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Actions of the group of homeomorphisms of the circle on surfaces

by

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Abstract. We describe all the group morphisms from the group of orientation-preserving homeomorphisms of the circle to the group of homeomorphisms of the annulus or of the torus.

1. Introduction. For a compact manifold M, we denote by $\mathrm{Homeo}(M)$ the group of homeomorphisms of M and by $\mathrm{Homeo}_0(M)$ the connected component of the identity of this group (for the compact-open topology). By a theorem by Fischer (see [6] and [3]), the group $\mathrm{Homeo}_0(M)$ is simple: the study of this group cannot be reduced to the study of other groups. One natural way to have a better understanding of this group is to look at its automorphism group. In this direction, Whittaker proved the following theorem.

THEOREM (Whittaker [16]). *Given two compact manifolds M and N and any group isomorphism $\varphi : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$, there exists a homeomorphism $h : M \to N$ such that φ is the map $f \mapsto h \circ f \circ h^{-1}$.*

This theorem was generalized by Filipkiewicz [5] to groups of diffeomorphisms by using a powerful theorem by Montgomery and Zippin which characterizes Lie groups among locally compact groups. The idea of the proof of Whittaker's theorem is the following: We see the manifold M algebraically by considering the subgroup $G_x \subset \mathrm{Homeo}(M)$ consisting of the homeomorphisms which fix the point x in M. One proves that, for any x in M, there exists a unique y_x in N such that $\varphi(G_x)$ is the group of homeomorphisms of N which fix the point y_x. Define h by $h(x) = y_x$. Then we check that h is a homeomorphism and that it satisfies the conclusion of the theorem. However, Whittaker's proof crucially uses the fact that we have a

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group isomorphism and a priori cannot be easily generalized to the cases of group morphisms. Here is a conjecture in this case.

Conjecture 1.1. *For a compact manifold M, every group morphism* $\mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(M)$ *is either trivial or induced by conjugacy by a homeomorphism.*

This conjecture is confirmed in [12] in the case of the circle. It is also proved in [9] in the case of groups of diffeomorphisms of the circle. We may also be interested in morphisms from $\mathrm{Homeo}_0(M)$ to $\mathrm{Homeo}_0(N)$, for another manifold N. This kind of questions are addressed in [9] for diffeomorphisms in the case where N is a circle or the real line. The following conjecture looks attainable.

Conjecture 1.2. *Let S be a closed surface different from the sphere. Denote by $S - D$ the closed surface S with one open disc removed. Denote by* $\mathrm{Homeo}_0(S - D)$ *the identity component of the group of homeomorphisms of $S - D$ with support contained in the interior of this surface. Every group morphism from* $\mathrm{Homeo}_0(S - D)$ *to* $\mathrm{Homeo}_0(S)$ *is induced by an embedding of $S - D$ in S.*

This conjecture is a consequence of a remarkable result by Hurtado [8] in the case of diffeomorphism groups, but remains open in the case of homeomorphism groups.

In this article, we investigate the case of group morphisms from the group $\mathrm{Homeo}_0(\mathbb{S}^1)$, which is also the group of orientation-preserving homeomorphisms of the circle, to the group $\mathrm{Homeo}(S)$ of homeomorphisms of a compact orientable surface S. By simplicity of $\mathrm{Homeo}_0(\mathbb{S}^1)$, such a morphism φ is either one-to-one or trivial. Moreover, as $\mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$ is countable, any morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$ is trivial. Hence, the image of φ is contained in $\mathrm{Homeo}_0(S)$. By a theorem by Rosendal and Solecki, any group morphism from the group of orientation-preserving homeomorphisms of the circle to a separable group is continuous (see [15, Theorem 4 and Proposition 2]): the group morphisms under consideration are continuous. If the surface S is different from the sphere, the torus, the closed disc or the closed annulus, then any compact subgroup of $\mathrm{Homeo}_0(S)$ is trivial by a theorem by Kerékjártó (see [2]). Therefore, $\mathrm{Homeo}_0(S)$ does not contain a subgroup isomorphic to $\mathrm{SO}(2)$ whereas $\mathrm{Homeo}_0(\mathbb{S}^1)$ does and the morphism φ is necessarily trivial. In what follows, we study the remaining cases.

In the second section, we state our classification theorem for actions of $\mathrm{Homeo}_0(\mathbb{S}^1)$ on the torus and on the annulus. Any action will be obtained by gluing actions which preserve a lamination by circles and actions which are transitive on open annuli. In the third section, we describe the continuous actions of the group of compactly-supported homeomorphisms of the real line on the real line or on the circle; this description is useful for the classification theorem and interesting in its own right. The fourth and fifth sections are devoted respectively to the proof of the classification theorem in the case of the closed annulus and in the case of the torus. Finally, in the last section we discuss the case of the sphere and of the closed disc.

2. Description of the actions. All the actions of $\mathrm{Homeo}_0(\mathbb{S}^1)$ will be obtained by gluing elementary actions of this group on the closed annulus. In this section, we will first describe these elementary actions before describing some model actions to which any continuous action will be conjugate.

The easiest action of $\mathrm{Homeo}_0(\mathbb{S}^1)$ on the closed annulus $\mathbb{A} = [0,1] \times \mathbb{S}^1$ preserves the foliation by circles of this annulus:

$$p : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{A}), \quad f \mapsto \big((r,\theta) \mapsto (r, f(\theta))\big).$$

The second elementary action we want to describe is a little more complex. Let $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ be the projection. For $\theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, we denote by $\tilde{\theta}$ a lift of θ, i.e. a point of $\mathbb{R}$ which projects on θ. For a homeomorphism f of the circle, we denote by $\tilde{f}$ a lift of f, i.e. an element of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of $\mathbb{R}$ which commute with integral translations so that $\pi \circ \tilde{f} = f \circ \pi$. The second elementary action is given by

$$a_- : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{A}), \quad f \mapsto \big((r,\theta) \mapsto (\tilde{f}(\tilde{\theta}) - \tilde{f}(\tilde{\theta} - r), f(\theta))\big).$$

Notice that the number $\tilde{f}(\tilde{\theta}) - \tilde{f}(\tilde{\theta} - r)$ does not depend on the lifts $\tilde{\theta}$ and $\tilde{f}$ chosen and belongs to $[0,1]$. Notice also that the map

$$[0,1] \to [0,1], \quad r \mapsto \tilde{f}(\tilde{\theta}) - \tilde{f}(\tilde{\theta} - r),$$

is a homeomorphism.

An analogous action is given by

$$a_+ : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{A}), \quad f \mapsto \big((r,\theta) \mapsto (\tilde{f}(\tilde{\theta} + r) - \tilde{f}(\tilde{\theta}), f(\theta))\big).$$

Notice that a_+ and a_- are conjugate via the homeomorphism of $\mathbb{A}$ given by $(r,\theta) \mapsto (1-r,\theta)$, which is orientation-reversing, and via the orientation-preserving homeomorphism of $\mathbb{A}$ given by $(r,\theta) \mapsto (r,\theta+r)$. We now describe another way to see the action a_- (and a_+). We see the torus $\mathbb{T}^2$ as the product $\mathbb{S}^1 \times \mathbb{S}^1$. Let

$$a_{\mathbb{T}^2} : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{T}^2), \quad f \mapsto \big((x,y) \mapsto (f(x), f(y))\big).$$

It is easily checked that this defines a morphism. This action leaves the diagonal $\{(x,x) \mid x \in \mathbb{S}^1\}$ invariant. The action obtained by cutting along the diagonal is conjugate to a_-. More precisely, define

$$h : \mathbb{A} \to \mathbb{T}^2, \quad (r,\theta) \mapsto (\theta, \theta - r).$$

Then, for any f in $\mathrm{Homeo}_0(\mathbb{S}^1)$,

$$h \circ a_-(f) = a_{\mathbb{T}^2}(f) \circ h.$$

Let G_{θ_0} be the group of homeomorphisms of the circle which fix a neighbourhood of the point θ_0. For a point θ_0 of the circle, the image by a_- of G_{θ_0} leaves the sets $\{(r,\theta_0) \mid r \in [0,1]\}$ and $\{(r,\theta_0+r) \mid r \in [0,1]\}$ globally invariant. On each connected component of the complement of the union of these two sets with the boundary of the annulus, the action of G_{θ_0} is transitive. This is most easily seen by using the action $a_{\mathbb{T}^2}$ and the fact that, on $\mathbb{S}^1 - \{\theta_0\}$, the group G_{θ_0} is transitive on pairs of points (x,y) such that $x < y$ (where the order is induced by the orientation of the circle).

Let us now describe the model actions on the closed annulus which are obtained by gluing the above actions. Take a non-empty compact subset $K \subset [0,1]$ which contains 0 and 1, and a map $\lambda : [0,1] - K \to \{-1,1\}$ which is constant on each connected component of $[0,1]-K$. Let us now define an action $\varphi_{K,\lambda}$ of $\mathrm{Homeo}_0(\mathbb{S}^1)$ on the closed annulus $\mathbb{A}$. To f in $\mathrm{Homeo}_0(\mathbb{S}^1)$, we associate a homeomorphism $\varphi_{K,\lambda}(f)$ of $\mathbb{A}$ defined as follows. If $r \in K$, we associate to $(r,\theta) \in \mathbb{A}$ the point $(r, f(\theta))$. If r belongs to a connected component (r_1,r_2) of the complement of K and if $\lambda((r_1,r_2)) = \{-1\}$, we associate to $(r,\theta) \in \mathbb{A}$ the point

$$\left((r_2-r_1)\left(\tilde{f}(\theta) - \tilde{f}\left(\theta - \frac{r-r_1}{r_2-r_1}\right)\right) + r_1, f(\theta)\right).$$

This last map is obtained by conjugating the homeomorphism $a_-(f)$ with the map $(r,\theta) \mapsto (\zeta(r),\theta)$ where ζ is the unique linear orientation-preserving homeomorphism $[0,1] \to [r_1,r_2]$. If r belongs to a connected component (r_1,r_2) of the complement of K and if $\lambda((r_1,r_2)) = \{1\}$, we associate to $(r,\theta) \in \mathbb{A}$ the point

$$\left((r_2-r_1)\left(\tilde{f}\left(\theta + \frac{r-r_1}{r_2-r_1}\right) - \tilde{f}(\theta)\right) + r_1, f(\theta)\right).$$

This last map is also obtained after renormalizing $a_+(f)$ on the interval (r_1,r_2). This defines a continuous morphism $\varphi_{K,\lambda} : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{A})$. To construct an action on the torus, it suffices to identify the point $(0,\theta)$ of $\mathbb{A}$ with $(1,\theta)$. We denote by $\varphi^{\mathbb{T}^2}_{K,\lambda}$ the continuous action on the torus obtained this way. By shrinking one of the boundary components (respectively both boundary components) of the annulus to a point (respectively to points), one obtains an action on the closed disc (respectively on the sphere) that we denote by $\varphi^{\mathbb{D}^2}_{K,\lambda}$ (respectively $\varphi^{\mathbb{S}^2}_{K,\lambda}$).

The main theorem of this article is the following:

Theorem 2.1. *Any non-trivial action of the group* $\mathrm{Homeo}_0(\mathbb{S}^1)$ *on the closed annulus is conjugate to one of the actions* $\varphi_{K,\lambda}$. *Any non-trivial action of* $\mathrm{Homeo}_0(\mathbb{S}^1)$ *on the torus is conjugate to one of the actions* $\varphi^{\mathbb{T}^2}_{K,\lambda}$.

In particular, any action of $\mathrm{Homeo}_0(\mathbb{S}^1)$ on the torus admits an invariant circle. By analogy with this theorem, one is tempted to formulate the following conjecture:

Conjecture 2.2. *Any non-trivial action of the group* $\mathrm{Homeo}_0(\mathbb{S}^1)$ *on the sphere* (*respectively on the closed disc*) *is conjugate to one of the actions* $\varphi^{\mathbb{S}^2}_{K,\lambda}$ (*respectively* $\varphi^{\mathbb{D}^2}_{K,\lambda}$).

Notice that this theorem does not directly describe the conjugacy classes of such actions, as two actions $\varphi_{K,\lambda}$ and $\varphi_{K',\lambda'}$ may be conjugate even though $K \neq K'$ or $\lambda \neq \lambda'$. Now, let $K \subset [0,1]$ and $K' \subset [0,1]$ be two compact sets which contain $\{0,1\}$. Let $\lambda : [0,1] - K \to \{-1,1\}$ and $\lambda' : [0,1] - K' \to \{-1,1\}$ be constant on each connected component of their domains of definition. The following theorem characterizes when the actions $\varphi_{K,\lambda}$ and $\varphi_{K',\lambda'}$ are conjugate.

Proposition 2.3. *The following statements are equivalent:*

- *the actions* $\varphi_{K,\lambda}$ *and* $\varphi_{K',\lambda'}$ *are conjugate;*
- *either there exists an orientation-preserving homeomorphism* $h : [0,1] \to [0,1]$ *such that* $h(K) = K'$ *and* $\lambda' \circ h = \lambda$ *except on a finite number of connected components of* $[0,1] - K$, *or there exists a decreasing homeomorphism* $h : [0,1] \to [0,1]$ *such that* $h(K) = K'$ *and* $\lambda' \circ h = -\lambda$ *except on a finite number of connected components of* $[0,1] - K$.

Proof. We begin by proving that the second statement implies the first.

If there exists an orientation-preserving homeomorphism h which maps K onto K', then the actions $\varphi_{K,\lambda}$ and $\varphi_{K',\lambda \circ h^{-1}}$ are conjugate. Indeed, denote by $\hat{h}$ the homeomorphism $[0,1] \to [0,1]$ which coincides with h on K and which, on each connected component (r_1, r_2) of the complement of K, is the unique orientation-preserving linear homeomorphism $(r_1, r_2) \to h((r_1, r_2))$. Then the actions $\varphi_{K,\lambda}$ and $\varphi_{K',\lambda\circ h^{-1}}$ are conjugate via the homeomorphism

$$\mathbb{A} \to \mathbb{A}, \qquad (r, \theta) \mapsto (\hat{h}(r), \theta).$$

Similarly, suppose that h is an orientation-reversing homeomorphism of $[0,1]$ which maps K onto K'. As above, we denote by $\hat{h}$ the homeomorphism obtained from h by mapping linearly each connected component of the complement of K onto a connected component of the complement of K'. Then the actions $\varphi_{K,\lambda}$ and $\varphi_{K',-\lambda\circ h^{-1}}$ are conjugate via the homeomorphism

$$\mathbb{A} \to \mathbb{A}, \qquad (r, \theta) \mapsto (\hat{h}(r), \theta).$$

It then suffices to use the following lemma to complete the proof of the converse in the proposition.

Lemma 2.4. *Let $K \subset [0,1]$ be a compact subset which contains 0 and 1. If $\lambda : [0,1] - K \to \{-1,1\}$ and $\lambda' : [0,1] - K \to \{-1,1\}$ are continuous maps which are equal except on one connected component (r_1, r_2) of the complement of K, then the actions $\varphi_{K,\lambda}$ and $\varphi_{K,\lambda'}$ are conjugate.*

Proof. This comes from the fact that the actions a_+ and a_- are conjugate via the homeomorphism of the annulus $(r,\theta) \mapsto (r, \theta + r)$. More precisely, suppose that $\lambda((r_1,r_2)) = \{1\}$. Then $\varphi_{K,\lambda}$ and $\varphi_{K,\lambda'}$ are conjugate via the homeomorphism h equal to the identity on the Cartesian product of the complement of (r_1,r_2) with the circle, and equal to $(r,\theta) \mapsto \big(r, \theta + \frac{r-r_1}{r_2-r_1}\big)$ on $(r_1,r_2) \times \mathbb{S}^1$. ∎

Let us now establish the other implication. Suppose that there exists a homeomorphism g which conjugates the actions $\varphi_{K,\lambda}$ and $\varphi_{K',\lambda'}$. Now, for any angle α, if we denote by R_α the rotation of angle α, then

$$g \circ \varphi_{K,\lambda}(R_\alpha) = \varphi_{K',\lambda'}(R_\alpha) \circ g.$$

The homeomorphism $\varphi_{K,\lambda}(R_\alpha)$ is the rotation of angle α of $\mathbb{A}$. This means that, for any point $(r,\theta) \in \mathbb{A}$ and any $\alpha \in \mathbb{S}^1$,

$$g((r,\theta+\alpha)) = g((r,\theta)) + (0,\alpha).$$

In particular, g permutes the leaves of the foliation of $\mathbb{A}$ by circles $\{r\} \times \mathbb{S}^1$. Fix now $\theta_0 \in \mathbb{S}^1$. Then g maps the set $K \times \{\theta_0\}$ of fixed points of $\varphi_{K,\lambda}(G_{\theta_0})$ (that is, fixed under every element of this group) to the set $K' \times \{\theta_0\}$ of fixed points of $\varphi_{K',\lambda'}(G_{\theta_0})$. From this and the above, we deduce that for any $r \in K$ and any angle θ, we have $g(r,\theta) = (h(r),\theta)$, where $h : K \to K'$ is a homeomorphism. Moreover, if g is orientation-preserving, then so is h, and if g is orientation-reversing, then so is h. We can extend h to a homeomorphism of $[0,1]$ which maps K onto K'. Notice that, for a connected component (r_1,r_2) of the complement of K in $[0,1]$, the homeomorphism g maps $(r_1,r_2) \times \mathbb{S}^1$ onto $(r_1',r_2') \times \mathbb{S}^1$, where $(r_1',r_2') = h((r_1,r_2))$ is a connected component of the complement of K' in $[0,1]$.

It suffices now to establish that the condition on the maps λ and λ' is satisfied for the homeomorphism h. Suppose, to simplify the proof, that g, and hence h, is orientation-preserving; the case where g is orientation-reversing can be treated similarly. Suppose for contradiction that there exists a sequence $((r_{1,n}, r_{2,n}))_{n\in\mathbb{N}}$ of connected components of $[0,1] - K$ such that:

- $\lambda((r_{1,n}, r_{2,n})) = 1$;
- $\lambda'((r_{1,n}', r_{2,n}')) = -1$, where $(r_{1,n}', r_{2,n}') = h((r_{1,n}, r_{2,n}))$;
- the sequence $(r_{1,n})_{n\in\mathbb{N}}$ is monotone and converges to a real number r_∞.

We will prove that either the curve $\{(r,\theta_0) \mid r > r_\infty\}$ (if the sequence $(r_{1,n})_{n\in\mathbb{N}}$ is decreasing) or $\{(r,\theta_0) \mid r < r_\infty\}$ (if $(r_{1,n})_{n\in\mathbb{N}}$ is increasing) is mapped by g onto a curve which accumulates on $\{r_\infty\} \times \mathbb{S}^1$, which is impossible. The hypotheses $\lambda((r_{1,n}, r_{2,n})) = -1$ and $\lambda((r'_{1,n}, r'_{2,n})) = 1$ for any n would lead to the same contradiction.

To achieve this, it suffices to prove that, for any positive integer n, the homeomorphism g maps $\{(r, \theta_0) \mid r_{1,n} < r < r_{2,n}\}$ onto $\big\{\big(r, \theta_0 + \frac{r-r'_{1,n}}{r'_{2,n}-r'_{1,n}}\big) \mid r'_{1,n} < r < r'_{2,n}\big\}$. Observe that the restriction of the group $\varphi_{K,\lambda}(G_{\theta_0})$ of homeomorphisms to $(r_{1,n}, r_{2,n}) \times \mathbb{S}^1$ has two invariant simple curves, $\{(r, \theta_0) \mid r_{1,n} < r < r_{2,n}\}$ and $\big\{\big(r, \theta_0 - \frac{r-r_{1,n}}{r_{2,n}-r_{1,n}}\big) \mid r_{1,n} < r < r_{2,n}\big\}$, on which the action is transitive. The action of this group is also transitive on the two connected components of the complement of these two curves, which are open sets. Likewise, the restriction of the group $\varphi_{K',\lambda'}(G_{\theta_0})$ to $(r'_{1,n}, r'_{2,n})$ has two invariant simple curves,

$$\{(r, \theta_0) \mid r'_{1,n} < r < r'_{2,n}\} \quad \text{and} \quad \left\{\left(r, \theta_0 + \frac{r - r'_{1,n}}{r'_{2,n} - r'_{1,n}}\right) \,\middle|\, r'_{1,n} < r < r'_{2,n}\right\},$$

on which the action is transitive, and the action of this group is transitive on the two connected components of the complement of these two curves, which are open sets. Therefore, g maps $\{(r, \theta_0) \mid r_{1,n} < r < r_{2,n}\}$ onto one of the displayed curves; let us find now onto which one.

We fix the orientation of the circle induced by the orientation of $\mathbb{R}$ and the covering map $\mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. This orientation gives rise to an order on $\mathbb{S}^1 - \{\theta_0\}$. Take f in G_{θ_0} different from the identity such that, for any $x \neq \theta_0$, we have $f(x) \geq x$. Then, for any $r \in (r_{1,n}, r_{2,n})$,

$$p_1 \circ \varphi_{K,\lambda}(f)(r, \theta_0) \geq r,$$

where $p_1 : \mathbb{A} = [0,1] \times \mathbb{S}^1 \to [0,1]$ is the projection, and the restriction of $\varphi_{K,\lambda}(f)$ to $\{(r, \theta_0) \mid r_{1,n} < r < r_{2,n}\}$ is different from the identity. Likewise, for any $r \in (r'_{1,n}, r'_{2,n})$,

$$p_1 \circ \varphi_{K',\lambda'}(f)(r, \theta_0) \leq r, \qquad p_1 \circ \varphi_{K',\lambda'}(f)\left(r, \theta_0 + \frac{r - r'_{1,n}}{r'_{2,n} - r'_{1,n}}\right) \geq r$$

and the restrictions of $\varphi_{K',\lambda'}(f)$ to $\{(r, \theta_0) \mid r'_{1,n} < r < r'_{2,n}\}$ and to $\big\{\big(r, \theta_0 + \frac{r-r'_{1,n}}{r'_{2,n}-r'_{1,n}}\big) \mid r'_{1,n} < r < r'_{2,n}\big\}$ are different from the identity. Moreover, the map

$$(r_{1,n}, r_{2,n}) \to (r'_{1,n}, r'_{2,n}), \qquad r \mapsto p_1 \circ g(r, \theta_0),$$

is strictly increasing, as g was supposed to be orientation-preserving. This implies what we wanted to prove. ∎

3. Continuous actions of $\mathrm{Homeo}_c(\mathbb{R})$ **on the line.** Let $\mathrm{Homeo}_c(\mathbb{R})$ denote the group of compactly supported homeomorphisms of $\mathbb{R}$. In this section, we prove the following theorem.

Theorem 3.1. *Let* $\psi : \mathrm{Homeo}_c(\mathbb{R}) \to \mathrm{Homeo}(\mathbb{R})$ *be a continuous group morphism whose image has no fixed point. Then there exists a homeomorphism* h *of* $\mathbb{R}$ *such that, for any* $f \in \mathrm{Homeo}_c(\mathbb{R})$,

$$\psi(f) = h \circ f \circ h^{-1}.$$

Remark 3.2. This theorem is true without the continuity hypothesis. However, we just need the above statement in this article.

Remark 3.3. This theorem also holds in the case of groups of diffeomorphisms, i.e. any continuous action of the group of compactly supported C^r diffeomorphisms on the real line is topologically conjugate to the inclusion. The proof in this case is the same.

This theorem enables us to describe any continuous action of the group $\mathrm{Homeo}_c(\mathbb{R})$ on the real line, as it suffices to consider the action on each connected component of the complement of the fixed point set of the action.

Proof of Theorem 3.1. During this proof, for a subset $A \subset \mathbb{R}$, we will denote by G_A the group of compactly supported homeomorphisms which pointwise fix a neighbourhood of A, and by $F_A \subset \mathbb{R}$ the closed set of fixed points of $\psi(G_A)$ (that is, fixed under every element of this group). Let us begin by sketching the proof. We will first prove that, for every non-trivial compact interval I (that is, one with non-empty interior), the set F_I is compact and non-empty. As a consequence, for any real x, the closed set F_x is non-empty. Then we will prove that each F_x is a single point. Once this is proved, we define $h : \mathbb{R} \to \mathbb{R}$ by $\{h(x)\} = F_x$. Then h is a homeomorphism as required. Let us give the details.

Notice that, as the group $\mathrm{Homeo}_c(\mathbb{R})$ is simple (see [6]), the image of ψ is contained in the group of orientation-preserving homeomorphisms of $\mathbb{R}$ (otherwise there would exist a non-trivial group morphism $\mathrm{Homeo}_c(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$) and ψ is one-to-one.

Lemma 3.4. *For every non-trivial compact interval* $I \subset \mathbb{R}$*, the closed set* F_I *is non-empty.*

Proof. Take a non-zero vector field $X : \mathbb{R} \to \mathbb{R}$ supported in I. The flow of X defines a morphism

$$\mathbb{R} \to \mathrm{Homeo}_c(\mathbb{R}), \qquad t \mapsto \varphi^t.$$

Assume for the moment that the set F of fixed points of the subgroup $\{\psi(\varphi^t) \mid t \in \mathbb{R}\}$ of $\mathrm{Homeo}(\mathbb{R})$ is non-empty. Notice that this set is not $\mathbb{R}$ as ψ is one-to-one. Since each φ^t commutes with any element in G_I, for any g in $\psi(G_I)$ we obtain $g(F) = F$. Moreover, as any element of G_I can be joined to the identity by a continuous path in G_I, and ψ is continuous, any connected component of F is invariant under $\psi(G_I)$. The upper and lower extremities of these intervals which lie in $\mathbb{R}$ are then fixed points of the group $\psi(G_I)$. This proves the lemma.

It remains to prove that the set F is non-empty. Suppose otherwise. Then, for any real x, the map

$$\mathbb{R} \to \mathbb{R}, \quad t \mapsto \psi(\varphi^t)(x),$$

is a homeomorphism. Indeed, if it were not onto, the supremum or the infimum of the image would provide a fixed point for $(\psi(\varphi^t))_{t\in\mathbb{R}}$. If it were not one-to-one, there would exist $t_0 \neq 0$ such that $\psi(\varphi^{t_0})(x) = x$. Then, for any positive integer n, $\psi(\varphi^{t_0/2^n})(x) = x$, and by continuity of ψ, for any real t, $\psi(\varphi^t)(x) = x$ and x would be fixed under $(\psi(\varphi^t))_{t\in\mathbb{R}}$.

Fix a real number x_0. Let $T_{x_0} : G_I \to \mathbb{R}$ be defined by

$$\psi(f)(x_0) = \psi(\varphi^{T_{x_0}(f)})(x_0).$$

The map T_{x_0} is a group morphism as, for any f and g in G_I,

$$\begin{aligned}\psi(\varphi^{T_{x_0}(fg)})(x_0) &= \psi(fg)(x_0) = \psi(f)\psi(\varphi^{T_{x_0}(g)})(x_0)\\ &= \psi(\varphi^{T_{x_0}(g)})\psi(f)(x_0) = \psi(\varphi^{T_{x_0}(g)+T_{x_0}(f)})(x_0).\end{aligned}$$

However, G_I, which is isomorphic to $\mathrm{Homeo}_c(\mathbb{R}) \times \mathrm{Homeo}_c(\mathbb{R})$, is a perfect group: any element of it can be written as a product of commutators. Therefore, the morphism T_{x_0} is trivial. As x_0 is any point in $\mathbb{R}$, we deduce that the restriction of ψ to G_I is trivial, which is impossible as ψ is one-to-one. ∎

REMARK 3.5. If $\psi(\mathrm{Homeo}_c(\mathbb{R})) \subset \mathrm{Homeo}_c(\mathbb{R})$, this lemma can be proved without the continuity hypothesis. Indeed, let f in $\mathrm{Homeo}_c(\mathbb{R})$ be supported in I. One of the connected components of the set of fixed points of $\psi(f)$ is of the form $(-\infty, a]$ for some a in $\mathbb{R}$. This interval is necessarily invariant under the group $\psi(G_I)$ which commutes with f. Hence, a is a fixed point for the group $\psi(G_I)$.

In the proof, we will often use the following elementary result:

LEMMA 3.6. *Let I and J be disjoint compact non-empty intervals. For any g in* $\mathrm{Homeo}_c(\mathbb{R})$*, there exist $g_1 \in G_I$, $g_2 \in G_J$ and $g_3 \in G_I$ such that*

$$g = g_1 g_2 g_3.$$

Proof. Let $g \in \mathrm{Homeo}_c(\mathbb{R})$. Let $h_1 \in G_I$ map the interval $g(I)$ onto an interval in the same connected component of $\mathbb{R} - J$ as I. Let $h_2 \in G_J$ be equal to $g^{-1} \circ h_1^{-1}$ on a neighbourhood of $h_1 \circ g(I)$. Then $h_2 \circ h_1 \circ g \in G_I$. It then suffices to take $g_1 = h_1^{-1}$, $g_2 = h_2^{-1}$ and $g_3 = h_2 \circ h_1 \circ g$. ∎

Before stating the next lemma, observe that, for non-trivial compact intervals I, the sets F_I are pairwise homeomorphic by an orientation-preserving homeomorphism. Indeed, let I and J be two such intervals. Then there exists λ in $\mathrm{Homeo}_c(\mathbb{R})$ such that $\lambda(I) = J$. Then $\lambda G_I \lambda^{-1} = G_J$. Taking the image under ψ, we obtain $\psi(\lambda)\psi(G_I)\psi(\lambda)^{-1} = \psi(G_J)$ and therefore $\psi(\lambda)(F_I) = F_J$.

LEMMA 3.7. *For every non-trivial compact interval $I \subset \mathbb{R}$, the closed set F_I is compact.*

Proof. Suppose for contradiction that there exists a sequence $(a_k)_{k\in\mathbb{N}}$ in F_I which tends to ∞ (if we suppose that it tends to $-\infty$, we obtain an analogous contradiction). Choose a compact interval J disjoint from I. By the remark before the lemma, there exists a sequence $(b_k)_{k\in\mathbb{N}}$ in F_J which tends to ∞. Take positive integers n_1, n_2 and n_3 such that $a_{n_1} < b_{n_2} < a_{n_3}$. Fix $x_0 < a_{n_1}$. Then for any $g_1 \in G_I$, $g_2 \in G_J$ and $g_3 \in G_I$,

$$\psi(g_1)\psi(g_2)\psi(g_3)(x_0) < a_{n_3}.$$

Lemma 3.6 implies that

$$\overline{\{\psi(g)(x_0) \mid g \in \mathrm{Homeo}_c(\mathbb{R})\}} \subset (-\infty, a_{n_3}].$$

The greatest element of the left-hand set is a fixed point of the image of ψ; but this is impossible as this image was supposed to have no fixed point. ∎

LEMMA 3.8. *The closed sets F_x, where $x \in \mathbb{R}$, are non-empty, compact, pairwise disjoint and have empty interior.*

Proof. Notice that if an interval J is contained in an interval I, then $G_I \subset G_J$ and so $\psi(G_I) \subset \psi(G_J)$. Therefore $F_J \subset F_I$. Now, for any finite family $(I_n)_n$ of intervals whose intersection has non-empty interior, the intersection of the closed sets F_{I_n} contains the non-empty closed set $F_{\bigcap_n I_n}$. Fix now x in $\mathbb{R}$. Notice that

$$F_x = \bigcap_I F_I,$$

where the intersection is taken over all compact intervals I whose interior contains x. By compactness, this set is not empty and it is compact.

Take $x \neq y$ in $\mathbb{R}$. If F_x and F_y were not disjoint, the group generated by $\psi(G_x)$ and $\psi(G_y)$ would have a fixed point p. However, by Lemma 3.6, the groups G_x and G_y generate $\mathrm{Homeo}_c(\mathbb{R})$. Hence, p would be a fixed point of the group $\psi(\mathrm{Homeo}_c(\mathbb{R}))$, a contradiction.

Now, let us prove that each F_x has empty interior. Of course, given $x, y \in \mathbb{R}$, if h in $\mathrm{Homeo}_c(\mathbb{R})$ sends x to y, then $\psi(h)(F_x) = F_y$. Therefore, the sets F_x are pairwise homeomorphic. If they had non-empty interiors, there would exist uncountably many pairwise disjoint open intervals of the real line, which is not the case. ∎

LEMMA 3.9. *For any $x_0 \in \mathbb{R}$ and any connected component C of the complement of F_{x_0}, there exists $y \neq x_0$ such that $F_y \cap C \neq \emptyset$.*

Proof. Let (a_1, a_2) be a connected component of the complement of F_{x_0}. It can happen that either $a_1 = -\infty$ or $a_2 = \infty$.

Let us prove that there exists $y_0 \neq x_0$ such that $F_{y_0} \cap (a_1, a_2) \neq \emptyset$. Suppose for contradiction that, for any $y \neq x_0$, $F_y \cap (a_1, a_2) = \emptyset$. For any real z_1 and z_2, choose h_{z_1,z_2} in $\mathrm{Homeo}_c(\mathbb{R})$ such that $h_{z_1,z_2}(z_1) = z_2$. We claim that the non-empty open sets $\psi(h_{x_0,y})((a_1, a_2))$, for $y \in \mathbb{R}$, are pairwise disjoint, which is impossible. Indeed, suppose that $\psi(h_{x_0,y_1})((a_1, a_2)) \cap \psi(h_{x_0,y_2})((a_1, a_2)) \neq \emptyset$ for some $y_1 \neq y_2$. As the union of the closed sets F_y is invariant under ψ, for $i = 1, 2$, when a_i is finite, $\psi(h_{x_0,y_1})^{-1} \circ \psi(h_{x_0,y_2})(a_i) \notin (a_1, a_2)$, and $\psi(h_{x_0,y_2})^{-1} \circ \psi(h_{x_0,y_1})(a_i) \notin (a_1, a_2)$ so that $\psi(h_{x_0,y_1})(a_i) = \psi(h_{x_0,y_2})(a_i)$. But this last equality cannot hold as the left-hand side belongs to F_{y_1} and the right-hand side to F_{y_2}, and we observed that these two closed sets are disjoint. ∎

Lemma 3.10. *Each F_x contains only one point.*

Proof. Suppose that F_x contains two points $p_1 < p_2$. By Lemma 3.9, there exists $y \neq x$ such that F_y has a point in the interval (p_1, p_2). Take a point $r < p_1$. Then, for any g_1 in G_x, g_2 in G_y and g_3 in G_x, we have

$$\psi(g_1) \circ \psi(g_2) \circ \psi(g_3)(r) < p_2.$$

By Lemma 3.6, this implies that

$$\{\psi(g)(r) \mid g \in \mathrm{Homeo}_c(\mathbb{R})\} \subset (-\infty, p_2].$$

The supremum of the left-hand set provides a fixed point for the action ψ, a contradiction. ∎

Take $f \in \mathrm{Homeo}_c(\mathbb{R})$ and $x \in \mathbb{R}$. Then the homeomorphism $\psi(f)$ sends the only point $h(x)$ in F_x to the only point $h(f(x))$ in $F_{f(x)}$. This implies that $\psi(f) \circ h = h \circ f$. Thus, it suffices to use Lemma 3.11 below to complete the proof of Theorem 3.1. ∎

Lemma 3.11. *Define $h(x)$ as the only point in the set F_x. Then the map h is a homeomorphism.*

Proof. By Lemma 3.8, h is one-to-one.

Fix $x_0 \in \mathbb{R}$ and let us prove that h is continuous at x_0. Take a compactly supported C^1 vector field $\mathbb{R} \to \mathbb{R}$ which does not vanish on a neighbourhood of x_0, and denote by $(\varphi^t)_{t \in \mathbb{R}}$ the flow of this vector field. Then, for any time t, $h(\varphi^t(x_0)) = \psi(\varphi^t)(h(x_0))$, which proves that h is continuous at x_0.

Finally, let us prove that h is onto. Notice that the interval $h(\mathbb{R})$ is invariant under the action ψ. Hence, if $\sup(h(\mathbb{R})) < \infty$ (respectively $\inf(h(\mathbb{R})) > -\infty$), then the point $\sup(h(\mathbb{R}))$ (respectively $\inf(h(\mathbb{R}))$) would be a fixed point of the action ψ, a contradiction. ∎

Proposition 3.12. *Any action of the group* $\mathrm{Homeo}_c(\mathbb{R})$ *on the circle has a fixed point.*

Hence, the description of the actions of $\mathrm{Homeo}_c(\mathbb{R})$ on the circle is given by an action of $\mathrm{Homeo}_c(\mathbb{R})$ on the real line which is homeomorphic to the circle minus one point.

Proof. As $H_b^2(\mathrm{Homeo}_c(\mathbb{R}), \mathbb{Z}) = \{0\}$ (see [11]), the bounded Euler class of this action necessarily vanishes so that this action admits a fixed point (see [7] for the bounded Euler class of an action on the circle and its properties). ∎

Finally, let us recall a result of [12] which also uses the bounded Euler class.

Proposition 3.13. *Any non-trivial action of the group* $\mathrm{Homeo}_0(\mathbb{S}^1)$ *on the circle is a conjugacy by a homeomorphism of the circle.*

4. Actions on the annulus. This section is devoted to the proof of Theorem 2.1 in the case of morphisms with values in the group of homeomorphisms of the annulus. Fix a morphism $\varphi : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{A})$. Recall that, by [15, Theorem 4 and Proposition 2], φ is necessarily continuous.

First, as the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ is simple (see [6]), φ is either trivial or one-to-one. We assume in the rest of this section that it is one-to-one. Recall that the induced group morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{A})/\mathrm{Homeo}_0(\mathbb{A})$ is not one-to-one as the source is uncountable and the target is countable. Therefore, it is trivial and the image of φ is contained in $\mathrm{Homeo}_0(\mathbb{A})$.

Now, recall that the subgroup of rotations of the circle is continuously isomorphic to the topological group $\mathbb{S}^1$. The image of this subgroup under φ is a compact subgroup of the group of homeomorphisms of the closed annulus which is continuously isomorphic to $\mathbb{S}^1$. It is known that such a subgroup is conjugate to the rotation subgroup $\{(r, \theta) \mapsto (r, \theta + \alpha) \mid \alpha \in \mathbb{S}^1\}$ of the group of homeomorphisms of the annulus (see [2]). This is the only place in this proof where we really need the continuity hypothesis. After possibly conjugating φ, we may suppose from now on that, for any angle α, the morphism φ sends the α-rotation of the circle to the α-rotation of the annulus.

4.1. An invariant lamination. Our goal in this section is to construct the set K as in Theorem 2.1.

Fix $\theta_0 \in \mathbb{S}^1$. Recall that G_{θ_0} is the group of homeomorphisms of $\mathbb{S}^1$ which fix a neighbourhood of θ_0. Denote by $F_{\theta_0} \subset \mathbb{A}$ the closed set of fixed points of $\varphi(G_{\theta_0})$ (i.e. fixed under every homeomorphism in this group). The action φ induces actions on the two boundary circles of the annulus. The action of $\varphi(G_{\theta_0})$ on each of these circles has a fixed point by Proposition 3.13. Therefore, the set F_{θ_0} is non-empty. For any angle θ, let $\alpha = \theta - \theta_0$. Then $G_\theta = R_\alpha G_{\theta_0} R_\alpha^{-1}$, where R_α denotes the α-rotation of the circle. Therefore, $\varphi(G_\theta) = R_\alpha \varphi(G_{\theta_0}) R_\alpha^{-1}$, where by abuse of notation, R_α denotes the α-rotation of the annulus, and $F_\theta = R_\alpha(F_{\theta_0})$. Let

$$B = \bigcup_{\theta \in \mathbb{S}^1} F_\theta = \bigcup_{\alpha \in \mathbb{S}^1} R_\alpha(F_{\theta_0}).$$

This set is of the form $K \times \mathbb{S}^1$, where K is the image of F_{θ_0} under the projection $\mathbb{A} = [0,1] \times \mathbb{S}^1 \to [0,1]$. The set K is compact and contains 0 and 1. Let $\mathcal{F}$ be the lamination of B by the circles $\{r\} \times \mathbb{S}^1$, where $r \in K$.

LEMMA 4.1. *Each leaf of the lamination $\mathcal{F}$ is preserved by the action φ.*

Proof. Fix an angle θ. Let us prove that the orbit of any $x \in F_\theta$ under φ is contained in $\{R_\alpha(x) \mid \alpha \in \mathbb{S}^1\} \subset \mathbb{A}$. First, for any orientation-preserving homeomorphism f of the circle which fixes θ, the homeomorphism $\varphi(f)$ pointwise fixes the set F_θ. Indeed, the homeomorphism f is the uniform limit of homeomorphisms which pointwise fix a neighbourhood of θ, and the claim results from the continuity of φ. Any orientation-preserving homeomorphism g of the circle can be written $g = R_\beta f$, where f is a homeomorphism which fixes θ. Now, any point x in F_θ is fixed under $\varphi(f)$ and sent to a point in $\{R_\alpha(x) \mid \alpha \in \mathbb{S}^1\} \subset \mathbb{A}$ under the rotation $R_\beta = \varphi(R_\beta)$, which proves the lemma. ∎

LEMMA 4.2. *Each closed set F_θ intersects each leaf of the lamination $\mathcal{F}$ in exactly one point. Moreover, the map h which to any $(r,\theta) \in K \times \mathbb{S}^1 \subset \mathbb{A}$ associates the only point of F_θ on the leaf $\{r\} \times \mathbb{S}^1$ is a homeomorphism of $K \times \mathbb{S}^1$. This homeomorphism conjugates the restriction of the action $\varphi_{K,\lambda}$ to $K \times \mathbb{S}^1$ with the restriction of φ to $K \times \mathbb{S}^1$ for any continuous map $\lambda : [0,1] - K \to \{-1,1\}$. Moreover, this homeomorphism is of the form $(r,\theta) \mapsto (r, \eta(r) + \theta)$ where $\eta : K \to \mathbb{S}^1$ is a continuous function.*

Proof. By Lemma 4.1, the action φ preserves each set $\{r\} \times \mathbb{S}^1$ and induces an action on it. By Proposition 3.13, the restriction of this action to a subgroup of the form G_θ has exactly one fixed point (this action is non-trivial as the rotation subgroup acts non-trivially). This implies the first statement of the lemma.

Take a sequence $(r_n)_{n \in \mathbb{N}}$ in K which converges to $r \in K$. Then, as $h(r_n, \theta_0) \in (\{r_n\} \times \mathbb{S}^1) \cap F_{\theta_0}$, the accumulation points of $(h(r_n,\theta_0))_n$ belong to $(\{r\} \times \mathbb{S}^1) \cap F_{\theta_0}$; hence there is only one accumulation point, $h(r,\theta_0)$. For any angle θ, we have $h(r,\theta) = R_{\theta-\theta_0} \circ h(r,\theta_0)$. This implies that h is continuous. This map is one-to-one: if $h(r,\theta) = h(r',\theta')$, then this point belongs to $\{r\} \times \mathbb{S}^1 = \{r'\} \times \mathbb{S}^1$ so that $r = r'$, and $R_{\theta-\theta_0} \circ h(r,\theta_0) = R_{\theta'-\theta_0} \circ h(r,\theta_0)$ so that $\theta = \theta'$. The map h is onto by definition of the set $B = K \times \mathbb{S}^1$

which is the union of the F_θ. As this map is defined on a compact set, it is a homeomorphism.

Take any orientation-preserving homeomorphism f of the circle with $f(\theta)=\theta'$. Notice that $G_{\theta'}=fG_\theta f^{-1}$. Therefore $\varphi(G_{\theta'})=\varphi(f)\varphi(G_\theta)\varphi(f)^{-1}$ and $\varphi(f)(F_\theta)=F_{\theta'}$. So, for any $(r,\theta)\in K\times\mathbb{S}^1$, as the action φ preserves each leaf of the lamination $\mathcal{F}$, the homeomorphism $\varphi(f)$ sends $h(r,\theta)$, which is the unique point in $F_\theta\cap(\{r\}\times\mathbb{S}^1)$, to the unique point in $F_{\theta'}\cap(\{r\}\times\mathbb{S}^1)$, which is $h(r,\theta')$. This implies that

$$\varphi(f)\circ h(r,\theta)=h\circ\varphi_{K,\lambda}(f)(r,\theta).$$

Now, denote by $\eta(r)$ the second projection of $h(r,0)$. As F_θ is the image of F_0 under the θ-rotation, we have $h(r,\theta)=h(r,0)+(0,\theta)$ and $h(r,\theta)=(r,\eta(r)+\theta)$. ∎

4.2. Action outside the lamination. In this section, we study the action φ on each connected component of $\mathbb{A}-K\times\mathbb{S}^1$. Let $A=[r_1,r_2]\times\mathbb{S}^1$ be the closure of such a component. By the last subsection, A is invariant under φ. This subsection is dedicated to the proof of the following proposition.

PROPOSITION 4.3. *The restriction φ^A of the action φ to A is conjugate to a_+ (or equivalently to a_-) via an orientation-preserving homeomorphism.*

Proof. Notice that, for any $\theta\in\mathbb{S}^1$, the action $\varphi^A_{|G_\theta}$ admits no fixed point in the interior of A by definition of $K\times\mathbb{S}^1$; we will often use this fact.

Let us begin by sketching the proof. We show that, for any $\theta\in\mathbb{S}^1$, the morphism $\varphi^A_{|G_\theta}$ can be lifted to a morphism $\tilde{\varphi}^A_\theta$ from G_θ to the group $\mathrm{Homeo}_{\mathbb{Z}}([r_1,r_2]\times\mathbb{R})$ of homeomorphisms of the closed band which commute with all translations $(r,x)\mapsto(r,x+n)$, where n is an integer. Moreover, this group morphism can be chosen to have a bounded orbit. We will then find a continuum $\widetilde{L_\theta}$ with empty interior which touches both boundary components of the band and is invariant under $\tilde{\varphi}^A_\theta$. Then we prove that the sets $L_\theta=\pi(\widetilde{L_\theta})$, where $\pi:[r_1,r_2]\times\mathbb{R}\to A$ is the projection, are pairwise disjoint and are simple paths which join the two boundary components of the annulus A. We see that the group $\varphi(G_\theta\cap G_{\theta'})$, for $\theta\neq\theta'$, has a unique fixed point $a(\theta,\theta')$ on L_θ. This last map a turns out to be continuous and allows us to build a conjugacy between φ^A and a_+.

The following lemma is necessary to build the invariant sets L_θ.

For g in $\mathrm{Homeo}_0(A)$, we denote by $\tilde{g}$ the lift of g to $\mathrm{Homeo}_{\mathbb{Z}}([r_1,r_2]\times\mathbb{R})$ (this means that $\pi\circ\tilde{g}=g\circ\pi$) with $\tilde{g}((r_1,0))\in[r_1,r_2]\times[-1/2,1/2)$. Denote by $D\subset\mathbb{R}^2$ the fundamental domain $[r_1,r_2]\times[-1/2,1/2]$ for the action of $\mathbb{Z}$ on $[r_1,r_2]\times\mathbb{R}$.

LEMMA 4.4. *The map* $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathbb{R}_+$ *which associates to any* f *in* $\mathrm{Homeo}_0(\mathbb{S}^1)$ *the diameter of the image under* $\widetilde{\varphi(f)}$ *(or equivalently under any lift of* $\varphi(f)$*) of the fundamental domain* D*, is bounded.*

REMARK 4.5. The continuity of φ is not used in the proof of this lemma.

Proof. For a group G generated by a finite set S and for any g in G, we denote by $l_S(g)$ the word length of g with respect to S, which is the minimal number of factors necessary to write g as a product of elements of $S \cup S^{-1}$, where S^{-1} is the set of inverses of elements in S. In order to prove the lemma, we need the following result which can be easily deduced from [14, Lemma 4.4], inspired from [1].

LEMMA 4.6. *There exist constants* $C > 0$ *and* $C' \in \mathbb{R}$ *such that, for any sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\mathrm{Homeo}_0(\mathbb{S}^1)$*, there exists a finite set* $S \subset \mathrm{Homeo}_0(\mathbb{S}^1)$ *such that:*

- *each* f_n *belongs to the group generated by* S*;*
- $l_S(f_n) \leq C\log(n) + C'$ *for all* $n \in \mathbb{N}$*.*

Let us now prove Lemma 4.4. Suppose for contradiction that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Homeo}_0(\mathbb{S}^1)$ such that, for any integer n,

$$\mathrm{diam}(\widetilde{\varphi(f_n)}(D)) \geq n.$$

We then apply Lemma 4.6 to this sequence to obtain a finite subset S of $\mathrm{Homeo}_0(\mathbb{S}^1)$ such that the conclusion of the lemma holds. Let $f_n = s_{1,n}\dots s_{w_n,n}$, where $w_n \leq C\log(n) + C'$ and the $s_{i,j}$'s are in $S \cup S^{-1}$. We now prove that this implies that the diameter of $\widetilde{\varphi(f_n)}(D)$ grows at most at a logarithmic speed, which contradicts the hypothesis on $(f_n)_n$.

Denote by M the maximum of $\|\widetilde{\varphi(s)}(x) - x\|$, where s varies over $S \cup S^{-1}$ and x varies over $\mathbb{R}^2$ (or equivalently over the compact set D) and where $\|\cdot\|$ denotes the euclidean norm. Then for any x and y in D and any integer n,

$$\begin{aligned}
&\big\|\widetilde{\varphi(s_{1,n})}\dots\widetilde{\varphi(s_{w_n,n})}(x) - \widetilde{\varphi(s_{1,n})}\dots\widetilde{\varphi(s_{w_n,n})}(y)\big\| \\
&\qquad\leq \big\|\widetilde{\varphi(s_{1,n})}\dots\widetilde{\varphi(s_{w_n,n})}(x) - x\big\| + \big\|\widetilde{\varphi(s_{1,n})}\dots\widetilde{\varphi(s_{w_n,n})}(y) - y\big\| \\
&\qquad\quad + \|x - y\|.
\end{aligned}$$

But for any z in D, we have

$$\begin{aligned}
&\big\|\widetilde{\varphi(s_{1,n})}\dots\widetilde{\varphi(s_{w_n,n})}(z) - z\big\| \\
&\quad\leq \sum_{k=1}^{w_n-1} \big\|\widetilde{\varphi(s_{k,n})}\dots\widetilde{\varphi(s_{w_n,n})}(z) - \widetilde{\varphi(s_{k+1,n})}\dots\widetilde{\varphi(s_{w_n,n})}(z)\big\| \leq (w_n - 1)M.
\end{aligned}$$

Hence

$$\operatorname{diam}\left(\widetilde{\varphi(s_{1,n})}\dots\widetilde{\varphi(s_{w_n,n})}(D)\right) = \operatorname{diam}(\widetilde{\varphi(f_n)}(D)) \\ \leq 2(w_n-1)M + \operatorname{diam}(D),$$

contrary to the hypothesis on $(f_n)_n$. ∎

Let θ_0 be a point of the circle.

LEMMA 4.7. *There exists a group morphism* $\tilde{\varphi}^A_{\theta_0} : G_{\theta_0} \to \mathrm{Homeo}_{\mathbb{Z}}([r_1,r_2] \times \mathbb{R})$ *such that:*

- *for any homeomorphism f in G_{θ_0}, $\Pi \circ \tilde{\varphi}^A_{\theta_0}(f) = \varphi^A(f)$, where Π : $\mathrm{Homeo}_{\mathbb{Z}}([r_1,r_2]\times\mathbb{R}) \to \mathrm{Homeo}_0(A)$ is the projection;*
- *the subset $\{\tilde{\varphi}^A_{\theta_0}(f)((r_1,0)) \mid f \in G_{\theta_0}\}$ of the band $[r_1,r_2]\times\mathbb{R}$ is bounded.*

Moreover, the morphism $\tilde{\varphi}^A_{\theta_0}$ is continuous.

REMARK 4.8. The continuity of φ is not necessary for the first part of this lemma. However, we will use it to simplify the proof.

Proof of Lemma 4.7. As G_{θ_0} is contractible and $\Pi : \mathrm{Homeo}_{\mathbb{Z}}([r_1,r_2]\times\mathbb{R}) \to \mathrm{Homeo}_0(A)$ is a covering, there exists a (unique) continuous map η : $G_{\theta_0} \to \mathrm{Homeo}_{\mathbb{Z}}([r_1,r_2]\times\mathbb{R})$ which lifts $\varphi^A_{|G_{\theta_0}}$ and sends the identity to the identity. Then the map

$$G_{\theta_0} \times G_{\theta_0} \to \mathrm{Homeo}_{\mathbb{Z}}([r_1,r_2]\times\mathbb{R}), \quad (f,g) \mapsto \eta(fg)^{-1}\eta(f)\eta(g),$$

is continuous and its image is contained in the discrete space of integral translations; hence it is constant and η is a group morphism. Two group morphisms which lift $\varphi^A_{|G_{\theta_0}}$ differ by a morphism $G_{\theta_0} \to \mathbb{Z}$. However, as G_{θ_0} is simple (hence perfect), the latter morphism is trivial and $\eta = \tilde{\varphi}^A_{\theta_0}$. The action $\varphi^A_{|G_{\theta_0}}$ has fixed points on the boundary of the annulus A. Hence, as G_{θ_0} is path-connected and $\eta(\mathrm{Id}) = \mathrm{Id}$, the action η has fixed points on the boundary of $[r_1,r_2]\times\mathbb{R}$ and any orbit on one of these boundary components is bounded. ∎

Let

$$F = \overline{\bigcup_{f \in G_{\theta_0}} \tilde{\varphi}^A_{\theta_0}(f)([r_1,r_2]\times(-\infty,0])}.$$

By the above two lemmas, there exists $M > 0$ such that $F \subset [r_1,r_2] \times (-\infty, M]$. Moreover, the closed set F is invariant under the action $\tilde{\varphi}^A_{\theta_0}$. Denote by U the connected component of the complement of $F \cup \{r_1,r_2\}\times\mathbb{R}$ which contains the open subset $(r_1,r_2)\times(M,\infty)$. By construction, the open set U is invariant under $\tilde{\varphi}^A_{\theta_0}$ (the interior of a fundamental domain far on the right must be sent into U by any homeomorphism in the image of $\tilde{\varphi}^A_{\theta_0}$, by the above two lemmas). Consider the topological space B which is the disjoint union of $[r_1, r_2] \times \mathbb{R}$ with a point $\{\infty\}$ and for which a neighbourhood basis of ∞ is given by the sets of the form $[r_1, r_2] \times (A, \infty) \cup \{\infty\}$. Now, consider the prime end compactification of $U \subset B$ (see [11]). The space of prime ends of the simply connected open set U, on which there is a natural action ψ of the group G_{θ_0} induced by $\tilde{\varphi}^A_{\theta_0}$, is homeomorphic to a circle. By the following lemma, the action ψ is continuous.

LEMMA 4.9. *Let W be a simply connected relatively compact open subset of the plane. Denote by $B(W)$ the space of prime ends of W. The map*

$$t : \mathrm{Homeo}(\overline{W}) \to \mathrm{Homeo}(B(W))$$

which, to a homeomorphism f of $\overline{W}$, associates the induced homeomorphism on the space of prime ends of W, is continuous.

Proof. As t is a group morphism and $B(W)$ is homeomorphic to the circle, it suffices to prove that, for any prime end ξ in $B(W)$, the map

$$\mathrm{Homeo}(\overline{W}) \to B(W), \qquad f \mapsto t(f)(\xi),$$

is continuous at the identity. Fix such a prime end ξ. Denote by $V_1 \supset V_2 \supset \cdots$ a prime chain which defines the prime end ξ. If we denote by $\tilde{V}_n$ the space of prime points of W which divide V_n, then the $\tilde{V}_n$'s are a neighbourhood basis of ξ (see [11, Section 3]). Fix n and $p > n$. If the uniform distance between $f \in \mathrm{Homeo}(\overline{W})$ and the identity is smaller than the distance between the frontier $\mathrm{Fr}_W(V_n)$ of V_n in W and the frontier $\mathrm{Fr}_W(V_p)$ of V_p in W, then $f(\mathrm{Fr}_W(V_n))$ does not meet $\mathrm{Fr}_W(V_p)$. By [11, Lemma 4], if f is sufficiently close to the identity, then $f(V_p) \subset V_n$ and $t(f)(\tilde{V}_p) \subset \tilde{V}_n$. This implies that, for f in such a neighbourhood, the point $t(f)(\xi)$ belongs to $\tilde{V}_n$, which is what we wanted to prove. ■

Take a prime end ξ of U. The *principal set* of ξ is the set of points p in B, called *principal points* of ξ, such that there exists a prime chain $V_1 \supset V_2 \supset \cdots$ defining ξ such that the sequence of frontiers of V_n in U converges for the Hausdorff topology to the single-point set $\{p\}$. This set is compact and connected. Consider the subset of prime ends of U whose principal set contains a point of $\{r_1, r_2\} \times \mathbb{R} \cup \{\infty\}$. This set is invariant under ψ. Denote by I a connected component of the complement of this set (the complement is non-empty by [11] because there exists an arc $[0, \infty) \to U$ which converges as $t \to \infty$ to a point which belongs to the frontier of U but not to $\{r_1, r_2\} \times \mathbb{R} \cup \{\infty\}$). By path-connectedness of G_{θ_0} and continuity of ψ, the interval I is invariant under ψ. Let ψ' be the restriction of ψ to I.

LEMMA 4.10. *The action ψ' has no fixed point. Therefore, the interval I is open.*

Proof. Suppose for contradiction that the action ψ' has a fixed point ξ. Then the principal set of the prime end ξ would provide a compact subset of $(r_1, r_2) \times \mathbb{R}$ which is invariant under the action $\tilde{\varphi}^A_{\theta_0}$, contradicting Lemma 4.11 below. ∎

Lemma 4.11. *The action $\tilde{\varphi}^A_{\theta_0}$ has no non-empty compact connected invariant set contained in $(r_1, r_2) \times \mathbb{R}$.*

Proof. Suppose for contradiction that the action $\tilde{\varphi}^A_{\theta_0}$ admits such an invariant set. Consider the unbounded component of the complement of this set. The action $\tilde{\varphi}^A_{\theta_0}$ induces an action η on the space P of prime ends of this set, which is homeomorphic to a circle. By Proposition 3.12, η has a fixed point.

If the set of fixed points of η has non-empty interior, then since accessible prime ends are dense in this set (see [11]), there exists an accessible prime end which is fixed under η. Therefore, the only point in the principal set of this prime end, which is contained in the interior of $[r_1, r_2] \times \mathbb{R}$, is fixed under the action $\tilde{\varphi}^A_{\theta_0}$.

Suppose now that the set of fixed points of η has empty interior. Take a connected component of the complement of this set and an endpoint e of this interval. Take a closed interval J of the circle whose interior contains θ_0. Denote by G_J the subgroup of $\mathrm{Homeo}_0(\mathbb{S}^1)$ consisting of the homeomorphisms which pointwise fix a neighbourhood of J. Then, according to Section 3 in which we describe the continuous actions of $\mathrm{Homeo}_c(\mathbb{R})$ on $\mathbb{R}$, the set of fixed points of $\psi(G_J)$ contains a non-trivial closed interval $\mathcal{J}$ which contains e (not necessarily in its interior). As accessible prime ends are dense in P, this closed interval contains an accessible prime end. Then, as the principal set of this prime end reduces to a point p, this point is fixed under the group $\tilde{\varphi}^A_{\theta_0}(G_J)$. As a result, the group $\tilde{\varphi}^A_{\theta_0}(G_J)$ of homeomorphisms of $[r_1, r_2] \times \mathbb{R}$ admits a non-empty set H_J of fixed points which is contained in the closure C_J of the union of the principal sets of prime ends in $\mathcal{J}$, which is compact. Moreover, H_J is contained in the interior of $[r_1, r_2] \times \mathbb{R}$. For any closed interval J' whose interior contains θ_0 and which is contained in J, the set $H_{J'}$ of fixed points of $\tilde{\varphi}^A_{\theta_0}(G_{J'})$ which are contained in C_J is non-empty. Moreover, if an interval J'' is contained in J', then $H_{J''} \subset H_{J'}$. By compactness, the intersection of those sets, which is contained in the set of fixed points of $\tilde{\varphi}^A_{\theta_0}$, is non-empty, a contradiction. ∎

Our next goal is to prove that the interval I corresponds to an embedded open interval L_{θ_0} in $(r_1, r_2) \times \mathbb{R}$.

By the description of continuous actions of $\mathrm{Homeo}_c(\mathbb{R})$ on $\mathbb{R}$ with no global fixed points (see Section 3), the action ψ is transitive on the open interval I. Hence, all the prime ends in I are accessible, by density of accessible prime ends. Moreover, by this same description, for any $\theta \neq \theta_0$ on the circle, there is a unique prime end $e_\theta \in I$ which is fixed under $\psi_{|G_{\theta_0} \cap G_\theta}$, and I is the union of these prime ends. For any $\theta \neq \theta_0$ on the circle, denote by $\tilde{x}_\theta$

the unique point in the principal set of the prime end e_θ. For any f in G_{θ_0} which sends $\theta \neq \theta_0$ to θ', we have $\tilde{\varphi}^A_{\theta_0}(f)(\tilde{x}_\theta) = \tilde{x}_{\theta'}$, as $G_{\theta'} = fG_\theta f^{-1}$. Let $\widetilde{L_{\theta_0}} = \{\tilde{x}_\theta \mid \theta \in \mathbb{S}^1 - \{\theta_0\}\}$ and $L_{\theta_0} = \pi(\widetilde{L_{\theta_0}})$.

Denote by $\lim_{\theta \to \theta_0^+} \tilde{x}_\theta$ (respectively $\lim_{\theta \to \theta_0^-} \tilde{x}_\theta$) the set of points which are limits of sequences $(\tilde{x}_{\theta_n})_{n \geq 1}$, where $(\theta_n)_{n \geq 1}$ is a sequence in $\mathbb{S}^1 - \{\theta_0\}$ such that $\lim_{n \to \infty} \theta_n = \theta_0$ and, for n sufficiently large, $\theta_n > \theta_0$ (respectively $\theta_n < \theta_0$).

Lemma 4.12. *The map*

$$\mathbb{S}^1 - \{\theta_0\} \to (r_1, r_2) \times \mathbb{R}, \qquad \theta \mapsto \tilde{x}_\theta,$$

is one-to-one and continuous. Moreover, the sets $\lim_{\theta \to \theta_0^+} \tilde{x}_\theta$ *and* $\lim_{\theta \to \theta_0^-} \tilde{x}_\theta$ *each contain exactly one point of the boundary* $\{r_1, r_2\} \times \mathbb{R}$.

Proof. Take $\theta' \neq \theta$ and suppose for contradiction that $\tilde{x}_\theta = \tilde{x}_{\theta'}$. Take $\theta'' \neq \theta$ in the same connected component of $\mathbb{S}^1 - \{\theta_0, \theta\}$ as θ' such that $\tilde{x}_{\theta''} \neq \tilde{x}_\theta$. Consider a homeomorphism f of the circle in $G_{\theta_0} \cap G_\theta$ which sends θ' to θ''. Then the homeomorphism $\tilde{\varphi}^A_{\theta_0}(f)$ of the band fixes $\tilde{x}_\theta$ and sends $\tilde{x}_{\theta'}$ to $\tilde{x}_{\theta''}$, which is impossible. Thus the map considered is one-to-one.

Now, let us prove that it is continuous. Take a smooth vector field on the circle which vanishes only on a small connected neighbourhood N of θ_0. Denote by $(h^t)_{t \in \mathbb{R}}$ the one-parameter group generated by this vector field. Fix $\theta_1 \in \mathbb{S}^1 - N$. For any $\theta \in \mathbb{S}^1 - N$, denote by $t(\theta)$ the unique time t such that $h^t(\theta_1) = \theta$. The map $\theta \mapsto t(\theta)$ is then a homeomorphism $\mathbb{S}^1 - N \to \mathbb{R}$. Now, the relation $\tilde{x}_\theta = \varphi(h^{t(\theta)})(\tilde{x}_{\theta_1})$ and the continuity of φ imply that the map $\theta \mapsto \tilde{x}_\theta$ is continuous, as the neighbourhood N can be taken arbitrarily small.

Lemma 4.11 implies that the intersection of each of the sets $\lim_{\theta \to \theta_0^+} \tilde{x}_\theta$ and $\lim_{\theta \to \theta_0^-} \tilde{x}_\theta$ with $\{r_1, r_2\} \times \mathbb{R}$ is non-empty. Indeed, otherwise, these limit sets would provide a non-empty compact connected invariant set for the action $\tilde{\varphi}^A_{\theta_0}$.

It remains to prove that the intersections $\lim_{\theta \to \theta_0^+} \tilde{x}_\theta \cap (\{r_1, r_2\} \times \mathbb{R})$ and $\lim_{\theta \to \theta_0^-} \tilde{x}_\theta \cap (\{r_1, r_2\} \times \mathbb{R})$ are single points. Suppose for instance that $\lim_{\theta \to \theta_0^+} \tilde{x}_\theta \cap (\{r_1\} \times \mathbb{R})$ contains at least two points. Recall that, by Proposition 3.13, the restriction of the action φ to $\{r_1\} \times \mathbb{S}^1$ is a conjugacy by a homeomorphism $\mathbb{S}^1 \to \{r_1\} \times \mathbb{S}^1$. Therefore, the action $\varphi^A_{|G_{\theta_0}}$ fixes a point p and is transitive on $\{r_1\} \times \mathbb{S}^1 - \{p\}$. Moreover, as any orbit of $\tilde{\varphi}^A_{\theta_0}$ is bounded by Lemmas 4.4 and 4.7, this last action pointwise fixes the set $\pi^{-1}(\{p\})$ and is transitive on each connected component of $\{r_1\} \times \mathbb{R} - \pi^{-1}(\{p\})$. Therefore, the intersection $\lim_{\theta \to \theta_0^+} \tilde{x}_\theta \cap (\{r_1\} \times \mathbb{R})$, which is closed and invariant under $\tilde{\varphi}^A_{\theta_0}$, contains two distinct lifts $\tilde{p}$ and $\tilde{p}'$ of p. Take a sequence $(\tilde{x}_{\theta_n})_{n \in \mathbb{N}}$ in $\widetilde{L_{\theta_0}}$, where $\theta_n \to \theta_0^+$, which converges to $\tilde{p}$, and a sequence $(\tilde{x}_{\theta'_n})_{n \in \mathbb{N}}$ in $\widetilde{L_{\theta_0}}$, where $\theta'_n \to \theta_0^+$, which converges to $\tilde{p}'$. Taking a subsequence if necessary, we may suppose that the sequences $(\theta_n)_n$ and $(\theta'_n)_n$ are monotone and that, for any n, the angle θ'_n lies between θ_n and θ_{n+1}. Take f in G_{θ_0} which, for any n, sends θ_n to θ'_n. Then, for any n, $\tilde{\varphi}^A_{\theta_0}(f)$ sends $\tilde{x}_{\theta_n}$ to $\tilde{x}_{\theta'_n}$. By continuity, this homeomorphism sends $\tilde{p}$ to $\tilde{p}'$. This is impossible as these points are fixed under the action $\tilde{\varphi}^A_{\theta_0}$. ∎

Note that we do not know for the moment that the points $\lim_{\theta\to\theta_0^+}\tilde{x}_\theta$ and $\lim_{\theta\to\theta_0^-}\tilde{x}_\theta$ lie on different boundary components of A.

For any $\theta \neq \theta_0$, we define $L_\theta = R_{\theta-\theta_0}(L_{\theta_0})$. Notice that, for any θ, the set L_θ is invariant under the action $\varphi^A_{|G_\theta}$. Indeed, $G_\theta = R_{\theta-\theta_0}G_{\theta_0}R^{-1}_{\theta-\theta_0}$, which implies that $\varphi^A(G_\theta) = R_{\theta-\theta_0}\varphi^A(G_{\theta_0})R^{-1}_{\theta-\theta_0}$ and, as L_{θ_0} is invariant under the action $\varphi^A_{|G_{\theta_0}}$, the claim follows. Moreover, for an orientation-preserving homeomorphism f of the circle which sends an angle θ to another angle θ', the homeomorphism $\varphi(f)$ maps the set L_θ onto $L_{\theta'}$. To prove this, use the fact that f can be written as the composition of a homeomorphism which fixes θ with a rotation.

LEMMA 4.13. *The sets L_θ are pairwise disjoint. Moreover, there exists a homeomorphism from $[0,1]$ onto the closure of L_{θ_0} which maps $(0,1)$ onto L_{θ_0}.*

Proof. By using rotations, it suffices to prove that $L_{\theta_0} \cap L_\theta = \emptyset$ for any $\theta \neq \theta_0$. Remember that the restriction of the action ψ' to $G_\theta \cap G_{\theta_0}$ has one fixed point e_θ and is transitive on each connected component of $I-\{e_\theta\}$. Denote by E_θ the subset of I consisting of the prime ends whose only point in the principal set belongs to $\pi^{-1}(L_\theta)$. As this set is invariant under $\psi_{|G_\theta\cap G_{\theta_0}}$, it is either empty, or the one-point set $\{e_\theta\}$, or one of the connected components of $I-\{e_\theta\}$, or the closure of such a component or I. In the last case, we would have $\widetilde{L_{\theta_0}} \subset \pi^{-1}(L_\theta)$ and $L_{\theta_0} \subset L_\theta$. Using homeomorphisms of the circle which fix θ_0 and send θ to another angle θ', we see that $L_{\theta_0} \subset L_{\theta'}$ for any θ'. This is impossible as the intersection of the closure of L_{θ_0} with the boundary of A is a two-point set which should be invariant under any rotation. The case where E_θ is a half-line (open or closed) leads to a similar contradiction by looking at one of the limit sets of L_{θ_0}. Hence, for any $\theta \neq \theta_0$, the intersection $L_{\theta_0} \cap L_\theta = L_{\theta_0} \cap R_{\theta-\theta_0}(L_{\theta_0})$ contains at most one point, the point $x_\theta = \pi(\tilde{x}_\theta)$.

This implies that any leaf of the form $\{r\} \times \mathbb{S}^1 \subset A$ contains at most two points of L_{θ_0}. We claim that if one of these leaves contains two points of L_{θ_0}, then no leaf contains exactly one point of this set.

Take a point x_{θ_1} in L_{θ_0} which belongs to the leaf $\{r\} \times \mathbb{S}^1$. Suppose that there exists another point of L_{θ_0} on this leaf. This implies that $x_{\theta_1} \in L_{\theta_1}$.

Using homeomorphisms in G_{θ_0} which send θ_1 to another point $\theta \neq \theta_0$ of the circle, we see that $x_\theta \in L_\theta$ for any $\theta \neq \theta_0$. Hence, each circular leaf which meets L_{θ_0} contains exactly two points of this set.

If x is another point of L_{θ_0} on the leaf $\{r\} \times \mathbb{S}^1$, then there exists α in $\mathbb{S}^1 - \{0\}$ such that $R_\alpha(x) = x_{\theta_1} \in L_{\alpha+\theta_0}$. Therefore, $\alpha + \theta_0 = \theta_1$ and necessarily $x = R_{\theta_0-\theta_1}(x_{\theta_1}) = x_{2\theta_0-\theta_1}$. Hence, the map $\theta \mapsto p_1(x_\theta)$, where $p_1 : [r_1, r_2] \times \mathbb{R} \mapsto [r_1, r_2]$ is the projection, is strictly monotone (as it is one-to-one) on $(\theta_0, \theta_0 + 1/2]$ and on $[\theta_0 + 1/2, \theta_0 + 1)$. Moreover, if this map were not globally one-to-one, the circular leaf which contains $x_{\theta_0+1/2}$ would contain only one point, which is impossible. Hence, this map is strictly monotone. This also implies that the sets L_θ for $\theta \neq \theta_0$ are disjoint from L_{θ_0}. The monotonicity of the map $\theta \mapsto p_1(x_\theta)$ combined with Lemma 4.12 implies the second part of the lemma. ∎

Now, we can complete the proof of the proposition. By Theorem 3.1, for any $\theta \in \mathbb{S}^1$ and $r \in (0,1)$ the action $\varphi^A_{|G_\theta \cap G_{\theta+r}}$ admits a unique fixed point $a(r,\theta)$ in L_θ. Moreover, for any $\theta \in \mathbb{S}^1$, the map

$$(0,1) \to A, \qquad r \mapsto a(r,\theta),$$

is one-to-one, continuous and extends to a continuous map $[0,1] \to A$ which allows us to define $a(0,\theta)$, which is the fixed point of $\varphi^A_{|G_\theta}$ on one boundary component of A, and $a(1,\theta)$, which is the fixed point of $\varphi^A_{|G_\theta}$ on the other boundary component of A. By Lemma 4.13, a is one-to-one. Moreover, for every $\alpha \in \mathbb{S}^1$ and every $(r,\theta) \in \mathbb{A} = [0,1] \times \mathbb{S}^1$, we have $a(r, \theta + \alpha) = R_\alpha(a(r,\theta))$. Indeed, $R_\alpha(a(r,\theta))$ is the only fixed point of $\varphi^A_{|G_{\theta+\alpha} \cap G_{\theta+\alpha+r}} = R_\alpha \varphi^A_{|G_\theta \cap G_{\theta+r}} R_\alpha^{-1}$ in $L_{\theta+\alpha} = R_\alpha(L_\theta)$. This implies that a is continuous, so it is a homeomorphism $\mathbb{A} \to A$. It remains to prove that it defines a conjugacy with a_+. Take a homeomorphism f and a point $(r,\theta) \in \mathbb{A}$. Notice that

$$fG_\theta \cap G_{\theta+r} f^{-1} = G_{f(\theta)} \cap G_{f(\theta+r)} = G_{f(\theta)} \cap G_{f(\theta)+(\tilde{f}(\theta+r)-\tilde{f}(\theta))}$$

and $\varphi^A(f)(L_\theta) = L_{f(\theta)}$. Therefore, $\varphi^A(f)(a(r,\theta)) = a(a_+(f)(r,\theta))$. The action φ^A is conjugate to a_+. As a_+ and a_- are conjugate both by an orientation-preserving homeomorphism and by an orientation-reversing one, the proposition is proved. ∎

4.3. Global conjugacy. In this section we end the proof of Theorem 2.1 in the case of actions on an annulus.

By Lemma 4.2, our action φ, restricted to the union $K \times \mathbb{S}^1$ of the F_θ where θ varies over the circle, is conjugate to the restriction of $\varphi_{K,\lambda}$ to $K \times \mathbb{S}^1$ by a homeomorphism of the form $(r,\theta) \mapsto (r, \eta(r)+\theta)$ for any λ. Conjugating φ by a homeomorphism of the annulus of the form $(r,\theta) \mapsto (r, \hat{\eta}(r) + \theta)$, where $\hat{\eta}$ is a continuous function equal to η on K, we may assume from now on that $\varphi = \varphi_{K,\lambda}$ on $K \times \mathbb{S}^1$ for any λ.

Moreover, for each connected component $(r_1^i, r_2^i) \times \mathbb{S}^1$ of the complement of $K \times \mathbb{S}^1$, the restriction of this action to $[r_1^i, r_2^i] \times \mathbb{S}^1$ is conjugate to a_+ via an orientation-preserving homeomorphism $g_i : [0,1] \times \mathbb{S}^1 \to [r_1^i, r_2^i] \times \mathbb{S}^1$, by Proposition 4.3.

We now define a particular continuous function $\lambda_0 : [0,1] - K \to \{-1, 1\}$ such that φ is conjugate to φ_{K,λ_0}. For any index i, denote by d_i^+ the diameter of $g_i(\{(r,0) \mid 0 \leq r \leq 1\})$ and by d_i^- the diameter of $g_i(\{(r,-r) \mid 0 \leq r \leq 1\})$. If $d_i^+ \leq d_i^-$, we define λ_0 to be identically 1 on (r_1^i, r_2^i). Otherwise, i.e. if $d_i^+ > d_i^-$, we define the map λ_0 to be identically -1 on (r_1^i, r_2^i).

Now, let us define a conjugacy between φ and φ_{K,λ_0}. Notice that, by the above, for any connected component $(r_1^i, r_2^i) \times \mathbb{S}^1$ of the complement of $K \times \mathbb{S}^1$, there exists an orientation-preserving homeomorphism h_i of $[r_1^i, r_2^i] \times \mathbb{S}^1$ such that, for any orientation-preserving homeomorphism f of the circle,

$$h_i \circ \varphi_{K,\lambda_0}(f)_{|[r_1^i, r_2^i] \times \mathbb{S}^1} = \varphi(f)_{|[r_1^i, r_2^i] \times \mathbb{S}^1} \circ h_i.$$

Then, for any $(r, \theta) \in [r_1^i, r_2^i] \times \mathbb{S}^1$ and any $\alpha \in \mathbb{S}^1$, we have $h_i(r, \theta + \alpha) = h_i(r, \theta) + (0, \alpha)$. (Recall that we have supposed at the beginning of this section that the morphism φ sends the α-rotation of the circle to the α-rotation of the annulus.)

Now, denote by $h : \mathbb{A} \to \mathbb{A}$ the map which is the identity on $K \times \mathbb{S}^1$, and h_i on the connected component $(r_1^i, r_2^i) \times \mathbb{S}^1$ of the complement of $K \times \mathbb{S}^1$. It is clear that h is a bijection and $h \circ \varphi_{K,\lambda_0}(f) = \varphi(f) \circ h$ for all f in $\mathrm{Homeo}_0(\mathbb{S}^1)$. Moreover, h commutes with rotations.

It remains to prove that h is continuous. As h commutes with rotations, it suffices to prove that the map

$$\eta : [0,1] \to \mathbb{A}, \qquad r \mapsto h(r, 0),$$

is continuous. Notice first that, for a connected component (r_1^i, r_2^i) of the complement of K, we have $\lim_{r \to r_1^{i+}} \eta(r) = h_i(r_1, 0)$. This last point is the only point of F_0 on $\{r_1\} \times \mathbb{S}^1$ and is therefore equal to $\eta(r_1) = (r_1, 0)$. The map η is also left continuous at r_2^i. It now suffices to establish that, for any sequence $((r_1^n, r_2^n))_{n \in \mathbb{N}}$ of connected components of the complement of K such that $(r_1^n)_{n \in \mathbb{N}}$ is monotone and converges to a point $r_\infty \in K$, for any sequence $(r_n)_{n \in \mathbb{N}}$ of real numbers such that $r_n \in (r_1^n, r_2^n)$ for any n, the sequence $(\eta(r_n))_{n \in \mathbb{N}}$ converges to $\eta(r_\infty) = (r_\infty, 0)$.

LEMMA 4.14. *For n sufficiently large, one of the curves $h([r_1^n, r_2^n] \times \{0\})$ and $h\big(\big\{\big(r, \theta - \frac{r - r_1^n}{r_2^n - r_1^n}\big) \mid r_1^n \leq r \leq r_2^n\big\}\big)$ is homotopic to $[r_1^n, r_2^n] \times \{0\}$ with fixed extremities in $[r_1^n, r_2^n] \times \mathbb{S}^1$. We denote this curve by c_n.*

Proof. Suppose for contradiction that there exists a strictly increasing map $\sigma : \mathbb{N} \to \mathbb{N}$ such that, for any $n \in \mathbb{N}$, neither $h([r_1^{\sigma(n)}, r_2^{\sigma(n)}] \times \{0\})$ nor $h\big(\{(r, \theta - \frac{r - r_1^{\sigma(n)}}{r_2^{\sigma(n)} - r_1^{\sigma(n)}}) \mid r_1^{\sigma(n)} \leq r \leq r_2^{\sigma(n)}\}\big)$ is homotopic to $[r_1^n, r_2^n] \times \{0\}$. Then one of these two curves, which we denote by γ_n, admits a lift $\widetilde{\gamma}_n$ such that $\widetilde{\gamma}_n \cap \{r_1^{\sigma(n)}\} \times \mathbb{R} = \{(r_1^{\sigma(n)}, 0)\}$ and $\widetilde{\gamma}_n \cap \{r_2^{\sigma(n)}\} \times \mathbb{R} = \{(r_2^{\sigma(n)}, k_n)\}$, where k_n is an integer with $|k_n| \geq 2$. Taking a subsequence if necessary, we can suppose that either $k_n \geq 2$ for any n, or $k_n \leq -2$ for any n. To simplify notation, suppose that $k_n \geq 2$ for any n.

We now need an intermediate result. Let $(\tilde{x}_n)_{n\in\mathbb{N}}$ and $(\tilde{y}_n)_{n\in\mathbb{N}}$ be sequences in $[0,1] \times \mathbb{R}$ converging respectively to $\tilde{x}_\infty$ and $\tilde{y}_\infty$ such that, for any n, $\tilde{x}_n$ and $\tilde{y}_n$ belong to the curve $\widetilde{\gamma}_n$ and are not endpoints of this curve. We suppose that $\tilde{x}_\infty \neq \tilde{y}_\infty$ and that the points $\tilde{x}_\infty$ and $\tilde{y}_\infty$ are not limit points of the endpoints of $\widetilde{\gamma}_n$. For any integer n, there exist unique real r_n, $r'_n \in (0,1)$ such that $x_n = \pi(\tilde{x}_n)$ (respectively $y_n = \pi(\tilde{y}_n)$) is the unique fixed point of the group $\varphi(G_0 \cap G_{r_n})$ (respectively $\varphi(G_0 \cap G_{r'_n})$) on the curve γ_n. We claim that, for any strictly increasing map $s : \mathbb{N} \to \mathbb{N}$, if $(r_{s(n)})_{n\in\mathbb{N}}$ and $(r'_{s(n)})_{n\in\mathbb{N}}$ converge respectively to R and R', then $R \neq R'$. Moreover, R and R' are different from 0 and 1.

To prove this claim, suppose for contradiction that there exists a map s such that $R = R'$. We may suppose (by extracting a subsequence and by changing the roles of x_n and y_n if necessary) that $r_{s(n)} < r'_{s(n)}$ for any n. Then the set of points x in $\gamma_{s(n)}$ for which there exists $r_{s(n)} \leq r \leq r'_{s(n)}$ such that x is fixed under the action $\varphi_{|G_0 \cap G_r}$ (this is also the projection of the set of points on $\widetilde{\gamma}_{s(n)}$ between $\tilde{x}_{s(n)}$ and $\tilde{y}_{s(n)}$) defines a sequence of paths which converge to an interval contained in $\{r_\infty\} \times \mathbb{S}^1$. This interval is the projection of the interval whose endpoints are $\tilde{x}_\infty$ and $\tilde{y}_\infty$; it has non-empty interior and is necessarily pointwise fixed by $\varphi_{|G_0 \cap G_R}$. Indeed, each f in $G_0 \cap G_R$ fixes the points between $r_{s(n)}$ and $r'_{s(n)}$ for n sufficiently large, since it pointwise fixes a neighbourhood of R by definition of G_R. Therefore $\varphi(f)$ fixes the projection of the set of points on $\widetilde{\gamma}_{s(n)}$ between $\tilde{x}_{s(n)}$ and $\tilde{y}_{s(n)}$, for n sufficiently large. Hence, it also pointwise fixes the projection of the interval between $\tilde{x}_\infty$ and $\tilde{y}_\infty$ on $\{r_\infty\} \times \mathbb{R}$. However, there exists no such non-trivial interval, as the group $\varphi(G_0 \cap G_R)$ fixes pointwise no non-trivial interval of $\{r_\infty\} \times \mathbb{S}^1$, a contradiction.

Let us now establish for instance that $R \neq 1$. The set of points in $\gamma_{s(n)}$ which are fixed under $\varphi_{|G_0 \cap G_r}$ with $r_{s(n)} \leq r \leq 1$ is a path which admits a lift which joins $\tilde{x}_{s(n)}$ to either $(r_2^{\sigma(s(n))}, k)$ or $(r_1^{\sigma(s(n))}, k)$. Taking a subsequence if necessary, this sequence of sets converges to a non-trivial interval in $\{r_\infty\} \times \mathbb{S}^1$. This interval is necessarily pointwise fixed by $\varphi_{|G_0 \cap G_R}$, a contradiction.

Let us come back to our proof. There exists a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ in $[0,1]\times\mathbb{R}$ converging to $(r_\infty,1)$ such that, for any n, $\tilde{x}_n$ belongs to the curve $\widetilde{\gamma}_n$ and is not an endpoint of this curve. Let $x_n=\pi(\tilde{x}_n)$. For any n, there exists a unique $r_n\in(0,1)$ such that x_n is the unique fixed point of the group $\varphi(G_0\cap G_{r_n})$ on γ_n. For any n, take a point $\tilde{y}_n$ on $\widetilde{\gamma}_n$ such that $(\tilde{y}_n)_n$ converges to a point $\tilde{p}$ not of the form (r_∞,k), where k is an integer. Then $\widetilde{p}$ is not a limit point of a sequence of endpoints of $\widetilde{\gamma}_n$. As above, let $r'_n\in(0,1)$ be associated to $y_n=\pi(\tilde{y}_n)$. Consider a strictly increasing map $s:\mathbb{N}\to\mathbb{N}$ such that $(r_{s(n)})_{n\in\mathbb{N}}$ and $(r'_{s(n)})_{n\in\mathbb{N}}$ converge respectively to R and R'. Take f in G_0 which sends R to R'. Let $z_n=\varphi(f)(x_n)$. Then, for any n, the real number associated to $z_{s(n)}$ is necessarily $f(r_{s(n)})$, and $(f(r_{s(n)}))_n$ converges to R'. By the claim above, $(z_{s(n)})_n$ has the same limit as $(y_{s(n)})_n$. By continuity, this homeomorphism of the annulus sends $(r_\infty,0)$ to $p=\pi(\tilde{p})$. This is impossible as $(r_\infty,0)$ is fixed under the group $\varphi(G_0)$. ■

LEMMA 4.15. *The diameter of the curve c_n tends to* 0 *as $n\to\infty$.*

Proof. This proof is similar to the previous one. Suppose that $\operatorname{diam} c_n\not\to 0$. Denote by $\tilde{c}_n$ the lift of c_n with origin at $(r_1^{\sigma(n)},0)$. There exists a subsequence $\tilde{c}_{s(n)}$ which converges (in the Hausdorff topology) to a non-trivial interval. As the projection of this interval on the annulus is invariant under $\varphi_{|G_0}$, this compact set projects onto the whole circle $\{r_\infty\}\times\mathbb{S}^1$. Therefore, there exists a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$, where $\tilde{x}_n\in\tilde{c}_{s(n)}$, which converges to a point of the form (r_∞,k) with $k\neq 0$ (which is not the limit of a sequence of endpoints of $\tilde{c}_{s(n)}$). There also exists a sequence $(\tilde{y}_n)_{n\in\mathbb{N}}$, where $\tilde{y}_n\in\tilde{c}_{s(n)}$, which converges to a point $\tilde{p}$ not of the form (r_∞,k), where k is an integer. As in the proof of the above lemma, let $r_n\in(0,1)$ (respectively r'_n) be associated to $x_n=\pi(\tilde{x}_n)$ (respectively $y_n=\pi(\tilde{y}_n)$). Taking subsequences if necessary, we may suppose that $(r_n)_n$ and $(r'_n)_n$ converge respectively to R and R'. As in the above proof, $R\neq R'$ and both are different from 0 and 1. Take f in G_0 which sends R to R'. Then $\varphi(f)$ sends $(r_\infty,0)$ to $p=\pi(\tilde{p})\neq(r_\infty,0)$, which is impossible as $(r_\infty,0)$ is fixed under the group $\varphi(G_0)$. ■

Now, let us prove that $c_n=h([r_1^n,r_2^n]\times\{0\})$ for n sufficiently large, which proves the continuity of the map η by the above lemma (because the endpoints of these curves converge to $\eta(r_\infty)$). Denote by c'_n the curve among $h([r_1^n,r_2^n]\times\{0\})$ and $h\big(\big\{\big(r,\theta-\frac{r-r_1^n}{r_2^n-r_1^n}\big)\ \big|\ r_1^n\le r\le r_2^n\big\}\big)$ which is not equal to c_n. As the homotopy class of c'_n is not the homotopy class of $[r_1^n,r_2^n]\times\{0\}$, the diameter of c'_n is bounded from below by $1/2$. Hence, for n sufficiently large (say for $n\ge N$), $\operatorname{diam} c_n<\operatorname{diam} c'_n$.

Fix now $n\ge N$. The orientation of the circle defines an order on $\mathbb{S}^1-\{0\}$. Take $f\neq\mathrm{Id}$ in G_0 such that $f(\theta)\ge\theta$ for any $\theta\neq 0$. Then the restriction of $\varphi(f)$ to $h([r_1^n, r_2^n] \times \{0\})$ and to $h\big(\big\{\big(r, \theta - \frac{r - r_1^n}{r_2^n - r_1^n}\big) \mid r_1^n \leq r \leq r_2^n\big\}\big)$ defines orientations on both curves (such that an arc from a point x to its image under $\varphi(f)$ is positively oriented). If $d_n^+ \leq d_n^-$, the interior of c_n is $g_n((0,1) \times \{0\})$. Indeed, this is the only simple curve in $(r_1^n, r_2^n) \times \mathbb{S}^1$ with the following properties:

- It is invariant under $\varphi_{|G_0}$.
- It is oriented from the point $(r_1^n, 0)$ to $(r_2^n, 0)$.

Moreover, $(r_1^n, r_2^n) \times \{0\}$ is the only simple curve in $(r_1^n, r_2^n) \times \mathbb{S}^1$ with the following properties:

- It is invariant under the action φ_{K,λ_0} restricted to G_0.
- It is oriented from $(r_1^n, 0)$ to $(r_2^n, 0)$.

Finally, as h is continuous on $[r_1^n, r_2^n] \times \mathbb{S}^1$ and $h \circ \varphi_{K,\lambda_0} = \varphi \circ h$, we have $h([r_1^n, r_2^n] \times \{0\}) = c_n$. If $d_n^- < d_n^+$, the interior of c_n is $g_n(\{(r, -r) \mid 0 < r < 1\})$. Indeed, this is the only simple curve in $(r_1^n, r_2^n) \times \mathbb{S}^1$ with the following properties:

- It is invariant under $\varphi_{|G_0}$.
- It is oriented from $(r_2^n, 0)$ to $(r_1^n, 0)$.

Moreover, $(r_1^n, r_2^n) \times \{0\}$ is the only simple curve in $(r_1^n, r_2^n) \times \mathbb{S}^1$ with the following properties:

- It is invariant under φ_{K,λ_0} restricted to G_0.
- It is oriented from $(r_2^n, 0)$ to $(r_1^n, 0)$.

Moreover, as h is continuous on $[r_1^n, r_2^n] \times \mathbb{S}^1$ and $h \circ \varphi_{K,\lambda_0} = \varphi \circ h$, we have $h([r_1^n, r_2^n] \times \{0\}) = c_n$.

5. Case of the torus. Let $\varphi : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}_0(\mathbb{T}^2)$ be a one-to-one group morphism (if such a group morphism is not one-to-one, it is trivial). Recall that such a morphism is continuous by [15, Theorem 4 and Proposition 2]. In this section, we prove the following theorem and its corollary:

Theorem 5.1. *For any point x of the circle, the image under φ of the group G_x admits a global fixed point.*

Corollary 5.2. *The action φ admits an invariant essential circle, so the study of this action reduces to the study of an action on an annulus.*

Here, *essential* means non-separating, so that the surface obtained by cutting along this curve is an annulus. Using the first part of Theorem 2.1 which was proved in the previous section, the corollary implies directly the second part of the theorem. Let us first see why this theorem implies the corollary.

Proof of Corollary 5.2. First, the image under φ of the group $\mathbb{S}^1$ of rotations of the circle is conjugate to the subgroup of rotations of the torus of the form

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{T}^2, \quad (\theta_1, \theta_2) \mapsto (\theta_1, \theta_2 + \alpha),$$

where $\alpha \in \mathbb{S}^1$. Therefore, after possibly conjugating, we may suppose that φ sends the α-rotation of the circle to the rotation $(\theta_1, \theta_2) \mapsto (\theta_1, \theta_2 + \alpha)$.

Fix a point x_0 on the circle. If p is a fixed point for the group $\varphi(G_{x_0})$, then the essential circle $\{\varphi(R_\alpha)(p) \mid \alpha \in \mathbb{S}^1\}$ is invariant under the action φ. The proof of this last claim is similar to the proof of Lemma 4.1. ∎

Proof of Theorem 5.1. Let us begin by sketching the proof. First, we prove that the diameters of the images of the fundamental domain $[0,1]^2$ under lifts of homeomorphisms in the image of φ are uniformly bounded. Then we prove that the restriction of φ to a subgroup of the form G_x lifts to a group morphism $\widetilde{\varphi}_x$ of G_x into $\mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2)$, the group of homeomorphisms of $\mathbb{R}^2$ which commute with integral translations. We prove that $\widetilde{\varphi}_x$ can be chosen to have a bounded orbit. Using these facts, we find a connected subset of $\mathbb{R}^2$ with empty interior which is invariant under the action $\widetilde{\varphi}_x$. Using prime ends, we can prove that the action $\varphi_{|G_x}$ has a fixed point on the projection of this connected set.

For g in $\mathrm{Homeo}_0(\mathbb{T}^2)$, we denote by $\tilde{g}$ the lift of g to $\mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2)$ (this means that $\pi \circ \tilde{g} = g \circ \pi$ where $\pi : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the projection) with $\tilde{g}(0) \in [-1/2, 1/2) \times [-1/2, 1/2)$. Denote by $D \subset \mathbb{R}^2$ the fundamental domain $[0,1]^2$ for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$.

LEMMA 5.3. *The map* $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathbb{R}_+$ *which associates to any* f *in* $\mathrm{Homeo}_0(\mathbb{S}^1)$ *the diameter of the image under* $\widetilde{\varphi(f)}$ (*or equivalently under any lift of* $\varphi(f)$) *of the fundamental domain* D *is bounded.*

Proof. The proof is almost identical to that of Lemma 4.4. ∎

Let x_0 be a point of the circle.

LEMMA 5.4. *There exists a group morphism* $\widetilde{\varphi}_{x_0} : G_{x_0} \to \mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2)$ *such that:*

- *for any homeomorphism* f *in* G_{x_0}, *we have* $\Pi \circ \widetilde{\varphi}_{x_0}(f) = \varphi(f)$, *where* $\Pi : \mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2) \to \mathrm{Homeo}_0(\mathbb{T}^2)$ *is the projection;*
- *the subset* $\{\widetilde{\varphi}_{x_0}(f)(0) \mid f \in G_{x_0}\}$ *is bounded.*

Moreover, $\widetilde{\varphi}_{x_0}$ *is continuous.*

Proof. Let $G = \varphi(G_{x_0})$. Observe that the map

$$G \times G \to \mathbb{Z}^2, \quad (f,g) \mapsto \widetilde{fg}^{-1}(\tilde{f}(\tilde{g}(0))),$$

defines a 2-cocycle on the group G (see [4] or [7] for more about cohomology of groups). Moreover, by Lemma 5.3, this cocycle is bounded. However, as

G is isomorphic to $\mathrm{Homeo}_c(\mathbb{R})$, the cohomology group $H_b^2(G,\mathbb{Z}^2)$ is trivial (see [10] and [13]). This implies that there exists a bounded map $b : G \to \mathbb{Z}^2$ such that

$$\forall f, g \in G, \quad (\widetilde{fg})^{-1}(\tilde{f}(\tilde{g}(0))) = b(f) + b(g) - b(fg).$$

It then suffices to take for $\widetilde{\varphi}_{x_0}$ the composition of $\varphi_{|G_{x_0}}$ with

$$G \to \mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2), \quad f \mapsto \tilde{f} + b(f).$$

For this action, the orbit of 0 is bounded by construction. It now suffices to prove that this action is continuous. As G_{x_0} is contractible and $\Pi : \mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2) \to \mathrm{Homeo}_0(\mathbb{T}^2)$ is a covering, there exists a (unique) continuous map $\eta : G_{x_0} \to \mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2)$ which lifts $\varphi_{|G_{x_0}}$ and sends the identity to the identity. Then the map

$$G_{x_0} \times G_{x_0} \to \mathrm{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2), \quad (f,g) \to \eta(fg)^{-1}\eta(f)\eta(g),$$

is continuous and its image is contained in the discrete space of integral translations; hence it is constant and η is a group morphism. Two group morphisms which lift $\varphi_{|G_{x_0}}$ differ by a group morphism $G_{x_0} \to \mathbb{Z}^2$. However, as the group G_{x_0} is simple (hence perfect), such a group morphism is trivial and $\eta = \widetilde{\varphi}_{x_0}$. ∎

Note that this proof can be adapted to prove Lemma 4.7. However, the proof of Lemma 4.7 is specific to the annulus: it uses the action on the boundary components of the annulus.

We can now complete the proof of Theorem 5.1. Denote by F the closure of

$$\bigcup_{f \in G_{x_0}} \widetilde{\varphi}_{x_0}(f)((-\infty, 0] \times \mathbb{R}).$$

By the above two lemmas, there exists $M > 0$ such that $F \subset (-\infty, M] \times \mathbb{R}$. Denote by U the connected component of the complement of F which contains the open subset $(M, \infty) \times \mathbb{R}$. Denote by U' the image of U under the projection $p_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and by ψ_{x_0} the action on the annulus $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ defined by

$$\forall f \in \mathrm{Homeo}_0(\mathbb{S}^1), \quad \psi_{x_0}(f) \circ p_2 = p_2 \circ \widetilde{\varphi}_{x_0}(f).$$

Notice that $p_1 \circ \psi_{x_0} = \varphi_{|G_{x_0}} \circ p_1$, where $p_1 : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = \mathbb{T}^2$.

By construction, the open set U' is invariant under the action ψ_{x_0} (a fundamental domain far on the right must be sent into U' by any homeomorphism in the image of ψ_{x_0}, by the above two lemmas). Now, this action can be extended to the set of prime ends of U', giving a continuous action ψ of the group G_{x_0} (isomorphic to $\mathrm{Homeo}_c(\mathbb{R})$) on the topological space of prime ends of U', which is homeomorphic to $\mathbb{S}^1$.

By Proposition 3.12, this last action has a fixed point. Moreover, for any closed interval I whose interior contains x_0, the set of fixed points of the action of G_I contains an open interval and hence an accessible prime end. Therefore, the intersection of the set F_I of fixed points of the action $\psi_{x_0|G_I}$ with $[0, M] \times \mathbb{R}/\mathbb{Z}$ is non-empty. Hence, the set of fixed points of ψ_{x_0}, which is the intersection of the F_I's, is non-empty. Thus Theorem 5.1 is proved. ∎

6. Case of the sphere and of the closed disc. In this section, we discuss Conjecture 2.2. The following first step toward this conjecture was communicated to me by Kathryn Mann.

PROPOSITION 6.1 (Mann). *Fix a morphism* $\varphi : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}_0(\mathbb{S}^2)$ (*respectively* $\varphi : \mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}_0(\mathbb{D}^2)$). *Then the action* φ *has exactly two global fixed points on the sphere* (*respectively one global fixed point on the closed disc*).

Proof. The case of the disc is almost identical to the case of the sphere and is left to the reader.

Identify the sphere with $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. By a theorem by Kerékjártó (see [2]), the restriction of ψ to the group of rotations of $\mathbb{S}^1 \subset \mathrm{Homeo}_0(\mathbb{S}^1)$ is topologically conjugate to the action of an $\mathbb{S}^1$ Lie subgroup of SO(3). In other words, it is conjugate to an action of the form

$$\begin{aligned}&\mathbb{S}^1 \to \mathrm{Homeo}_0(\mathbb{S}^2),\\ &\theta \mapsto (x, y, z) \mapsto \big(\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y, z\big).\end{aligned}$$

Hence the action of the circle induced by φ has exactly two fixed points, which we denote by N and S. We now prove that the set $\{N, S\}$ is preserved by any element of the φ-image of $\mathrm{Homeo}_0(\mathbb{S}^1)$. Consider the subset $A \subset \mathrm{Homeo}_0(\mathbb{S}^1)$ consisting of the homeomorphisms which commute with a non-trivial finite order rotation of the circle. Then any element of $\varphi(A)$ preserves the set of fixed points of the φ-image of a non-trivial finite order rotation. This last set is equal to $\{N, S\}$. By the following lemma, each element of $\varphi(\mathrm{Homeo}_0(\mathbb{S}^1))$ preserves $\{N, S\}$.

LEMMA 6.2. *The set* A *generates the group* $\mathrm{Homeo}_0(\mathbb{S}^1)$, *i.e. any homeomorphism in* $\mathrm{Homeo}_0(\mathbb{S}^1)$ *can be written as a product of elements of* A.

Now, ψ restricted to $\{N, S\}$ induces a morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathbb{Z}/2$. As the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ is simple, this morphism is trivial, so Proposition 6.1 is proved. ∎

Proof of Lemma 6.2. By the fragmentation lemma (see [3, Theorem 1.2.3]), any homeomorphism in $\mathrm{Homeo}_0(\mathbb{S}^1)$ can be written as a product of homeomorphisms each supported in an interval of length smaller than 1/6 (where the length of the circle is 1). Moreover, any homeomorphism supported in the interior of an interval $I \subsetneq \mathbb{S}^1$ can be written as a commutator

$$f = f_1 f_2 f_1^{-1} f_2^{-1},$$

where f_1 and f_2 are homeomorphisms of the circle supported in I (see [14, Lemma 4.6]). Thus it suffices to prove that any commutator of homeomorphisms supported in the same interval I of length smaller than 1/6 can be written as a product of elements of A.

Let f_1 and f_2 be two homeomorphisms supported in I. Let R_θ be the θ-rotation of the circle. For $i = 1, 2$, define $g_{i|I} = f_i$, $g_{i|R_{1/2}(I)} = R_{1/2} f_i R_{1/2}^{-1}$, and $g_i(x) = x$ elsewhere. Notice that the homeomorphisms g_1 and g_2 commute with $R_{1/2}$ and hence belong to A. Take a homeomorphism h in A which commutes with all order 3 rotations such that $h(R_{1/2}(I)) \cap R_{1/2}(I) = \emptyset$ and $h_{|I} = \mathrm{Id}_I$ (such a homeomorphism h exists as I is short enough). Then the homeomorphism $[g_1, hg_2h^{-1}]$ is equal to $[g_1, g_2] = [f_1, f_2]$ on I, to $[\mathrm{Id}, hg_2h^{-1}] = \mathrm{Id}$ on $h(R_{1/2}(I))$, to $[g_1, \mathrm{Id}] = \mathrm{Id}$ on $R_{1/2}(I)$, and to the identity elsewhere. Hence

$$[f_1, f_2] = [g_1, hg_2h^{-1}]$$

and Lemma 6.2 is proved. ■

It is now natural to try to adapt the proof of Section 4 to prove Conjecture 2.2. As in the case of the annulus, we can find an invariant lamination by circles but there is a problem when this lamination does not accumulate on one of the global fixed points of this action: one has to study the actions of the group of orientation-preserving homeomorphisms of the circle on the open annulus or on the half-open annulus such that the groups of the form G_θ have no fixed point in the interior of these surfaces. If we try to adapt the proof of Subsection 4.2, we are confronted with a problem: Lemma 4.4 is false in this case (it is easy to find counter-examples) and it seems difficult to adapt it to the new situation. However, this lemma seems to be the only problematic step for the proof of this conjecture.

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