}^{p} \|w_i\|\, \big\|F'(x^*)^{-1}\big(F'(x_0 + \theta_i(y_0 - x_0)) - F'(x^*)\big)\big\| \\ &\quad \le L_0 \sum_{i=1}^{p} \|w_i\|\big((1-\theta_i)\|x_0 - x^*\| + \theta_i\|y_0 - x^*\|\big) \\ &\quad \le L_0 \sum_{i=1}^{p} \|w_i\|\big(1 - \theta_i + \theta_i g_1(\|x_0 - x^*\|)\big)\|x_0 - x^*\| \\ &\quad = f_2(\|x_0 - x^*\|) < f_2(r) < 1. \end{aligned} \tag{2.18}$$

It follows from (2.18) that $A_0^{-1} \in L(Y,X)$ and

$$\|A_0^{-1}F'(x^*)\| \le \frac{1}{1-f_2(\|x_0-x^*\|)}. \tag{2.19}$$

Hence, z_0 is well defined by method (1.2) for $n=0$. Then, using (2.2), (2.16) (for $y_0 = x_0$), (2.17) and (2.19) we get

$$\begin{aligned} \|z_0-x^*\| &\le \|y_0-x^*\| + \frac{|\beta|M\|y_0-x^*\|}{1-f_2(\|x_0-x^*\|)} \\ &= \left(1+\frac{|\beta|M}{1-f_2(\|x_0-x^*\|)}\right)(\|y_0-x^*\|) \\ &\le \left(1+\frac{|\beta|M}{1-f_2(\|x_0-x^*\|)}\right)g_1(\|x_0-x^*\|)\|x_0-x^*\| \\ &= g_2(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r, \end{aligned} \tag{2.20}$$

which shows (2.11) for $n=0$ and $z_0 \in U(x^*,r)$.

Next we show that $B_0^{-1} \in L(Y,X)$. Let $z_0 = y_0$, $g_1 = g_2$, and $f_2 = f_3$, in (2.18). Then by (2.4) we obtain

$$\|F'(x^*)^{-1}(B_0 - F'(x^*))\| \le f_3(\|x_0-x^*\|) < f_3(r) < 1. \tag{2.21}$$

It follows that $B_0^{-1} \in L(Y,X)$ and

$$\|B_0^{-1}F'(x^*)\| \le \frac{1}{1-f_3(\|x_0-x^*\|)}. \tag{2.22}$$

Hence, x_1 is well defined by the third substep of method (1.2) for $n=0$. Then, using (2.4), (2.16) (for $z_0 = x_0$), (2.20) and (2.22), we get

$$\begin{aligned} \|x_1-x^*\| &\le \|z_0-x^*\| + \frac{|\gamma|M\|z_0-x^*\|}{1-f_3(\|x_0-x^*\|)} \\ &= \left(1+\frac{|\gamma|M}{1-f_3(\|x_0-x^*\|)}\right)(\|z_0-x^*\|) \\ &\le \left(1+\frac{|\gamma|M}{1-f_3(\|x_0-x^*\|)}\right)g_2(\|x_0-x^*\|)\|x_0-x^*\| \\ &= g_3(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r, \end{aligned} \tag{2.23}$$

which shows (2.12) for $n=0$ and $x_1 \in U(x^*,r)$. Hence by simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates (2.10)–(2.12). Using the estimate $|x_{k+1}-x^*| < |x_k - x^*| < r$, we deduce that $x_{k+1} \in U(x^*,r)$ and $\lim_{k\to\infty} x_k = x^*$.

To prove the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^*-y^*))\,d\theta$ for some $y^* \in \bar{U}(x^*,T)$ with $F(y^*)=0$. Using (2.6) we get

$$|F'(x^*)^{-1}(Q - F'(x^*))| \le \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|\,d\theta \le L_0\int_0^1 (1-\theta)|x^* - y^*|\,d\theta \le \frac{L_0}{2}T < 1. \tag{2.24}$$

It follows from (2.24) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. ∎

REMARK 2.2. 1. In view of (2.6) and the estimate

$$\|F'(x^*)^{-1}F'(x)\| = \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \le 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 1 + L_0\|x - x^*\|$$

condition (2.8) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or $M(t) = M = 2$, since $t \in [0, 1/L_0)$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then we can choose $P(x) = x + 1$.

3. The radius r_A was shown to be the convergence radius of Newton's method [5, 6]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad \text{for } n = 0, 1, 2, \ldots, \tag{2.25}$$

under the conditions (2.6) and (2.7). It follows from the definition of r that the convergence radius r of method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.25). As already noted in [5, 6], r_A is at least as large as the convergence ball given by Rheinboldt [31]

$$r_R = \frac{2}{3L}. \tag{2.26}$$

In particular, for $L_0 < L$ we have

$$r_R < r$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L} \to 0.$$

That is, our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [32].

4. It is worth noticing that method (1.2) does not change when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [1, 2, 3, 25, 26, 27]. Moreover, we can compute the computational order of convergence (COC) [15] defined by

$$\xi = \ln\left(\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}\right) / \ln\left(\frac{\|x_n-x^*\|}{\|x_{n-1}-x^*\|}\right)$$

or the approximate computational order of convergence [15]:

$$\xi_1 = \ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right) / \ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates of higher Fréchet derivatives of F.

3. Numerical examples. In this section we present two numerical examples. In both, we have taken the parameters $\alpha, \beta, \gamma, w_i, \theta_i, p, i = 1, \ldots, p$, as given in the introduction for method (1.6)–(1.8). We have taken $w_1 = \theta_1 = p = 1$ for methods (1.3)–(1.5) and $p = w_1 = 1$ and $\theta_1 = 0$ for method (1.9).

Example 3.1. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define a function F on D for $w = (x,y,z)^T$ by

$$F(w) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

Table 1. Parameters of methods (1.3)–(1.9): Example 3.1

parameters/ methods	$r = \bar{r}_3$	r_2	r_1	r_3	r_{f_2}	ξ
(1.3)	0.0667	0.1650	0.3249	0.1490	0.3828	0.993076
(1.4)	0.0667	0.1650	0.3249	0.1490	0.3828	3.655630
(1.5)	0.2962	0.3249	0.3249	0.3142	0.4079	3.655630
(1.6)	0.0667	0.1650	0.3249	0.1490	0.3828	1.753711
(1.7)	0.0636	0.1570	0.3249	0.3269	0.4079	1.994778
(1.8)	0.0291	0.3249	0.3249	0.3254	0.4079	2.004979
(1.9)	0.0667	0.1650	0.3249	0.1490	0.3828	3.639839

Then the Fréchet derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $L_0 = e - 1, L = e, M = 2$. The parameters are given in Table 1.

Example 3.2. Returning to the motivational example in the introduction, we have $L_0 = L = 146.6629073$, $M = 2$. The parameters are given in Table 2.

Table 2. Parameters of methods (1.3)–(1.9): Example 3.2

parameters/ methods	$r = \bar{r}_3$	r_2	r_1	r_3	r_{f_2}	ξ
(1.3)	0.0011	0.0026	0.0045	0.0048	0.0050	1.002163
(1.4)	0.0011	0.0026	0.0045	0.0048	0.0050	0.993283
(1.5)	0.0036	0.0045	0.0045	0.0042	0.0052	0.991749
(1.6)	0.0011	0.0026	0.0045	0.0048	0.0050	0.992117
(1.7)	0.0011	0.0024	0.0045	0.0048	0.0052	0.998581
(1.8)	0.0003	0.0024	0.0045	0.0048	0.0052	0.992479
(1.9)	0.0011	0.0026	0.0045	0.0048	0.0050	0.992479

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