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# Automorphisms of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$

by

**Paul Larson** and **Paul McKenney** (Oxford, OH)

**Abstract.** We study conditions on automorphisms of Boolean algebras of the form $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ (where $\lambda$ is an uncountable cardinal and $\mathcal{I}_\kappa$ is the ideal of sets of cardinality less than $\kappa$) which allow one to conclude that a given automorphism is trivial. We show (among other things) that every automorphism of $\mathcal{P}(2^\kappa)/\mathcal{I}_{\kappa^+}$ which is trivial on all sets of cardinality $\kappa^+$ is trivial, and that $\mathrm{MA}_{\aleph_1}$ implies both that every automorphism of $\mathcal{P}(\mathbb{R})/\mathtt{Fin}$ is trivial on a cocountable set and that every automorphism of $\mathcal{P}(\mathbb{R})/\mathtt{Ctble}$ is trivial.

**1. Introduction.** Given a set $X$ and an ideal $\mathcal{I}$ on $X$, an automorphism of $\mathcal{P}(X)/\mathcal{I}$ is said to be *trivial* if it is induced by a bijection between sets in $\mathcal{P}(X)\setminus\mathcal{I}$ (see Definition 2.1 below for a more precise formulation). In 1956, Walter Rudin [14] showed that if the Continuum Hypothesis holds, then the set of nontrivial automorphisms of $\mathcal{P}(\omega)/\mathtt{Fin}$ has cardinality $2^{\aleph_1}$. Around 1980, Saharon Shelah [15] showed that consistently all automorphisms of $\mathcal{P}(\omega)/\mathtt{Fin}$ are trivial. Boban Veličković [18] later proved from $\mathrm{OCA}+\mathrm{MA}_{\aleph_1}$, a weak fragment of the Proper Forcing Axiom, that all automorphisms of $\mathcal{P}(\lambda)/\mathtt{Fin}$ are trivial for all infinite cardinals $\lambda$. In the same paper, Veličković showed that the existence of nontrivial automorphisms of $\mathcal{P}(\omega)/\mathtt{Fin}$ is consistent with $\mathrm{MA}_{\aleph_1}$.

The possibilities for automorphisms of structures of the form $\mathcal{P}(\lambda)/\mathcal{I}$, for $\lambda$ an uncountable cardinal and $\mathcal{I}$ an ideal containing $\mathtt{Fin}$, seem to be much less understood than the case $\lambda=\omega$. For instance, it appears to be unknown whether ZFC proves that every automorphism of $\mathcal{P}(\lambda)/\mathtt{Fin}$ is trivial off of a countable subset of $\lambda$, or that every automorphism of $\mathcal{P}(\lambda)/\mathtt{Ctble}$ is trivial. Shelah and Steprāns [16] have recently shown, however, that for every $\lambda$ below the least strongly inaccessible cardinal, every automorphism of $\mathcal{P}(\lambda)/\mathtt{Fin}$ is trivial off of a subset of $\lambda$ of cardinality $2^{\aleph_0}$.

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Many questions about automorphisms of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are closely related to the question (due to Marian Turzański, and often called the *Katowice Problem*) of whether the Boolean algebras $\mathcal{P}(\omega)/\mathtt{Fin}$ and $\mathcal{P}(\omega_1)/\mathtt{Fin}$ can be isomorphic. There exists such an isomorphism if and only if there is an automorphism of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ which maps the equivalence class of some infinite set to the equivalence class of an infinite set of a different cardinality (analogous possibilities exist at higher cardinals). We call automorphisms where this does not happen *cardinality-preserving* (see Definition 2.7 for a more precise formulation).

In this paper we consider the ideals $\mathcal{I}_\kappa = \{X \mid |X| < \kappa\}$ for infinite cardinals $\kappa$. We prove (Theorem 3.4) than a cardinality-preserving automorphism of $\mathcal{P}(2^\kappa)/\mathcal{I}_{\kappa^+}$ (for any infinite cardinal $\kappa$) which is trivial on all sets of cardinality $\kappa^+$ is trivial. Assuming a weak fragment of Martin's Axiom, we prove (Theorem 4.4) the analogous result for automorphisms of $\mathcal{P}(\mathbb{R})/\mathtt{Fin}$ which are trivial on all countable sets. Assuming another fragment of Martin's Axiom, we show (Corollary 5.9) that every automorphism of $\mathcal{P}(\mathbb{R})/\mathtt{Ctble}$ is trivial, and also that every automorphism of $\mathcal{P}(\mathbb{R})/\mathtt{Fin}$ is trivial off of a countable set.

In Section 2 we prove several lemmas in a general setting that will be useful for work in later sections. In Section 3 we discuss almost-trivial automorphisms, and prove Theorem 3.4 (mentioned above). In Section 4 we introduce a weak fragment of Martin's Axiom, and use it to prove Theorem 4.4. Section 5 develops a necessary and sufficient condition for when an isomorphism between two countable, atomless subalgebras of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ can extend to a trivial automorphism, when $\kappa$ has uncountable cofinality. We use this to study automorphisms of $\mathcal{P}(\omega_1)/\mathtt{Ctble}$ when an uncountable $Q$-set exists. In Section 6 we give some conditions on automorphisms which imply the existence of fixed points. In Section 7 we develop a connection between ladder systems and nonfixed points of automorphisms of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, showing in particular that if there is an automorphism of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ whose set of ordinal fixed points is nonstationary, then $2^{\aleph_0} = 2^{\aleph_1}$. Finally, Section 8 contains a list of open questions.

**1.1. Notation.** We write $A \sim_\kappa B$ to indicate $|A \triangle B| < \kappa$. Given $\sigma \in 2^{<\omega}$, we write $N_\sigma$ for the set $\{x \in 2^\omega \mid \sigma \subset x\}$. Given a set $X$ and a cardinal $\kappa$, we write $\mathcal{I}_\kappa^X$ for the ideal of subsets of $X$ of cardinality less than $\kappa$. When there is no chance of confusion, we will drop the $X$ and just write $\mathcal{I}_\kappa$. We write $\mathtt{Fin}$ for $\mathcal{I}_{\aleph_0}$ and $\mathtt{Ctble}$ for $\mathcal{I}_{\aleph_1}$. If $\mathcal{I}$ is an ideal on a set $X$ and $A \subseteq X$, we write $[A]_\mathcal{I}$ for the equivalence class of $A$ in $\mathcal{P}(X)/\mathcal{I}$. When there is no chance of confusion, we will simply write $[X]$ instead. We write $[X]_\mathcal{I} \le [Y]_\mathcal{I}$ to mean that $X \setminus Y \in \mathcal{I}$, and $[X]_\mathcal{I} < [Y]_\mathcal{I}$ to mean that $[X]_\mathcal{I} \le [Y]_\mathcal{I}$ and $[X]_\mathcal{I} \ne [Y]_\mathcal{I}$.

## 2. Preliminaries

Definition 2.1. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals on sets $X$ and $Y$ respectively. A homomorphism $\pi : \mathcal{P}(X)/\mathcal{I} \to \mathcal{P}(Y)/\mathcal{J}$ is *trivial* if there is a function $f : Y \to X$ such that $\pi([A]_{\mathcal{I}}) = [f^{-1}(A)]_{\mathcal{J}}$ for all $A \subseteq X$. Similarly, if $Z \subseteq X$ then we say that $\pi$ is *trivial on $Z$* if there is a function $f : Y \to Z$ such that $\pi([A]_{\mathcal{I}}) = [f^{-1}(A)]_{\mathcal{J}}$ for all $A \subseteq Z$.

One gets equivalent definitions by allowing the domain of $f$ to be a subset of $Y$ with complement in $\mathcal{J}$. We say that such a function *witnesses* the triviality of $\pi$. We use inverse images to describe trivial homomorphisms since these are guaranteed to preserve the Boolean operations. The following lemma shows that we can often work with forward images instead.

Lemma 2.2. *Let $X$ and $Y$ be sets, let $\kappa$ be an infinite cardinal, and suppose that $f : Y \to X$ witnesses that $\pi : \mathcal{P}(X)/\mathcal{I}_\kappa \to \mathcal{P}(Y)/\mathcal{I}_\kappa$ is a trivial isomorphism. Then there are sets $E \subseteq X$ and $F \subseteq Y$ with $X \setminus E \in \mathcal{I}_\kappa^X$ and $Y \setminus F \in \mathcal{I}_\kappa^Y$ such that $f$ restricts to a bijection from $F$ to $E$. Moreover, $f^{-1} : E \to F$ witnesses that $\pi^{-1}$ is trivial.*

*Proof.* Suppose that $A = X \setminus \operatorname{ran} f$ has cardinality $\geq \kappa$. Then $[A]$ is nonzero, but $\pi([A]) = [f^{-1}(A)]$ is zero, a contradiction. Now suppose that there is a set $A \subseteq X$ such that $|f^{-1}(a)| \geq 2$ for all $a \in A$, and $|A| \geq \kappa$. Then $f^{-1}(A)$ has cardinality $\geq \kappa$. Let $f^{-1}(A) = B \cup C$ be a partition such that $f''(B) = f''(C) = A$ and $|B|, |C| \geq \kappa$. Then there is no $D$ such that $f^{-1}(D) \sim_\kappa B$, a contradiction of the fact that $[B]$ is in the range of $\pi$. Let $E = \{x \in X \mid |f^{-1}(x)| = 1\}$ and $F = f^{-1}(E)$. It follows that $f$ restricts to a bijection from $F$ to $E$, and $|X \setminus E|, |Y \setminus F| < \kappa$.

For the last part of the lemma, we want to see that for each $A \subseteq Y$, we have $\pi^{-1}([A]) = [(f^{-1})^{-1}(A)]$, i.e. that $\pi([(f^{-1})^{-1}(A)]) = [A]$. Now, $(f^{-1})^{-1}(A) = f''(A \cap F)$, so we want $\pi([f''(A \cap F)]) = [A]$. We see that $\pi([f''(A \cap F)]) = [f^{-1}(f''(A \cap F))]$. Since $f^{-1}(f''(A \cap F)) \cap F = A \cap F$, we have $[f^{-1}(f''(A \cap F))] = [A \cap F]$, which is the same as $[A]$. ∎

Definition 2.3. A Boolean algebra $\mathcal{B}$ is *$<\kappa$-complete* if every subset $\mathcal{A}$ of $\mathcal{B}$ with cardinality $< \kappa$ has a least upper bound.

Remark 2.4. The Boolean algebra $\mathcal{P}(X)/\mathcal{I}_\kappa$ is $<\operatorname{cf}\kappa$-complete. In particular, if $\mathcal{A}$ is a family of subsets of $X$, and $|\mathcal{A}| < \operatorname{cf}\kappa$, then $[\bigcup \mathcal{A}]$ is a least upper bound for the set $\{[A] \mid A \in \mathcal{A}\}$.

It is a well-known open question (asked by Marian Turzański) whether the Boolean algebras $\mathcal{P}(\omega_1)/\mathtt{Fin}$ and $\mathcal{P}(\omega)/\mathtt{Fin}$ can consistently be isomorphic. In many of the arguments in this paper, we must allow for the possibility that such an isomorphism exists (as well as analogous isomorphisms at other cardinals). We record here some facts that we use to deal with this possibility. The following theorem was proved by Balcar and Frankiewicz [1] in the case $\lambda = \omega$ and $\mu = \omega_1$; their proof gives the general version below.

THEOREM 2.5 (Balcar, Frankiewicz). *Suppose $\kappa \leq \lambda < \mu$, and $\kappa$ is regular. If $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ and $\mathcal{P}(\mu)/\mathcal{I}_\kappa$ are isomorphic, then $\lambda = \kappa$ and $\mu = \kappa^+$.*

We will also make use of the following fact, where $\mathfrak{d}$ is the minimal cardinality of a set $X$, consisting of functions from $\omega$ to $\omega$, such that every such function is dominated everywhere by a member of $X$.

THEOREM 2.6 (Balcar, Frankiewicz). *If $\mathcal{P}(\omega)/\mathtt{Fin}$ and $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are isomorphic, then $\mathfrak{d} = \omega_1$.*

Finally, we make the following definition.

DEFINITION 2.7. A homomorphism $\pi : \mathcal{P}(X)/\mathcal{I} \to \mathcal{P}(Y)/\mathcal{J}$ is *cardinality-preserving* if for every $A \subseteq X$, there is some $B \subseteq Y$ such that $|A| = |B|$ and $\pi([A]) = [B]$.

REMARK 2.8. An isomorphism $\pi : \mathcal{P}(X)/\mathcal{I}_\kappa \to \mathcal{P}(Y)/\mathcal{I}_\kappa$ is cardinality-preserving if and only if for all $A \subseteq X$, and $B \subseteq Y$, if $|A|, |B| \geq \kappa$ and $\pi([A]) = [B]$ then $|A| = |B|$. By Theorem 2.5, for any pair of infinite cardinals $\kappa < \lambda$, there exists an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ which is not cardinality-preserving if and only if there is an isomorphism between $\mathcal{P}(\kappa)/\mathcal{I}_\kappa$ and $\mathcal{P}(\kappa^+)/\mathcal{I}_\kappa$.

In our first application of the notion of cardinality-preservation, we show that it allows one to lift automorphisms on Boolean algebras of the form $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ to ones of the form $\mathcal{P}(\lambda)/\mathcal{I}_\mu$ when $\kappa \leq \mu \leq \lambda$.

DEFINITION 2.9. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals on sets $X$ and $Y$ respectively, and that $\pi : \mathcal{P}(X)/\mathcal{I} \to \mathcal{P}(Y)/\mathcal{J}$ is a function. A *selector* for $\pi$ is a map $\pi^* : \mathcal{P}(X) \to \mathcal{P}(Y)$ such that $\pi([A]) = [\pi^*(A)]$ for all $A \subseteq X$.

REMARK 2.10. A *selector*, in the literature, often denotes a function which is constant on equivalence classes. Our definition does not make this requirement, and in fact we will often instead take selectors which form bijections between equivalence classes.

LEMMA 2.11. *Let $\kappa \leq \mu \leq \lambda$ be infinite cardinals, and let $\pi$ be an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. Suppose that at least one of the following holds:*

- *$\pi$ is cardinality-preserving;*
- *$\mu > \kappa^+$ and $\kappa$ is regular.*

*Then $\pi$ induces an automorphism $\pi_\mu$ of $\mathcal{P}(\lambda)/\mathcal{I}_\mu$. In particular, if $\mathcal{A}$ is a family of subsets of $\lambda$, $|\mathcal{A}| < \operatorname{cf} \mu$, and $\pi^*$ is a selector for $\pi$, then*

$$\pi^*\Big(\bigcup \mathcal{A}\Big) \sim_\mu \bigcup \{\pi^*(A) \mid A \in \mathcal{A}\}.$$

*Proof.* Our assumptions on $\pi$ (using Theorem 2.5 in the case where $\mu > \kappa^+$ and $\kappa$ is regular) imply that $\pi$ takes the subalgebra $\mathcal{I}_\mu/\mathcal{I}_\kappa$ into itself. It follows that if $\pi^*$ is a selector for $\pi$, then the map

$$\pi_\mu([A]_\mu) = [\pi^*(A)]_\mu$$

is well-defined and an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\mu$. The rest follows from the $<\operatorname{cf}\mu$-completeness of the Boolean algebra $\mathcal{P}(\lambda)/\mathcal{I}_\mu$. ∎

The function $\pi_\mu$ from the proof of Lemma 2.11 clearly does not depend on the choice of $\pi^*$. We make the following definition, which will be used in Section 3.

DEFINITION 2.12. Let $\kappa \le \mu \le \lambda$ be infinite cardinals, and let $\pi$ be an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. We let $\pi_\mu$ be the function on $\mathcal{P}(\lambda)/\mathcal{I}_\mu$ defined by setting $\pi_\mu([A]_\mu) = [\pi^*(A)]_\mu$ for each $A \subseteq \lambda$ and any selector $\pi^*$ for $\pi$.

Lemma 2.13 shows that every automorphism of a Boolean algebra of the form $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ is determined by how it acts on sets of cardinality $\kappa$.

LEMMA 2.13. *Let $\kappa \le \lambda$ and let $\pi$ and $\rho$ be automorphisms of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. Then if $\pi \ne \rho$, there is some $X \subseteq \lambda$ of cardinality $\kappa$ such that $\pi([X]) \ne \rho([X])$. Moreover, for each $X \subseteq \lambda$ such that $\pi([X]) \not\le \rho([X])$, there exists $Y \in [X]^\kappa$ with $\pi([Y]) \not\le \rho([Y])$.*

*Proof.* By composing with $\rho^{-1}$, we may assume that $\rho = \mathrm{id}$. Fix a bijective selector $\pi^*$ for $\pi$, and choose $X \subseteq \lambda$ such that $\pi([X]) \ne [X]$. Without loss of generality, by choosing between $X$ and its complement we may assume $\pi^*(X) \setminus X$ has cardinality $\ge \kappa$. Let

$$W = (\pi^*)^{-1}(\pi^*(X) \setminus X).$$

Then $|W| \ge \kappa$ and $|W \setminus X| < \kappa$. Let $Y$ be a subset of $W \cap X$ of cardinality $\kappa$. Then $\pi^*(Y)$ has cardinality at least $\kappa$, and

$$|\pi^*(Y) \setminus (\pi^*(X) \setminus X)| < \kappa.$$

It follows that $|\pi^*(Y) \setminus Y| \ge \kappa$, so $\pi([Y]) \not\le [Y]$. ∎

## 3. Almost trivial automorphisms of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$

DEFINITION 3.1. Given an automorphism $\pi$ of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, we define $\mathcal{T}(\pi)$ to be the ideal of subsets $A$ of $\lambda$ such that $\pi$ is trivial on $A$. We let $\mathcal{A}_\mu(\pi)$ be the ideal generated by $\mathcal{T}(\pi)$ and $\mathcal{I}_\mu$.

If $A \in \mathcal{A}_\mu(\pi)$, then we say $\pi$ is *$\mu$-almost trivial* on $A$. If $\pi$ is $\mu$-almost trivial on $\lambda$, we just say that $\pi$ is *$\mu$-almost trivial*. If $\pi$ is $\kappa^+$-almost trivial on a set $A$, then we just say that $\pi$ is almost trivial on $A$.

In Lemma 3.2 we show that if two automorphisms lift to the same automorphism, then they are the same off of a small set.

LEMMA 3.2. *Suppose that $\kappa < \mu \leq \lambda$ are infinite cardinals, with $\kappa$ and $\mu$ regular, and let $\pi$ and $\rho$ be automorphisms of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. Suppose that either $\mu > \kappa^+$, or both $\pi$ and $\rho$ are cardinality-preserving. If $\pi_\mu = \rho_\mu$ then there is some $A \in \mathcal{I}_\mu$ such that for all $X \subseteq \lambda \setminus A$, $\pi([X]) = \rho([X])$.*

*Proof.* By composing with $\rho^{-1}$, we may assume that $\rho = \mathrm{id}$. Let $\pi^*$ be a bijective selector for $\pi$. Our assumptions on $\pi$ imply that $\pi$ is a permutation of $\mathcal{I}_\mu$. Suppose that the conclusion of the lemma fails, so that for every $A \in \mathcal{I}_\mu$ there is some $X \subseteq \lambda$ disjoint from $A$ with $\pi([X]) \neq [X]$. We will show that $\pi_\mu$ is not the identity.

Fix some $A \in \mathcal{I}_\mu$. By our assumption, there is some $X$ disjoint from $A \cup \pi^*(A) \cup (\pi^*)^{-1}(A)$ with $\pi^*(X) \not\sim_\kappa X$. By choosing between $X$ and $\lambda \setminus (A \cup X)$, we may assume that $|\pi^*(X) \setminus X| \geq \kappa$. By Lemma 2.13, we may also assume that $X$ has cardinality $\kappa$. Applying this observation repeatedly, we may construct sets $A_\alpha$ ($\alpha < \mu$) in $\mathcal{I}_\mu$, and sets $X_\alpha$ ($\alpha < \mu$) of cardinality $\kappa$, such that:

- for all $\alpha < \mu$:
  - $X_\alpha \cap (A_\alpha \cup \pi^*(A) \cup (\pi^*)^{-1}(A)) = \emptyset$,
  - $|\pi^*(X_\alpha) \setminus X_\alpha| \geq \kappa$,
  - $A_\alpha \cup X_\alpha \cup \pi^*(A_\alpha) \cup (\pi^*)^{-1}(A_\alpha) \cup \pi^*(X_\alpha) \subseteq A_{\alpha+1}$;
- for all limit $\alpha < \mu$, we have $A_\alpha = \bigcup\{A_\beta \mid \beta < \alpha\}$.

For each $\alpha < \mu$, we have $|\pi^*(X_\alpha) \cap A_\alpha| < \kappa$, so $|\pi^*(X_\alpha) \setminus (A_{\alpha+1} \setminus A_\alpha)| < \kappa$. Let $X = \bigcup\{X_\alpha \mid \alpha < \mu\}$. Since, for all $\alpha < \mu$,

$$\begin{aligned} |(\pi^*(X_\alpha) \setminus X_\alpha) \cap (A_{\alpha+1} \setminus A_\alpha)| &\geq \kappa, \\ |\pi^*(X_\alpha) \setminus \pi^*(X)| &< \kappa, \\ X \cap (A_{\alpha+1} \setminus A_\alpha) &= X_\alpha, \end{aligned}$$

it follows that $|(\pi^*(X) \setminus X) \cap (A_{\alpha+1} \setminus A_\alpha)| \geq \kappa$ for every $\alpha < \mu$. Then $|\pi^*(X) \setminus X| \geq \mu$. ■

By applying Lemma 3.2 in the case where $\pi_\mu$ (recall Definition 2.12) is trivial, we obtain the following

THEOREM 3.3. *Suppose that $\kappa < \mu \leq \lambda$ are infinite cardinals with $\kappa$ and $\mu$ regular, and let $\pi$ be an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. Suppose that either $\pi$ is cardinality-preserving or $\mu > \kappa^+$. If $\pi_\mu$ is trivial, then $\pi$ is $\mu$-almost trivial.*

Theorem 3.4 below is one of the main results of the paper. The strategy used in its proof is reused in Section 4.

THEOREM 3.4. *Suppose that $\pi$ is an automorphism of $\mathcal{P}(2^\kappa)/\mathcal{I}_{\kappa^+}$, and that $\pi$ is trivial on every set of cardinality $\kappa^+$. Then $\pi$ is trivial.*

*Proof.* Let $\pi^*$ be a bijective selector for $\pi$, and for each $A \subseteq 2^\kappa$ of cardinality $\kappa^+$, choose a function $f_A : A \to \pi^*(A)$ such that for all $B \subseteq A$, $\pi^*(B) \sim_{\kappa^+} f''_A(B)$. By Lemma 2.2, each $f_A$ restricts to a bijection between subsets of $A$ and $\pi^*(A)$ whose complements (in $A$ and $\pi^*(A)$ respectively) have cardinality at most $\kappa$, and moreover, for every $B \subseteq \pi^*(A)$, we have $\pi^{-1}([B]) = [f_A^{-1}(B)]$.

Let $\langle x_\alpha \mid \alpha < 2^\kappa \rangle$ be an enumeration of $\mathcal{P}(\kappa)$. For each $\beta < \kappa$, let $R_\beta = \{\alpha < 2^\kappa \mid \beta \in x_\alpha\}$ and let $T_\beta = (\pi^*)^{-1}(R_\beta)$. For each $\gamma < 2^\kappa$, let $y_\gamma = \{\beta < \kappa \mid \gamma \in T_\beta\}$. Let $h : 2^\kappa \to 2^\kappa$ be such that for all $\gamma, \alpha < 2^\kappa$, if $y_\gamma = x_\alpha$, then $h(\gamma) = \alpha$ (1).

For each $\beta < \kappa$, and $A \in [2^\kappa]^{\kappa^+}$, let $G_{A,\beta}$ be the set of $\gamma \in A$ for which $\gamma \in T_\beta$ if and only if $f_A(\gamma) \in R_\beta$. Since for each such $\beta$ and $A$, we have

$$f_A^{-1}[R_\beta] \sim_{\kappa^+} (\pi^*)^{-1}(R_\beta \cap \pi^*(A)) \sim_{\kappa^+} (T_\beta \cap A),$$

it follows that $G_{A,\beta} \sim_{\kappa^+} A$. Then, for each $A \in [2^\kappa]^{\kappa^+}$,

$$H_A = \bigcap \{G_{A,\beta} \mid \beta < \kappa\} \sim_{\kappa^+} A.$$

For each $\gamma \in H_A$ and $\beta < \kappa$, we have

$$\beta \in y_\gamma \;\Leftrightarrow\; \gamma \in T_\beta \;\Leftrightarrow\; f_A(\gamma) \in R_\beta \;\Leftrightarrow\; \beta \in x_{f_A(\gamma)}.$$

Then, for each $\gamma \in H_A$, $y_\gamma = x_{f_A(\gamma)}$, so $h(\gamma) = f_A(\gamma)$. Since $H_A \sim_{\kappa^+} A$, we then have $h \restriction A \sim_{\kappa^+} f_A$.

It follows from this that

$$\left|\left\{\alpha < 2^\kappa \;\middle|\; |h^{-1}[\{\alpha\}]| \neq 1\right\}\right| \leq \kappa,$$

so for some $B, C \in [2^\kappa]^{\leq \kappa}$, $h \restriction (2^\kappa \setminus B)$ is a bijection between $2^\kappa \setminus B$ and $2^\kappa \setminus C$. Thus, the map $\rho([A]) = [h''(A)]$ defines a trivial automorphism of $\mathcal{P}(2^\kappa)/\mathcal{I}_{\kappa^+}$. By the above, we have $\pi([A]) = \rho([A])$ for all $A \subseteq 2^\kappa$ with cardinality $\leq \kappa^+$; hence, by Lemma 2.13, $\pi = \rho$. ∎

Remark 3.5. Theorem 3.4 contradicts the remark at the end of [18] which claims that $\mathrm{MA}_{\aleph_1}$ + OCA (which implies that $2^{\aleph_0} \geq \aleph_2$, and that all automorphisms of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are trivial) does not imply that all automorphisms of $\mathcal{P}(\omega_2)/\mathtt{Fin}$ are trivial. Combining Theorem 3.4 with the main result of [16], one deduces that if all automorphisms of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are trivial, then all automorphisms of $\mathcal{P}(\lambda)/\mathtt{Fin}$ are trivial, for all $\lambda$ below the least strongly inaccessible cardinal.

Remark 3.6. Theorems 3.3 and 3.4 show that if $\mu < \kappa$ are infinite cardinals and $\pi$ is an automorphism of $\mathcal{P}(2^\kappa)/\mathcal{I}_\mu$ which is trivial on all sets of cardinality $\kappa^+$, then $\pi$ is trivial.

(1) While it is not important for the current proof, we note (without any triviality condition on $\pi$) that $h''(A) \sim_{\kappa^+} \pi^*(A)$ for every $A$ in the smallest $\kappa$-complete subalgebra of $\mathcal{P}(2^\kappa)$ containing $\mathcal{I}_{\kappa^+}$ and the sets $T_\beta$ ($\beta < \kappa$).

We finish this section with facts about $\mathcal{T}(\pi)$ to be used in Section 4.

LEMMA 3.7. *Suppose that $\kappa \leq \lambda$ are infinite cardinals, and that $\pi$ is an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. Then $\mathcal{T}(\pi)$ is closed under unions of cardinality less than* $\operatorname{cf}\kappa$.

*Proof.* Fix a cardinal $\gamma < \operatorname{cf}\kappa$, and let $A_\delta$ $(\delta < \gamma)$ be sets in $\mathcal{T}(\pi)$. We may assume that each $A_\delta$ has cardinality $\geq \kappa$, and that the $A_\delta$'s are pairwise disjoint. Applying Lemma 2.2 (and possibly removing a set of cardinality less than $\kappa$ from each $A_\delta$), let $f_\delta : A_\delta \to \lambda$ $(\delta < \gamma)$ be injections such that for all $\delta < \gamma$ and $X \subseteq A_\delta$, $\pi([X]) = [f''_\delta(X)]$. Set $f = \bigcup\{f_\delta \mid \delta < \gamma\}$ and $A = \bigcup\{A_\delta \mid \delta < \gamma\}$. Fix a selector $\pi^*$ of $\pi$, and let $X \subseteq A$. Since $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ is $<\operatorname{cf}\kappa$-complete and $\pi$ is an automorphism, it follows that

$$\pi^*(X) \sim_\kappa \bigcup\{\pi^*(X \cap A_\delta) \mid \delta < \gamma\}.$$

Note also that

$$f''(X) = \bigcup\{f''(X \cap A_\delta) \mid \delta < \gamma\}.$$

Since $\pi^*(X \cap A_\delta) \sim_\kappa f''_\delta(X \cap A_\delta)$ for every $\delta < \gamma$, $\pi^*(X) \sim_\kappa f''(X)$. This shows that $A \in \mathcal{T}(\pi)$, as required. ∎

Theorem 3.3 and Lemma 3.7 give the following.

LEMMA 3.8. *Suppose that $\kappa < \lambda$ are infinite cardinals, and that $\pi$ is a cardinality-preserving automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. Then $\mathcal{A}_{\kappa^+}(\pi)$ is closed under unions of cardinality $\kappa$.*

**4. Automorphisms of $\mathcal{P}(\mathbb{R})/\mathtt{Fin}$.** In this section we define a cardinal characteristic of the continuum—the Cofinal Selection Number—and use it to show that a certain fragment of $\mathrm{MA}_{\aleph_1}$ (a consequence of $\operatorname{cov}(\mathcal{M}) > \aleph_1$, where $\operatorname{cov}(\mathcal{M})$ denotes the covering number for the ideal of meager sets) implies that any automorphism of $\mathcal{P}(\mathbb{R})/\mathtt{Fin}$ which is trivial on all countable sets is trivial. By Theorem 3.4, it suffices to prove this result with $\omega_1$ in place of $\mathbb{R}$; as this makes no essential difference in the proof, we work with $\mathbb{R}$. Veličković has shown [18] that $\mathrm{MA}_{\aleph_1}$ implies that any automorphism of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ which is trivial on all countable sets is trivial. His fragment of $\mathrm{MA}_{\aleph_1}$ is different, corresponding roughly to adding $\aleph_1$ many Cohen reals and then specializing an Aronszajn tree.

DEFINITION 4.1. Given $\Gamma \subseteq \mathcal{P}(2^\omega)$, we let $\mathrm{CSN}(\Gamma)$ be the smallest cardinality of a family $\mathcal{F} \subseteq (2^\omega)^\omega \times (2^\omega)^\omega$ such that:

(1) for every $(f, g) \in \mathcal{F}$, $\{f(n) \mid n < \omega\} \cup \{g(n) \mid n < \omega\}$ is dense in $2^\omega$,
(2) for all pairs $(f, g), (f', g')$ from $\mathcal{F}$, if $g \neq g'$, then

$$\{g(n) \mid n < \omega\} \cap \{g'(n) \mid n < \omega\} = \emptyset,$$

(3) for every $(f, g) \in \mathcal{F}$ and $n < \omega$, we have $f(n) \neq g(n)$,

(4) for every set $A \in \Gamma$, the set

$$\{(f,g) \in \mathcal{F} \mid \exists^\infty n < \omega \ \ |A \cap \{f(n), g(n)\}| = 1\}$$

has cardinality smaller than that of $\mathcal{F}$,

if such a family $\mathcal{F}$ exists. If no such family exists, we set $\mathrm{CSN}(\Gamma) = (2^{\aleph_0})^+$.

It is not hard to see that $\mathrm{CSN}(\mathcal{P}(2^\omega)) = (2^{\aleph_0})^+$ (condition (2) was included to make this the case).

Consider the poset $\mathbb{Q}$ with conditions $(\sigma, s)$, where $\sigma \in 2^{<\omega}$ and $s$ is a function mapping into 2, with domain $\{(\sigma{\restriction}n)^\frown\langle 1-\sigma(n)\rangle \mid n < \operatorname{dom}\sigma\}$. We define $(\sigma,s) \le (\tau,t) \Leftrightarrow \sigma \supseteq \tau \wedge s \supseteq t$. It is easy to see that $\mathbb{Q}$ is isomorphic to a dense subset of $\mathbb{C}\times\mathbb{C}$, where $\mathbb{C}$ is a Cohen forcing. Given $p = (\sigma, s) \in \mathbb{Q}$ we define

$$U_p = \bigcup\{N_\tau \mid \tau \in \operatorname{dom} s \wedge s(\tau) = 1\},$$
$$V_p = \bigcup\{N_\tau \mid \tau \in \operatorname{dom} s \wedge s(\tau) = 0\}$$

and, given $G \subseteq \mathbb{Q}$, we set

$$U_G = \bigcup\{U_p \mid p \in G\}, \qquad V_G = \bigcup\{V_p \mid p \in G\}.$$

Lemma 4.2. *Let $X = \{x_n \mid n < \omega\}$ and $Y = \{y_n \mid n < \omega\}$ be subsets of $2^\omega$ such that $X \cup Y$ is dense and $x_n \neq y_n$ for all $n < \omega$. Then if $G$ is $\mathbb{Q}$-generic, there are infinitely many $n < \omega$ such that $U_G$ contains exactly one of $x_n, y_n$.*

*Proof.* Given $p \in \mathbb{Q}$ we let $E_p$ be the set of $n \in \omega$ for which $U_p$ and $V_p$ each contain a member of $\{x_n, y_n\}$. We will show that for each $p \in \mathbb{Q}$, there exist $q \le p$ and $n \notin E_p$ with $n \in E_r$. Let $p \in \mathbb{Q}$ be given. Since $X \cup Y$ is dense, there must be some $n$ such that at least one of $x_n$ or $y_n$ is in $[\sigma_p]$. We consider the case $x_n \in [\sigma_p]$; the case $y_n \in [\sigma_p]$ can be handled similarly.

Suppose first that $y_n \notin [\sigma_p]$; then $y_n \supseteq \tau$ for some $\tau \in \operatorname{dom} s_p$. Let $\sigma_q$ be some extension of $\sigma_p$ such that $x_n \not\supseteq \sigma_q$; say $k$ is minimal such that $x_n(k) \neq \sigma_q(k)$. Let $\nu = (\sigma_q{\restriction}k)^\frown x_n(k)$. Define $s_q$ so that $s_q(\nu) = 1 - s_p(\tau)$.

Now suppose $y_n \in [\sigma_p]$. Then we may find $\sigma_q$ extending $\sigma_p$ such that neither of $x_n, y_n$ is in $[\sigma_q]$. Let $k$ and $\ell$ be minimal such that $x_n(k) \neq \sigma_q(k)$ and $y_n(\ell) \neq \sigma_q(\ell)$; let

$$\tau = (\sigma_q{\restriction}k)^\frown x_n(k), \qquad \nu = (\sigma_q{\restriction}\ell)^\frown y_n(\ell).$$

Define $s_q$ so that $s_q(\tau) = 0$ and $s_q(\nu) = 1$. ∎

Lemma 4.2 and the fact that $\mathbb{Q}$ is isomorphic to a dense subset of $\mathbb{C}\times\mathbb{C}$ give the following.

Corollary 4.3. $\mathrm{CSN}(\mathbf{\Sigma}^0_1) \ge \mathrm{cov}(\mathcal{M})$.

Theorem 4.4 is a variant of Theorem 3.4, restricted to the case $\kappa = \omega$. Theorem 4.4 assumes triviality on countable sets, instead of sets of cardinality $\aleph_1$ as in Theorem 3.4, at the cost of assuming a weak fragment of Martin's Axiom.

THEOREM 4.4. *Assume* $\mathrm{CSN}(\boldsymbol{\Delta}^1_1) > \omega_1$, *and let* $\pi$ *be a cardinality-preserving automorphism of* $\mathcal{P}(2^\omega)/\mathtt{Fin}$ *which is trivial on every countable subset of* $2^\omega$. *Then* $\pi$ *is trivial.*

*Proof.* Suppose that $\pi$ is nontrivial and let $\pi^*$ be a bijective selector for $\pi$. We may choose $\pi$ so that for all $\sigma, \tau \in 2^{<\omega}$, if $\sigma \subseteq \tau$ then $\pi^*(N_\sigma) \supseteq \pi^*(N_\tau)$, and if $\sigma \perp \tau$ then $\pi^*(N_\sigma) \cap \pi^*(N_\tau) = \emptyset$.

For each $x \in 2^\omega$ and $n < \omega$, there is a unique $\sigma \in 2^n$ such that $x$ is in $\pi^*(N_\sigma)$. Moreover, these $\sigma$'s form a branch through $2^\omega$, which we will call $h(x)$. Thus $h : 2^\omega \to 2^\omega$ is a function satisfying $h^{-1}(N_\sigma) = \pi^*(N_\sigma)$ for each $\sigma \in 2^{<\omega}$. By Lemma 2.11, it follows that $h^{-1}(B) \triangle \pi^*(B)$ is countable for every Borel set $B \subseteq 2^\omega$. In particular, $h$ is countable-to-one.

Let $Q$ be the set of $\sigma \in 2^{<\omega}$ such that $\pi$ is nontrivial on $N_\sigma$.

CLAIM 4.5. *$Q$ is a perfect tree.*

*Proof.* Clearly, every $\sigma \in Q$ has at least one extension in $Q$. Suppose that for some $\sigma \in Q$, there is exactly one $x \in [Q]$ which extends $\sigma$. Then $\pi$ is trivial on $N_\tau$ for every $\tau \supseteq \sigma$ with $\tau \not\subseteq x$; hence by Lemma 3.8 (and our assumption that $\pi$ is trivial on countable sets), $\pi$ is trivial on their union, i.e. $N_\sigma \setminus \{x\}$. But then clearly $\pi$ is trivial on $N_\sigma$. ∎

For each countable $A \subseteq 2^\omega$ we may fix a function $f_A : \pi^*(A) \to A$ such that for all $X \subseteq A$, we have $\pi([X]) = [f_A^{-1}(X)]$.

CLAIM 4.6. *Let* $A \subseteq 2^\omega$ *be countable. Then there is a countable set* $B \subseteq 2^\omega$ *with* $B \supseteq A$, *and an infinite set* $X \subseteq B \setminus A$, *such that* $h''X \cup f''_B X$ *is dense in* $[Q]$, *and for every* $x \in X$, $h(x) \neq f_B(x)$.

*Proof.* Suppose otherwise. Then there is a countable $A^* \subseteq 2^\omega$ such that for every countable $B \supseteq A^*$, there is some $\sigma \in Q$ such that the set of $x \in B \setminus A^*$ with $h(x) \neq f_B(x)$ and $N_\sigma \cap \{h(x), f_B(x)\} \neq \emptyset$ is finite. Pressing down, we may fix a $\sigma^* \in Q$ and a finite $F^* \subseteq 2^\omega$ such that for all $B$ in some stationary subset of $[2^\omega]^\omega$, if $x \in B \setminus A^* \cup F^*$, then whenever one of $h(x), f_B(x)$ is in $N_{\sigma^*}$, we have $h(x) = f_B(x)$. In particular, if $x \in \pi^*(N_{\sigma^*}) = h^{-1}(N_{\sigma^*})$, then $h(x) \in N_{\sigma^*}$ and so $h(x) = f_B(x)$ as long as $x \in B \setminus (A^* \cup F^*)$. It follows that $\pi$ is trivial on $\pi^*(N_{\sigma^*})$, a contradiction. ∎

By applying Claim 4.6 repeatedly, we may find a $\subseteq$-increasing sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ consisting of countable subsets of $2^\omega$, and infinite sets $X_\alpha \subseteq A_{\alpha+1} \setminus A_\alpha$, such that for every $\alpha < \omega_1$, $h''X_\alpha \cup f''_{A_{\alpha+1}} X_\alpha$ is dense in $[Q]$, and $h(x) \neq f_{A_{\alpha+1}}(x)$ for every $x \in X_\alpha$. Since $h$ is countable-to-one, we may thin the sequence if necessary so that the sets $h[X_\alpha]$ are disjoint for distinct $\alpha$. Applying the assumption that $\mathrm{CSN}(\mathbf{\Delta}^1_1) > \omega_1$ (and possibly thinning our sequence again), we may find a Borel set $B \subseteq [Q]$ such that for every $\alpha < \omega_1$, there are infinitely many $x \in X_\alpha$ such that $B$ contains exactly one of $h(x), f_{A_{\alpha+1}}(x)$. Hence,

$$(h^{-1}(B) \triangle f^{-1}_{A_{\alpha+1}}(B)) \cap (A_{\alpha+1} \setminus A_\alpha)$$

is infinite for every $\alpha < \omega_1$. As $f^{-1}_{A_{\alpha+1}}(B) \triangle (\pi^*(B) \cap A_{\alpha+1})$ is finite for each $\alpha$, it follows that $\pi^*(B) \triangle h^{-1}(B)$ is uncountable, a contradiction. ∎

REMARK 4.7. Since $\mathfrak{d} \geq \mathrm{cov}(\mathcal{M})$, if we replace "$\mathrm{CSN}(\mathbf{\Delta}^1_1) > \omega_1$" with "$\mathrm{cov}(\mathcal{M}) > \omega_1$" in Theorem 4.4, by Theorem 2.6 we can then remove the assumption that $\pi$ is cardinality-preserving. On the other hand, it follows from Theorem 2.5 that if $\pi$ and $\pi^{-1}$ are both trivial on every countable subset of $2^\omega$, then $\pi$ must be cardinality-preserving.

**5. Isomorphisms between countable subalgebras.** In [7, Theorem 3.1], Geschke showed that any isomorphism between countable subalgebras of $\mathcal{P}(\omega)/\mathtt{Fin}$ extends to a trivial automorphism, and in fact a trivial automorphism witnessed by a permutation of $\omega$. In this section we find a necessary and sufficient condition for an isomorphism between two countable, atomless subalgebras of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ to extend to a trivial automorphism, in the case where $\kappa$ has uncountable cofinality. We then use our result to examine automorphisms of $\mathcal{P}(\omega_1)/\mathtt{Ctble}$ when a $Q$-set exists.

Recall that any two countable, atomless Boolean algebras are isomorphic. In particular, if $\mathcal{A}$ is a countable, atomless Boolean algebra, then $\mathcal{A}$ is isomorphic to the Boolean algebra of clopen subsets of $2^\omega$, hence $\mathcal{A}$ is generated by elements $a_\sigma$ ($\sigma \in 2^{<\omega}$) such that $a_\sigma \wedge a_\tau = 0$ for $\sigma \perp \tau$, and $a_{\sigma^\frown 0} \vee a_{\sigma^\frown 1} = a_\sigma$. Note that in this case, if $\pi : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then the elements $b_\sigma = \pi(a_\sigma)$ ($\sigma \in 2^{<\omega}$) generate $\mathcal{B}$, and satisfy the same relations.

DEFINITION 5.1. If $\mathcal{A}$ is a countable, atomless Boolean subalgebra of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, then we say that a sequence $\bar{A} = \langle A_\sigma \mid \sigma \in 2^{<\omega}\rangle$ of subsets of $\lambda$ is a *nice sequence of representatives* for $\mathcal{A}$ if

- for every $\sigma \in 2^{<\omega}$, the sets $A_{\sigma^\frown 0}$ and $A_{\sigma^\frown 1}$ partition $A_\sigma$, and
- the sequence $[A_\sigma]$ ($\sigma \in 2^{<\omega}$) generates $\mathcal{A}$.

If $\bar{A}$ is a nice sequence of representatives for $\mathcal{A}$, and $x \in 2^\omega$, then we set

$$A_x = \bigcap\{A_\sigma \mid \sigma \subset x\}, \qquad X(\bar{A}) = \{x \in 2^\omega \mid A_x \neq \emptyset\}.$$

Note that if $\bar{A}$ is a nice sequence of representatives for $\mathcal{A}$, then $[A_\emptyset]$ is the top element of $\mathcal{A}$, and

$$A_\emptyset = \bigcup\{A_x \mid x \in X(\bar{A})\}.$$

The following lemma shows that this information serves as a sort of invariant for $\mathcal{A}$.

LEMMA 5.2. *Suppose that $\mathcal{A}$ is a countable, atomless Boolean subalgebra of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, where $\operatorname{cf}\kappa > \omega$, and that $\bar{A}$ and $\bar{B}$ are nice sequences of representatives for $\mathcal{A}$ with $A_\sigma \sim_\kappa B_\sigma$ for each $\sigma \in 2^{<\omega}$. Then there is an $S \in \mathcal{I}_\kappa$ such that for every $x \in 2^\omega$, $A_x \setminus S = B_x \setminus S$. Moreover, $X(\bar{A}) \sim_\kappa X(\bar{B})$.*

*Proof.* Let $S = \bigcup\{A_\sigma \triangle B_\sigma \mid \sigma \in 2^{<\omega}\}$; then $|S| < \kappa$, and for every $x \in 2^\omega$, we have $A_x \setminus S = B_x \setminus S$. Now, if $x \in X(\bar{A}) \triangle X(\bar{B})$, then it follows that $A_x \cup B_x \subseteq S$, since one of $A_x$ or $B_x$ must be empty. But for each $\alpha \in S$, there is at most one $x \in 2^\omega$ such that $\alpha \in A_x$, and likewise, there is at most one $y \in 2^\omega$ such that $\alpha \in B_y$. ■

Theorem 5.3 characterizes when an isomorphism between two countable, atomless subalgebras of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$ can extend to a trivial isomorphism.

THEOREM 5.3. *Let $\bar{A}$ and $\bar{B}$ be nice sequences of representatives for countable, atomless Boolean subalgebras of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, where $\operatorname{cf}\kappa > \omega$. Then the following are equivalent:*

(1) *There is a trivial isomorphism from $\mathcal{P}(A_\emptyset)/\mathcal{I}_\kappa$ to $\mathcal{P}(B_\emptyset)/\mathcal{I}_\kappa$ which sends $[A_\sigma]$ to $[B_\sigma]$ for every $\sigma \in 2^{<\omega}$.*
(2) *$X(\bar{A}) \sim_\kappa X(\bar{B})$, and there is some $S \in \mathcal{I}_\kappa$ such that for all $x \in 2^\omega$, $|A_x \setminus S| = |B_x \setminus S|$.*

*Proof.* Suppose that $f : E \to F$ is a bijection, where:

- $E \subseteq A_\emptyset$,
- $F \subseteq B_\emptyset$,
- $A_\emptyset \setminus E$ and $B_\emptyset \setminus F$ are both in $\mathcal{I}_\kappa$,
- $f''(A_\sigma \cap E) \sim_\kappa B_\sigma$ for all $\sigma \in 2^{<\omega}$.

Let $\bar{C} = \langle f''(A_\sigma) \mid \sigma \in 2^{<\omega}\rangle$. Then $\bar{B}$ and $\bar{C}$ satisfy the hypotheses of Lemma 5.2, so $X(\bar{C}) \sim_\kappa X(\bar{B})$. Since $f''(A_x \cap E) = C_x \cap F$ for all $x \in 2^\omega$, it follows that $X(\bar{C}) \sim_\kappa X(\bar{A})$ as well, so $X(\bar{A}) \sim_\kappa X(\bar{B})$. Now let $S$ be as given in Lemma 5.2, applied to $\bar{B}$ and $\bar{C}$, so that $C_x \setminus S = B_x \setminus S$ for all $x \in 2^\omega$. Expanding $S$ if necessary (but preserving its membership in $\mathcal{I}_\kappa$) we may assume that $S$ contains $A_\emptyset \setminus E$ and $B_\emptyset \setminus F$, and is closed under $f$ and $f^{-1}$. Then, for each $x \in 2^\omega$, $f$ maps $A_x \setminus S$ to $B_x \setminus S$. Since $f$ is a bijection, it follows that $|A_x \setminus S| = |B_x \setminus S|$.

Suppose now that $X(\bar{A}) \sim_\kappa X(\bar{B})$, and that there exists an $S \in \mathcal{I}_\kappa$ such that $|A_x \setminus S| = |B_x \setminus S|$ for every $x \in 2^\omega$. For each $x \in X(\bar{A}) \triangle X(\bar{B})$, either

$A_x = \emptyset$ or $B_x = \emptyset$, so

$$A_x \setminus S = B_x \setminus S = \emptyset \quad \text{and} \quad A_x \cup B_x \subseteq S.$$

It follows that if

$$E = \bigcup\{A_x \mid x \in X(\bar{A}) \cap X(\bar{B})\}, \qquad F = \bigcup\{B_x \mid x \in X(\bar{A}) \cap X(\bar{B})\},$$

then $A_\emptyset \setminus S \subseteq E$ and $B_\emptyset \setminus S \subseteq F$. For each $x \in X(\bar{A}) \cap X(\bar{B})$, choose a bijection $g_x : A_x \setminus S \to B_x \setminus S$. Since for $x \neq y$ we have $A_x \cap A_y = B_x \cap B_y = \emptyset$, it follows that $g = \bigcup_{x \in X(\bar{A}) \cap X(\bar{B})} g_x$ is a bijection from $E \setminus S$ to $F \setminus S$ such that $g''(A_\sigma \setminus S) = B_\sigma \setminus S$ for every $\sigma \in 2^{<\omega}$. Then $g$ is as desired. ∎

REMARK 5.4. If $X \subseteq \mathbb{R}$ has no isolated points, then $X \cap N_\sigma$ ($\sigma \in 2^{<\omega}$) forms a nice sequence of representatives for a countable, atomless subalgebra $\mathcal{A}_X$ of $\mathcal{P}(\mathbb{R})/\mathtt{Ctble}$, and the set $X(\bar{A})$, where each $A_\sigma$ is $X \cap N_\sigma$, is exactly equal to $X$. If $X$ and $Y$ are two such sets, then there is an isomorphism from $\mathcal{A}_X$ to $\mathcal{A}_Y$ sending $[X \cap N_\sigma]$ to $[Y \cap N_\sigma]$ for every $\sigma$. By Theorem 5.3, if $X \triangle Y$ is uncountable, then there does not exist a trivial isomorphism from $\mathcal{P}(X)/\mathtt{Ctble}$ to $\mathcal{P}(Y)/\mathtt{Ctble}$ sending each set $[X \cap N_\sigma]$ to $[Y \cap N_\sigma]$.

DEFINITION 5.5. A set $X \subseteq 2^\omega$ is a *$Q_B$-set* if for every $Y \subseteq X$, there is a Borel $B \subseteq 2^\omega$ such that $B \cap X = Y$.

The above is a weakening of the usual notion of a $Q$-set, where the set $B$ above is required to be $G_\delta$ and not just Borel. The reader can consult [12, 11] for properties of $Q$-sets. We note in particular that $\mathrm{MA}_\kappa$ implies all subsets of $\mathbb{R}$ of size $\kappa$ are $Q$-sets (this is credited to Silver in [9, p. 162]), and that the Ramsey forcing axiom $\mathcal{K}_4$ implies that all subsets of $\mathbb{R}$ of size $\omega_1$ are $Q$-sets [17]. As for the difference between $Q$-sets and $Q_B$-sets, Miller [10] showed that if $X \subseteq \mathbb{R}$ is a $Q_B$-set, then there is some $\alpha < \omega_1$ such that every subset of $X$ is relatively $\mathbf{\Sigma}^0_\alpha$, whereas it is consistent that for every $2 \leq \alpha < \omega_1$, there is an uncountable $X \subseteq \mathbb{R}$ such that $\alpha$ is the minimal ordinal for which every subset of $X$ is relatively $\mathbf{\Sigma}^0_\alpha$.

THEOREM 5.6. *Suppose there exists a $Q_B$-set $X \subseteq 2^\omega$ of cardinality $\lambda$, where $\operatorname{cf} \lambda > \omega$. Then the following are equivalent:*

(1) *There exists a cardinality-preserving, nontrivial automorphism of $\mathcal{P}(\lambda)/\mathtt{Ctble}$.*
(2) *There exists a $Q_B$-set $Y \subseteq 2^\omega$ such that $X \triangle Y$ is uncountable and, for every Borel $B \subseteq 2^\omega$, $|B \cap X| + \aleph_0 = |B \cap Y| + \aleph_0$.*

*Proof.* Assuming that (1) holds, we may fix a nontrivial cardinality-preserving isomorphism $\pi$ from $\mathcal{P}(X)/\mathtt{Ctble}$ to $\mathcal{P}(\lambda)/\mathtt{Ctble}$. Choose a sequence $\bar{A} = \langle A_\sigma \mid \sigma \in 2^{<\omega}\rangle$ of sets such that $A_\emptyset = \lambda$ and, for all $\sigma \in 2^{<\omega}$,

- $\pi([N_\sigma \cap X]) = [A_\sigma]$,
- $A_\sigma$ is the disjoint union of $A_{\sigma^\frown 0}$ and $A_{\sigma^\frown 1}$.

Since $X$ is a $Q_B$-set, every subset of $X$ is equal to $C \cap X$ for some Borel $C$. Since $\mathcal{P}(\lambda)/\mathtt{Ctble}$ is countably complete, $\pi([C \cap X]) = [D]$, where $D$ is the set built from $\bar{A}$ in the same way that $C$ is built from $\langle N_\sigma : \sigma \in 2^{<\omega}\rangle$. Since $\pi$ is an isomorphism, every subset of $\lambda$ is, up to a countable set, a member of the $\sigma$-algebra generated by the sets $A_\sigma$ ($\sigma \in 2^{<\omega}$). It follows that for each $y \in 2^\omega$, the set

$$A_y = \bigcap\{A_{y\restriction n} : n \in \omega\}$$

is countable, and that the set of $y \in 2^\omega$ for which $|A_y| \geq 2$ is countable. Let $Y$ be the set of $y \in 2^\omega$ for which $|A_y| = 1$, and for each $y \in Y$, let $h(y)$ be the unique element of $A_y$. Then $h$ is injective, $h''(Y)$ is a cocountable subset of $\lambda$, and $h^{-1}(A_\sigma) = N_\sigma \cap Y$ for each $\sigma \in 2^{<\omega}$. We claim that $Y$ is the desired set.

To see that $Y$ is a $Q_B$-set, fix $Z \subseteq Y$. Then there exists a $D$ in the $\sigma$-algebra generated by $\bar{A}$ such that $D \triangle h''(Z)$ is countable. Then $h^{-1}(D) \triangle Z$ is countable, and $h^{-1}(D)$ is equal to $B \cap Y$ for a Borel set $B \subseteq 2^\omega$ which is built from the sets $N_\sigma$ in the same way that $D$ was built from the sets $A_\sigma$. Similarly, for each Borel set $B \subseteq 2^\omega$, letting $D$ be the set built from the sets $A_\sigma$ in the way that $B$ was built from the sets $N_\sigma$, we get $\pi([B \cap X]) = [D]$ and $h^{-1}(D) = B \cap Y$. Since $\lambda \setminus h''(Y)$ is countable,

$$|B \cap Y| + \aleph_0 = |D| + \aleph_0.$$

Since $\pi$ is cardinality-preserving,

$$|D| + \aleph_0 = |C \cap X| + \aleph_0.$$

Finally, if $X \triangle Y$ were a countable set $S$, we could fix a bijection $b : X \setminus S \to h''(Y \setminus S)$ by setting $b(x)$ to be $h(x)$. Then as above, for each Borel set $B \subseteq 2^\omega$, we have $\pi([B \cap X]) = [h''(B \cap Y)] = [b''(B \cap X)]$. Again applying the fact that $X$ is a $Q_B$-set, one shows that $\pi$ is trivial.

Now suppose that $Y \subseteq 2^\omega$ witnesses (2). Define $\pi : \mathcal{P}(X)/\mathtt{Ctble} \to \mathcal{P}(Y)/\mathtt{Ctble}$ by

$$\pi([C \cap X]) = [C \cap Y]$$

where $C$ ranges over the Borel sets. We claim that:

(i) $\pi$ is well-defined,
(ii) $\pi$ is an isomorphism, and
(iii) $\pi$ is cardinality-preserving and nontrivial.

All of these follow easily from our assumptions, except perhaps the claim that $\pi$ is nontrivial. Suppose then that $S$ and $T$ are countable subsets of $2^\omega$, and that $f$ is a bijection from $X \setminus S$ to $Y \setminus T$ such that $\pi([A]) = [f''(A \setminus S)]$ for all $A \subseteq X$. Since $X \triangle Y$ is uncountable, there are uncountably many $x \in X \setminus S$ such that $f(x) \neq x$. Moreover, we may fix incompatible $\sigma, \tau$ in $2^{<\omega}$

such that the set

$$Z = \{x \in N_\sigma \cap (X \setminus S) \mid f(x) \in N_\tau\}$$

is uncountable. Then there exists a Borel set $B$ such that $Z = B \cap X = (B \cap N_\sigma) \cap X$, so by the definition of $\pi$,

$$\pi([Z]) = [(B \cap N_\sigma) \cap Y].$$

On the other hand, $f''(Z) \subseteq N_\tau$, hence $f''(Z) \cap N_\sigma = \emptyset$. This contradicts our assumption that $\pi([Z]) = [f''(Z)]$. ■

REMARK 5.7. We do not know whether it is consistent with ZFC that there exists a nontrivial automorphism of $\mathcal{P}(\omega_1)/\texttt{Ctble}$. On the other hand, [6, Theorem 1] shows that the analogous result for Calkin algebras holds under the assumption that $2^{\omega_1} = \omega_2$. More precisely, let $\mathcal{B}$ denote the C*-algebra of bounded, linear operators on a Hilbert space of dimension $\omega_1$, and let $\mathcal{J}$ be its (closed, two-sided, $*$-) ideal of operators with separable range. Then $2^{\omega_1} = \omega_2$ implies there are $2^{\omega_2}$ many automorphisms of $\mathcal{B}/\mathcal{J}$.

Notice that condition (2) above fails whenever the union of two $Q_B$-sets of cardinality $\omega_1$ is also a $Q_B$-set. Combining this with Theorem 3.4, we obtain the following corollary.

COROLLARY 5.8. *Suppose that there is a $Q_B$-set of cardinality $\omega_1$, and the union of any two $Q_B$-sets of cardinality $\omega_1$ is a $Q_B$-set. Then every automorphism of $\mathcal{P}(\mathbb{R})/\texttt{Ctble}$ is trivial.*

The cardinal characteristic $\mathfrak{q}_0$ is the least cardinality of a subset of $2^\omega$ which is not a $Q$-set (see [2], for instance). Following [19], we let $\mathfrak{z}$ be the least cardinality of a subset of $2^\omega$ which is not a $Q_B$-set. Then $\mathfrak{q}_0$ is clearly at most $\mathfrak{z}$. We note that $\mathfrak{q}_0$ is at most $\mathfrak{d}$ (see [2]; we do not know if the same holds with $\mathfrak{z}$ in place of $\mathfrak{q}_0$) and that $\mathrm{MA}_{\aleph_1}$ implies $\mathfrak{q}_0 = 2^{\aleph_0}$, by the result of Silver mentioned after Definition 5.5.

Corollary 5.8 and Theorem 3.3 give the following.

COROLLARY 5.9. *If $\mathfrak{z} > \aleph_1$ then each of the following holds:*

- *Every automorphism of $\mathcal{P}(\mathbb{R})/\texttt{Ctble}$ is trivial.*
- *Every cardinality-preserving automorphism of $\mathcal{P}(\mathbb{R})/\texttt{Fin}$ is trivial on a cocountable set.*

In conjunction with the main result of [16], we see that $\mathfrak{z} > \aleph_1$ implies that every cardinality-preserving automorphism of $\mathcal{P}(\lambda)/\texttt{Fin}$ is trivial on a cocountable set, for every $\lambda$ less than the least strongly inaccessible cardinal. Veličković [18] has shown that $\mathrm{MA}_{\aleph_1}$ is consistent with the existence of a nontrivial automorphism of $\mathcal{P}(\omega)/\texttt{Fin}$.

REMARK 5.10. Corollary 5.8 applies to automorphisms $\pi$ which may not be induced by automorphisms of $\mathcal{P}(\mathbb{R})/\texttt{Fin}$. We do not know if the hypothesis of either of Theorem 4.4 and Corollary 5.8 implies the other. However, as $\mathrm{CSN}(\mathcal{P}(2^\omega)) = (2^{\aleph_0})^+$, $\mathrm{CSN}(\mathbf{\Delta}^1_1)$ is at least $\mathfrak{z}$.

**6. Fixed points.** If $\pi$ is an automorphism of a Boolean algebra of the form $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, then a *fixed point* of $\pi$ is a set $A \subseteq \lambda$ such that $\pi([A]) = [A]$. A fixed point $A$ is *nontrivial* if $A$ and $\lambda \setminus A$ both have cardinality at least $\kappa$. By the $<\mathrm{cf}\,\kappa$-completeness of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, the set of ($\sim_\kappa$-classes of) fixed points of such a $\pi$ is a (possibly trivial) $<\mathrm{cf}\,\kappa$-complete subalgebra of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$.

LEMMA 6.1. *Suppose that $\kappa \leq \lambda$ are infinite cardinals, and that $\pi$ is an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. Let $\pi^*$ be a selector for $\pi$. Let $\eta$ be an infinite regular cardinal not equal to* $\mathrm{cf}\,\kappa$, *and suppose that $\langle A_\alpha : \alpha < \eta\rangle$ is a sequence of subsets of $\lambda$ such that:*

(1) *for all $\alpha < \beta < \eta$,*
$$|(\pi^*(A_\alpha) \cup (\pi^*)^{-1}(A_\alpha)) \setminus A_\beta| < \kappa,$$
(2) *for all $\beta < \eta$,*
$$\Big|\bigcup\{A_\alpha \mid \alpha < \beta\} \setminus A_\beta\Big| < \kappa.$$

*Then $\bigcup\{A_\alpha : \alpha < \eta\}$ is a fixed point of $\pi$.*

*Proof.* Let $B = \bigcup\{A_\alpha \mid \alpha < \eta\}$. We want to see that $\pi^*(B) \sim_\kappa B$.

Suppose first that $|B \setminus \pi^*(B)| \geq \kappa$. We claim that there is some $\alpha < \eta$ such that $|A_\alpha \setminus \pi^*(B)| \geq \kappa$. If $\eta < \mathrm{cf}\,\kappa$, then this follows directly; on the other hand, if $\eta > \mathrm{cf}\,\kappa$, then (by the regularity of $\eta$) there is some $\alpha < \eta$ such that
$$\Big|\bigcup\{A_\beta \setminus \pi^*(B) \mid \beta < \alpha\}\Big| \geq \kappa,$$
in which case we have $|A_\alpha \setminus \pi^*(B)| \geq \kappa$ by (2). Now fix some $X \subseteq A_\alpha \setminus \pi^*(B)$ with cardinality exactly $\kappa$. Then
$$|(\pi^*)^{-1}(X) \setminus A_{\alpha+1}| < \kappa \quad \text{and} \quad |(\pi^*)^{-1}(X) \cap B| < \kappa,$$
hence $|(\pi^*)^{-1}(X)| < \kappa$, a contradiction.

Supposing instead that $|\pi^*(B) \setminus B| \geq \kappa$ we get $|B \setminus (\pi^*)^{-1}(B)| \geq \kappa$, and we can run the argument just given with $(\pi^*)^{-1}$ in place of $\pi^*$ to obtain another contradiction. ∎

Summarizing, we get the following. Part (3) uses Theorem 2.5. The proof of (4) breaks into two cases, one where $\mathrm{cf}\,\kappa$ is uncountable, and one where $\mathrm{cf}\,\kappa = \aleph_0$.

THEOREM 6.2. *Suppose that $\kappa \leq \lambda$ are infinite cardinals, and that $\pi$ is an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$.*

(1) *The set of $\sim_\kappa$-classes of fixed points $\pi$ is a $<\mathrm{cf}\,\kappa$-complete subalgebra of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$.*

(2) *If $\eta$ is a regular cardinal not equal to* $\operatorname{cf}\kappa$, *and* $\langle A_\alpha : \alpha < \eta\rangle$ *is a sequence of fixed points of* $\pi$ *such that for all* $\beta < \eta$,

$$\Big|\bigcup\{A_\alpha \mid \alpha < \beta\} \setminus A_\beta\Big| < \kappa,$$

*then* $\bigcup\{A_\alpha \mid \alpha < \eta\}$ *is a fixed point of* $\pi$.

(3) *If $\kappa$ is regular and $\lambda > \kappa$, then for every $A \subseteq \lambda$, there is a fixed point $B \subseteq \lambda$ such that $A \subseteq B$ and $|B| \le |A| + \kappa^+$.*

(4) *If $\pi$ is cardinality-preserving and $\kappa$ is uncountable, then for every $A \subseteq \lambda$, there is a fixed point $B \subseteq \lambda$ such that $A \subseteq B$ and $|B| = |A|$.*

Theorem 6.2 gives the following corollary.

Corollary 6.3. *If $\lambda > \kappa$ are infinite cardinals, and $\pi$ is an automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, then $\pi$ has nontrivial fixed points if at least one of the following holds:*

- *$\kappa$ is regular and $\lambda > \kappa^+$,*
- *$\pi$ is cardinality-preserving and $\kappa$ is uncountable.*

Remark 6.4. Chodounský, Dow, Hart and de Vries [4] have shown that if the algebras $\mathcal{P}(\omega)/\mathtt{Fin}$ and $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are isomorphic, then there exists a nontrivial automorphism of $\mathcal{P}(\omega)/\mathtt{Fin}$. A simpler, previously known argument (see [13]) uses the trivial automorphism of $\mathcal{P}(\omega)/\mathtt{Fin}$ induced by the (upwards or downwards) shift to establish that if $\mathcal{P}(\omega)/\mathtt{Fin}$ and $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are isomorphic, then there exists an automorphism of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ without nontrivial fixed points. An easy argument shows that every trivial automorphism of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ has a club of ordinal fixed points.

Section 7 considers ordinal fixed points of cofinality $\kappa$ for automorphisms of Boolean algebras of the form $\mathcal{P}(\kappa^+)/\mathcal{I}_\kappa$.

**7. Ladder systems.** A *ladder* on a limit ordinal $\alpha$ is a cofinal subset of $\alpha$ whose ordertype is the cofinality of $\alpha$ (we do not require here that the subset be closed). If $S$ is a set of limit ordinals, a *ladder system on $S$* is a sequence $\langle L_\alpha \mid \alpha \in S\rangle$ such that each $L_\alpha$ is a ladder on the corresponding $\alpha$. A ladder system $\langle L_\alpha \mid \alpha \in S\rangle$ satisfies *$\kappa$-uniformization* (for a given cardinal $\kappa$) if for every sequence of functions $f_\alpha : L_\alpha \to \kappa$ ($\alpha \in S$), there is a function $F : \sup(S) \to \kappa$ such that for all $\alpha \in S$,

$$\{\beta \in L_\alpha \mid F(\beta) \neq f_\alpha(\beta)\} \in \mathcal{I}_\kappa.$$

Here we show that the existence of a cardinality-preserving automorphism of $\mathcal{P}(\kappa^+)/\mathcal{I}_\kappa$ without ordinal fixed points of cofinality $\kappa$ (where $\kappa$ is a regular cardinal) gives rise to a ladder system on a club subset of $\kappa^+$ which satisfies 2-uniformization (which is easily seen to be equivalent to $\gamma$-uniformization for any $\gamma < \kappa$).

Given an automorphism $\pi$ of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$, for infinite cardinals $\kappa \le \lambda$, we say that $\beta \in \lambda$ is a *closure point* of $\pi$ if for all $\alpha < \beta$, $\pi([\alpha]) < [\beta]$ and $\pi^{-1}([\alpha]) < [\beta]$ (2). If $\lambda > \kappa$ and $\pi$ is cardinality-preserving, then the set of closure points of $\pi$ is a club subset of $\lambda$. It is easy to see that every closure point whose cofinality is not $\operatorname{cf}\kappa$ is a fixed point (this does not require that $\pi$ is cardinality-preserving, or that $\kappa$ is regular).

THEOREM 7.1. *Suppose that:*

- *$\kappa$ is a regular cardinal,*
- *$\pi$ is a cardinality-preserving automorphism of $\mathcal{P}(\kappa^+)/\mathcal{I}_\kappa$,*
- *$C$ is the set of closure points of $\pi$, and*
- *$S \subseteq \kappa^+$ is the set of $\alpha \in C$ which are not fixed points of $\pi$.*

*Then there exist $S_0$ and $S_1$ such that $S = S_0 \cup S_1$, and $S_0$ and $S_1$ each support a ladder system satisfying 2-uniformization.*

*Proof.* Let

$$S_0 = \{\alpha \in S \mid [\alpha] \not\le \pi([\alpha])\} \quad \text{and} \quad S_1 = \{\alpha \in S \mid \pi([\alpha]) \not\le [\alpha]\}.$$

Then clearly $S = S_0 \cup S_1$. Let $\pi^*$ be a bijective selector for $\pi$. For each $\alpha \in S_0$, let $L^0_\alpha = \alpha \setminus \pi^*(\alpha)$, and for each $\alpha \in S_1$, let $L^1_\alpha = \alpha \setminus (\pi^*)^{-1}(\alpha)$. Then for each $i \in \{0,1\}$ and each $\alpha \in S_i$,

- $|L^i_\alpha| = \kappa$,
- $|L^i_\alpha \cap \beta| < \kappa$ for all $\beta < \alpha$.

The second of these follows from the fact that $\alpha$ is a closure point of $\pi$, and from the fact that $|(\pi^*)^{-1}(L^0_\alpha) \cap \alpha| < \kappa$ in the case where $\alpha \in S_0$, and $|\pi^*(L^1_\alpha) \cap \alpha| < \kappa$ in the case where $\alpha \in S_1$. It follows that each $L^i_\alpha$ is a ladder on the corresponding $\alpha$.

Now suppose we are given 2-colorings $f^i_\alpha : L^i_\alpha \to 2$ for each pair $(\alpha, i)$ with $i \in \{0,1\}$ and $\alpha \in S_i$. For each such pair $(\alpha, i)$, let $a^i_\alpha = (f^i_\alpha)^{-1}(\{1\})$. For each $\alpha \in S_0$, define

$$b^0_\alpha = (\pi^*)^{-1}(a_\alpha) \cap \big((\pi^*)^{-1}(\alpha) \setminus \alpha\big),$$

and for each $\alpha \in S_1$, set

$$b^1_\alpha = \pi^*(a_\alpha) \cap (\pi^*(\alpha) \setminus \alpha).$$

Notice that, for each $i \in \{0,1\}$, we have $b^i_\alpha \cap b^i_\beta = \emptyset$ for distinct $\alpha, \beta \in S_i$.

Let $B_i = \bigcup\{b^i_\alpha \mid \alpha \in S_i\}$ for each $i < 2$. Define $A_0 = \pi^*(B_0)$ and $A_1 = (\pi^*)^{-1}(B_1)$. For each $i \in \{0,1\}$, let $F_i$ be the characteristic function of $A_i$. If $\alpha \in S_0$, then

$$B_0 \cap \big((\pi^*)^{-1}(\alpha) \setminus \alpha\big) = b^0_\alpha,$$

hence $A_0 \cap L_\alpha \sim_\kappa a_\alpha$. Similarly, if $\alpha \in S_1$, then

(2) See Subsection 1.1 for the meaning of the order $<$ in this context.

$$B_1 \cap (\pi^*(\alpha) \setminus \alpha) = b^1_\alpha,$$

so that $A_1 \cap L_\alpha \sim_\kappa a_\alpha$. It follows that $F_0 \restriction L_\alpha \sim_\kappa f_\alpha$ for all $\alpha \in S_0$, and $F_1 \restriction L_\alpha \sim_\kappa f_\alpha$ for all $\alpha \in S_1$. ∎

The following theorem of Devlin and Shelah then shows that the existence of an automorphism of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ without nontrivial ordinal fixed points implies that $2^{\aleph_0} = 2^{\aleph_1}$ (3).

THEOREM 7.2 (Devlin–Shelah [5]). *Suppose $\{S_\alpha : \alpha < \omega_1\}$ is such that:*

- *each $S_\alpha$ is a subset of $\omega_1$ supporting a ladder system satisfying 2-uniformization,*
- *the diagonal union of $\{S_\alpha \mid \alpha < \omega_1\}$ contains a club subset of $\omega_1$.*

*Then $2^{\aleph_0} = 2^{\aleph_1}$.*

After a simple modification of $\pi^*$, we see that $\pi^*$ moves the ladders $L_\alpha$ for $\alpha \in S_0$ to disjoint sets; and $(\pi^*)^{-1}$ moves $L_\alpha$ for $\alpha \in S_1$ to disjoint sets. It follows that they satisfy uniformization properties stronger than 2-uniformization (but not comparable, as far as we know, with $\kappa$-uniformization). For instance, they each satisfy the following property: for any partition of $S_i$ into sets $\{T_\alpha \mid \alpha < \gamma\}$ (for some $\gamma \leq \kappa^+$) there exist sets $\{K_\alpha \mid \alpha < \gamma\}$ such that:

- for all $\alpha < \gamma$ and all $\beta \in T_\alpha$, $|L_\beta \setminus K_\alpha| < \kappa$,
- for every sequence of functions $f_\alpha : K_\alpha \to 2$ $(\alpha < \gamma)$ there exists a function $F : \kappa^+ \to 2$ such that $F \restriction K_\alpha \sim_\kappa f_\alpha$ for each $\alpha < \gamma$.

**8. Open questions.** We collect here various open questions related to the material in this paper, some of which have been mentioned above, and some of which have been posed by others. First, we ask for various types of automorphisms.

QUESTION 1. Are any of the following consistent with ZFC?

(a) There exist an uncountable cardinal $\lambda$ and an automorphism of $\mathcal{P}(\lambda)/\mathtt{Fin}$ which is not trivial on any cocountable set.
(b) There exist an uncountable cardinal $\lambda$ and an automorphism of $\mathcal{P}(\lambda)/\mathtt{Fin}$ which is not trivial on any uncountable set. (By [16], $\lambda$ would have to be at most $2^{\aleph_0}$.)
(c) There exist an infinite cardinal $\kappa$ and a nontrivial automorphism of $\mathcal{P}(\kappa^+)/\mathcal{I}_\kappa$ which is trivial on all sets of cardinality $\kappa$.
(d) There exists an infinite cardinal $\kappa$ such that all automorphisms of $\mathcal{P}(\kappa)/\mathcal{I}_\kappa$ are trivial, but there is a nontrivial automorphism of $\mathcal{P}(\kappa^+)/\mathcal{I}_\kappa$.

(3) If $2^{\aleph_0} < 2^{\aleph_1}$, then all automorphisms of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are cardinality-preserving.

(e) There exist infinite cardinals $\kappa < \lambda$ and a nontrivial automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_{\kappa^+}$ which is trivial on all sets of cardinality $\kappa^+$. (By Theorem 3.4, $\lambda$ would have to be bigger than $2^\kappa$.)
(f) There exists an automorphism of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ with no nontrivial fixed points. (If this holds, then $2^{\aleph_0} = 2^{\aleph_1}$, by Theorems 7.1 and 7.2.)
(g) There exist uncountable cardinals $\kappa \leq \lambda$ and a nontrivial automorphism of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$. (What if $\kappa = \aleph_1$?)
(h) There exists an uncountable cardinal $\lambda$ and an outer automorphism of the Calkin algebra on the Hilbert space of dimension $\lambda$. (See Remark 5.7 or [6] for definitions and more information.)

We also ask about the Katowice Problem, Question 2(a) below, and its relation to automorphisms. The reader is referred to [8, 3] for more on the Katowice Problem and related questions. We note in particular that, in [3], Chodounský has constructed a model of ZFC where most of the known consequences of a positive answer to Question 2(a) hold.

Question 2.

(a) (Turzański) Is it consistent with ZFC that the Boolean algebras $\mathcal{P}(\omega)/\mathtt{Fin}$ and $\mathcal{P}(\omega_1)/\mathtt{Fin}$ are isomorphic?
(b) Is it consistent with ZFC that there is an isomorphism from $\mathcal{P}(\omega_1)/\mathtt{Fin}$ to $\mathcal{P}(\omega)/\mathtt{Fin}$ which is trivial on all countable sets?
(c) Is it consistent with ZFC that there exists an automorphism $\pi$ of $\mathcal{P}(\omega_1)/\mathtt{Fin}$ such that $\pi([A]) = [B]$ for no infinite $A \subset B \subseteq \omega_1$ with $\omega_1 \setminus B$ infinite?
(d) Does the existence of an isomorphism $\mathcal{P}(\omega)/\mathtt{Fin} \simeq \mathcal{P}(\omega_1)/\mathtt{Fin}$ imply that there is a nontrivial automorphism of $\mathcal{P}(\omega_1)/\mathtt{Ctble}$? (Since such an isomorphism implies there is an uncountable $Q$-set, by Theorem 5.6 it is enough to ask whether such an isomorphism implies there exist two uncountable $Q_B$-sets, with uncountable difference, which intersect the same Borel sets uncountably.)
(e) Does the existence of an isomorphism between $\mathcal{P}(\omega)/\mathtt{Fin}$ and $\mathcal{P}(\omega_1)/\mathtt{Fin}$ imply that $\mathfrak{z} = \aleph_1$?

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Paul Larson, Paul McKenney
Department of Mathematics
Miami University
Oxford, OH 45056, U.S.A.
E-mail: larsonpb@miamioh.edu
mckennp2@miamioh.edu