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On Bourgain's bound for short exponential sums and squarefree numbers

by

RAMON M. NUNES (Orsay)

1. Introduction. As usual, let

$$e(x) := e^{2i\pi x} \quad \text{for } x \in \mathbb{R}.$$

In a recent paper, Bourgain [2] proved a non-trivial bound for exponential sums such as

$$\sum_{\substack{n \le N \\ (n,q)=1}} e\left(\frac{a\overline{n}^2}{q}\right),$$

where $q > 1$ is an integer and $\bar{n}$ denotes the multiplicative inverse of n (mod q). His result holds in the range $N \ge q^{\epsilon}$ for an arbitrarily small, but fixed, $\epsilon > 0$. In his paper, Bourgain was interested in an application related to the size of fundamental solutions $\epsilon_D > 1$ to the Pell equation

$$t^2 - Du^2 = 1.$$

He followed the lead of Fouvry [3], who suggested that such an upper bound could help improve the lower bounds for the counting function

$$S^f(x,\alpha) := \#\{(\epsilon_D, D);\ 2 \le D \le x,\ D \text{ is not a square, and } \epsilon_D \le D^{1/2+\alpha}\}$$

for small values of α. In this article, we are interested in a different application of Bourgain's result (see Proposition 4.2 below) related to squarefree numbers in arithmetic progressions.

Let $X \ge 1$. Let a and q be coprime integers such that $q \ge 2$ and let

$$E(X,q,a) := \sum_{\substack{n \le X \\ n \equiv a \,(\mathrm{mod}\, q)}} \mu(n)^2 - \frac{6}{\pi^2} \prod_{p \mid q} \left(1 - \frac{1}{q^2}\right)^{-1} \frac{X}{q}. \tag{1.1}$$

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For fixed q, the last term is known to be asymptotically equivalent to

$$\frac{1}{\varphi(q)}\sum_{\substack{n\le X\\(n,q)=1}}\mu(n)^2$$

as $X\to\infty$. So $E(X;q,a)$ can be seen as an error term of the distribution of squarefree numbers in arithmetic progressions. One naturally has the trivial bound

$$|E(X,q,a)|\le X/q+1. \tag{1.2}$$

In a previous article, the author [5] proved

THEOREM 1.1. *There exists an absolute constant $C>0$, such that, for every $\epsilon>0$, we have*

$$\sum_{\substack{a\,(\mathrm{mod}\,q)\\(a,q)=1}}E(X,q,a)^2\sim C\prod_{p|q}(1+2p^{-1})^{-1}X^{1/2}q^{1/2} \tag{1.3}$$

for $X\to\infty$, uniformly for integers q satisfying $X^{31/41+\epsilon}\le q\le X^{1-\epsilon}$.

This theorem gives the asymptotic variance of the above mentioned distribution.

Inspired by an equivalent problem considered by Fouvry et al. [4, Theorem 1.5.], we study how $E(X,q,a)$ correlates with $E(X,q,\gamma(a))$ for suitable choices of $\gamma:\mathbb{Z}/q\mathbb{Z}\to\mathbb{Z}/q\mathbb{Z}$. It is natural to choose γ to be an affine linear map, i.e.

$$\gamma_{r,s}(a)=ra+s, \tag{1.4}$$

where $r,s\in\mathbb{Z}$, $r\neq 0$ are fixed. Thus our object of study is the correlation sum

$$C[\gamma_{r,s}](X,q):=\sum_{\substack{a\,(\mathrm{mod}\,q)\\a\neq 0,\gamma_{r,s}^{-1}(0)}}E(X,q,a)E(X,q,\gamma_{r,s}(a)) \tag{1.5}$$

for q prime. In [5], we already considered the case $s=0$, and we found that the correlation always existed for any non-zero value of r. In particular, there exists $C_r\neq 0$ such that for $X\to\infty$ and $X^{31/41+\epsilon}\le q\le X^{1-\epsilon}$, one has

$$C[\gamma_{r,0}](X,q)\sim C_r\Bigg(\sum_{\substack{a\,(\mathrm{mod}\,q)\\(a,q)=1}}E(X,q,a)^2\Bigg). \tag{1.6}$$

Our main result is the following theorem which exhibits a certain independence between the functions $a\mapsto E(X,q,a)$ and $a\mapsto E(X,q,\gamma_{r,s}(a))$ considered as random variables on $\mathbb{Z}/q\mathbb{Z}$, which agrees with our intuition that $E(X,q,a)$ and $E(X,q,\gamma(a))$ should be asymptotically independent random variables when γ is not a homothety.

THEOREM 1.2. *There exists an absolute $\delta > 0$ such that for every $\epsilon > 0$ and every integer $r \neq 0$, there exists $C_{\epsilon,r}$ such that*

$$|C[\gamma_{r,s}](X,q)| \le C_{\epsilon,r}\left(q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta} + \frac{X^{5/3+\epsilon}}{q} + \left(\frac{X}{q}\right)^2\right) \tag{1.7}$$

uniformly for $X \ge 2$, integers s and prime numbers $q \le X$ such that $q \nmid rs$.

A consequence of Theorems 1.1 and 1.2 is the following

COROLLARY 1.3. *For every $\epsilon > 0$ and $r \neq 0$, there exists a function $\Phi_{\epsilon,r} : \mathbb{R}^+ \to \mathbb{R}^+$, tending to zero at infinity, such that for every $X \ge 2$, every integer s and every prime q such that $q \nmid rs$ and $X^{7/9+\epsilon} \le q \le X^{1-\epsilon}$, one has*

$$|C[\gamma_{r,s}](X,q)| \le \Phi_{\epsilon,r}(X)\Big(\sum_{\substack{a \,(\mathrm{mod}\, q)\\ (a,q)=1}} E(X,q,a)^2\Big). \tag{1.8}$$

Inequality (1.8) shows a behavior different from (1.6) corresponding to the case where $q \mid s$. Here, as in [5], we give results that are true for a general $r \neq 0$, but in order to simplify the presentation, we give proofs that are only complete when r is squarefree (the case where $\mu(r) = 0$ implies a more difficult definition of the κ function in (4.10)).

2. Notation.

We define the Bernoulli polynomials $B_k(x)$ for $k \ge 1$, on $[0,1)$, in the following recursive way:

$$B_1(x) := x - 1/2, \qquad \frac{d}{dx}B_{k+1}(x) = B_k(x), \qquad \int_0^1 B_k(x)\,dx = 0.$$

We can extend these functions to periodic functions defined on the whole real line by setting

$$B_k(x) := B_k(\{x\}).$$

We further notice that $B_1(x)$ satisfies the relation

$$\lfloor x \rfloor = x - 1/2 - B_1(x), \tag{2.1}$$

and $B_2(x)$ satisfies

$$B_2(x) = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} \quad \text{for } 0 \le x \le 1. \tag{2.2}$$

In the course of the proof of Theorem 1.2 we will make repeated use of the multiplicative function

$$h(d) = \mu(d)^2 \prod_{p \mid d} (1 - 2p^{-2})^{-1}. \tag{2.3}$$

We also define the closely related product

$$C_2 = \prod_p \left(1 - \frac{2}{p^2}\right). \tag{2.4}$$

We denote, as usual, by $d(n)$ and $d_3(n)$ the classical binary and ternary divisor functions, respectively.

We write $\omega(n)$ for the number of primes dividing n.

We write $n \sim N$ as an alternative to $N < n \leq 2N$.

If S is a finite set, $\#S$ denotes its cardinality. If $I \subset R$ is an interval, $|I|$ denotes its length.

We use indistinguishably the notation $f = O(g)$ and $f \ll g$ when there is an absolute constant C such that $|f| \leq Cg$, on a certain domain of the variables which will be clear from the context, and the same for the symbols O_ϵ, O_r, $O_{\epsilon,r}$ and $\ll_\epsilon$, $\ll_r$, $\ll_{\epsilon,r}$, but with constants that may depend on the subscripted variables.

3. Initial steps. Let $X \geq 2$. Let $\gamma = \gamma_{r,s}$ be given by (1.4) and let q be a prime number $\leq X$ such that $q \nmid rs$.

We start by completing the sum defining $C[\gamma](X, q)$ (see (1.5)), and we bound trivially the additional terms. By (1.2), we see that

$$C[\gamma](X, q) = \sum_{a=0}^{q-1} E(X, q, a)E(X, q, \gamma(a)) + O\left(\left(\frac{X}{q}\right)^2\right). \tag{3.1}$$

In what follows, for simplification, we shall write

$$C(q) = \frac{6}{\pi^2}\left(1 - \frac{1}{q^2}\right)^{-1}. \tag{3.2}$$

As we develop the sum on the right-hand side of (3.1), we obtain

$$C[\gamma](X, q) = S[\gamma](X, q) - 2C(q)\frac{X}{q}\sum_{n \leq X} \mu(n)^2 + C(q)^2\frac{X^2}{q} + O\left(\frac{X^2}{q^2}\right), \tag{3.3}$$

where $S[\gamma](X, q)$ is defined by the double sum

$$S[\gamma](X, q) = \sum\sum_{\substack{n_1, n_2 \leq X \\ n_2 \equiv \gamma(n_1) \pmod{q}}} \mu(n_1)^2 \mu(n_2)^2. \tag{3.4}$$

We point out that $S[\gamma](X, q)$ is the only difficult term appearing in equation (3.3), since we have the well-known formula

$$\sum_{n \leq X} \mu^2(n) = \frac{6}{\pi^2}X + O(\sqrt{X}) = C(q)X + O\left(\frac{X}{q^2} + \sqrt{X}\right), \tag{3.5}$$

uniformly for $1 \le q \le X$. An asymptotic expansion of $S[\gamma](X,q)$ will be given in Proposition 5.1.

4. Useful lemmata. We start with a lemma concerning the multiplicative function $h(d)$ which follows easily from [5, Lemma 5.2]:

LEMMA 4.1. *Let $h(d)$ be as in (2.3) and let β be the multiplicative function defined by*

$$h(d) = \sum_{mn=d} \beta(m), \qquad d \ge 1.$$

Then

$$\sum_{m \ge M} \frac{\beta(m)}{m} \ll M^{-1/2}, \tag{4.1}$$

$$\sum_{m \le M} \beta(m) \ll M, \tag{4.2}$$

uniformly for every $M \ge 1$.

Proof. By [5, Lemma 5.2], $\beta(m)$ is supported on cubefree numbers, and if we write $m = ab^2$ with a, b squarefree and relatively prime, then

$$\beta(m) \ll d(a)/a^2.$$

In particular, $\beta(m) \ll 1$, and it is sufficient to prove (4.2). In order to prove (4.1), we notice that

$$\sum_{m \ge M} \frac{\beta(m)}{m} \ll \sum\sum_{ab^2 \ge M} \frac{d(a)}{a^3 b^2} \le M^{-1/2} \sum_{a \ge 1} \frac{d(a)}{a^{5/2}} \ll M^{-1/2}. \ \blacksquare$$

The next proposition is the main result of [2], and it is crucial to our proof.

PROPOSITION 4.2 (see [2, Proposition 4]). *There exist constants c, C, C' such that for every $N, q \ge 2$ and $1/\log 2N < \beta < 1/10$, there exists a subset $E_N \subset \{1, \dots, N\}$ (independent of q) satisfying*

$$|E_N| \le C'\beta \Big(\log \frac{1}{\beta}\Big)^C N \tag{4.3}$$

and such that, uniformly for $(a,q) = 1$,

$$\Big| \sum_{\substack{n \le N \\ n \notin E_N,\, (n,q)=1}} e\Big(\frac{a\bar{n}^2}{q}\Big) \Big| \le C'(\log 2N)^C N^{1-c\left(\beta \frac{\log N}{\log q}\right)^C}. \tag{4.4}$$

In fact we need the following corollary.

Corollary 4.3. *There exists an absolute $\delta > 0$ such that, for every $\epsilon > 0$, we have*

$$\sum_{\substack{n\le N\\(n,q)=1}} e\left(\frac{a\bar{n}^2}{q}\right) \ll_\epsilon N(\log q)^{-\delta},$$

uniformly for $N, q \ge 2$ and $N \ge q^\epsilon$.

Remark 4.4. More generally, we may consider the sum

$$\Sigma(I,q) = \sum_{\substack{n\in I\\(n,q)=1}} e\left(\frac{a\bar{n}^2}{q}\right)$$

where I is a general interval of length N (mod q). It is well-known that

$$\Sigma(I,q) \ll q^{1/2}\log q \tag{4.5}$$

for prime numbers q. Hence, (4.5) is non-trivial as soon as $N \ge q^{1/2+\epsilon}$ (for any $\epsilon > 0$). Obviously, Bourgain's result is much stronger than (4.5), but it only applies, roughly speaking, to intervals starting at 1.

Proof of Corollary 4.3. We use Proposition 4.2 and choose $\beta = (\log N)^{-\delta_1}$, where $\delta_1 = \min\left(\frac{1}{2}, \frac{1}{2C}\right)$. We add (4.3) and (4.4) to obtain

$$\sum_{n\le N,\,(n,q)=1} e\left(\frac{a\bar{n}^2}{q}\right) \ll N\frac{(\log\log N)^C}{(\log N)^{-\delta_1}} + N\frac{(\log N)^C}{\exp(c\epsilon^C(\log N)^{1/2})}.$$

The corollary now follows by taking, for example, $\delta = \delta_1/2$. ∎

Remark 4.5. Corollary 4.3 will be essential to the proof of Proposition 5.1, in which we use it for values of N which are roughly of size $\sqrt{X/q}$. Since we want to take q as large as $X^{1-\epsilon}$, it is important that Bourgain's result holds for N as small as q^ϵ.

The next lemma is similar in essence to many others to be found in the literature: see for example [7, Theorem 1], [1, Proposition 1.4] or [6, Theorem 3]. The proof, for instance, follows the lines of [1, Proposition 1.4].

Lemma 4.6. *Let $X \ge 1$ and let ℓ, r be integers such that r is squarefree. Let*

$$I(X,\ell,r) := \{u \in \mathbb{R}; u, ru+\ell \in (0,X)\}, \tag{4.6}$$

$$S(\ell,r) := \sum_{n\in I(X,\ell,r)} \mu(n)^2\mu(rn+\ell)^2. \tag{4.7}$$

Then, for every $r > 0$,

$$S(\ell,r) = f(\ell,r)|I(X,\ell,r)| + O_r\left(d_3(\ell)X^{2/3}(\log 2X)^{7/3}\right) \tag{4.8}$$

uniformly for $X \geq 2$ and integers ℓ, where

$$f(\ell, r) = C_2\Big(\prod_{p|r} \frac{p^2-1}{p^2-2}\Big)\Big(\prod_{\substack{p^2|\ell \\ p\nmid r}} \frac{p^2-1}{p^2-2}\Big)\kappa((\ell, r^2)), \tag{4.9}$$

where κ is the multiplicative function defined by

$$\kappa(p^\alpha) = \begin{cases} \dfrac{p^2-p-1}{p^2-1} & \text{if } \alpha = 1, \\ \dfrac{p^2-p}{p^2-1} & \text{if } \alpha = 2, \\ 0 & \text{if } \alpha \geq 3. \end{cases} \tag{4.10}$$

We recall that C_2 is defined in (2.4).

Proof. We start by defining

$$\sigma(n) = \prod_{p^2|n} p, \qquad n \neq 0,$$

and

$$\xi(n) = \sigma(n)\sigma(rn+\ell). \tag{4.11}$$

Notice that the right-hand side of (4.11) actually depends on ℓ and r, but since these numbers will be held fixed in the following calculations, we omit this dependence. Since $\xi(n)$ is an integer ≥ 1 and

$$\mu(n)^2\mu(rn+\ell)^2 = 1 \iff \xi(n) = 1,$$

we deduce that

$$S(\ell, r) = \sum_{n\in I(X,\ell,r)} \sum_{d|\xi(n)} \mu(d) = \sum_{d\geq 1} \mu(d)N_d(\ell, r), \tag{4.12}$$

where

$$N_d(\ell, r) = \#\{n \in I(X, \ell, r);\ \xi(n) \equiv 0 \pmod d\}.$$

Notice that, for fixed ℓ and r, the condition $p \mid \xi(n)$ only depends on the congruence class of n modulo p^2. We let

$$\begin{aligned} u_p(\ell, r) &:= \#\{0 \leq v \leq p^2-1;\ \xi(v) \equiv 0 \pmod p\}, \\ U_d(\ell, r) &:= \prod_{p|d} u_p(\ell, r). \end{aligned} \tag{4.13}$$

By the Chinese remainder theorem, we have

$$N_d(\ell, r) = U_d(\ell, r)\frac{|I(X, \ell, r)|}{d^2} + O(U_d(\ell, r)) \tag{4.14}$$

for every positive squarefree integer d. We also notice that if $(p,r)=1$ then $|u_p(\ell,r)|\le 2$, and $|u_p(\ell,r)|\le p^2$ in general. Therefore

$$U_d(\ell,r)\ll_r 2^{\omega(d)}.$$

Let $2\le y\le X$ be a parameter which will be chosen later to be a power of X. We multiply (4.14) by $\mu(d)$ and sum over $d\le y$ to obtain

$$\sum_{d\le y}\mu(d)N_d(\ell,r)=\sum_{d\le y}\mu(d)U_d(\ell,r)\frac{|I(X,\ell,r)|}{d^2}+O_r\Big(\sum_{d\le y}2^{\omega(d)}\Big). \tag{4.15}$$

By completing the first sum on the right-hand side of (4.15), we get

$$\sum_{d\le y}\mu(d)N_d(\ell,r)=\prod_p\bigg(1-\frac{u_p(\ell,r)}{p^2}\bigg)|I(X,\ell,r)|+O_r\bigg(\frac{X\log y}{y}+y\log y\bigg). \tag{4.16}$$

For large d, formula (4.14) is useless. Instead, we will estimate by different means the sum

$$N_{>y}(\ell,r):=\sum_{d>y}\mu(d)N_d(\ell,r),$$

from which we will deduce the result.

We notice that $d\,|\,\xi(n)$ if and only if there exist $j,k\ge 1$ such that $d=jk$, $j^2\,|\,n$ and $k^2\,|\,rn+\ell$. Moreover, since $n, rn+\ell<X$ we have $j,k<\sqrt{X}$. From this observation we deduce

$$\begin{aligned}|N_{>y}(\ell,r)| &= \Big|\sum_{y<d\le X}\mu(d)|\{n\in I(X,\ell,r);\ \xi(n)\equiv 0 \pmod d\}|\Big|\\ &\le \sum_{\substack{j,k\le\sqrt{X}\\ jk>y}}\big|\big\{n\in\mathbb{Z};\ 0<n, rn+\ell<X \text{ and } j^2\,|\,n,\ k^2\,|\,rn+\ell\big\}\big|\\ &= \sum_{\substack{j,k\le\sqrt{X}\\ jk>y}} N(j,k),\end{aligned} \tag{4.17}$$

say. We shall divide the possible values of j and k into sets of the form

$$\mathcal{B}(J,K):=\{(j,k)\in\mathbb{Z}^2;\ j\sim J,\ k\sim K\}.$$

We can use at most $O((\log X)^2)$ such sets since we are summing over $j,k\le X^{1/2}$. For every $J,K\ge 1$, let

$$\begin{aligned}\mathcal{N}(J,K) &:= \sum_{j\sim J,\,k\sim K} N(j,k)\\ &= \#\{(j,k,u,v);\ j\sim J,\ k\sim K,\ 0<j^2u,k^2v<X \text{ and } k^2v=rj^2u+\ell\}.\end{aligned} \tag{4.18}$$

By dyadic decomposition we can find $1 \le J, K \le X^{1/2}$ such that $JK \ge y/4$, and we have the upper bound

$$N_{>y}(\ell, r) \ll \mathcal{N}(J, K)(\log X)^2. \tag{4.19}$$

Finally, we estimate

$$\mathcal{N}(J,K) \le \sum_{k\sim K} \sum_{u \le XJ^{-2}} \sum_{\substack{j\sim J \\ j^2 ru \equiv -\ell \,(\mathrm{mod}\, k^2)}} 1.$$

For j, k relevant to the sum above, we write $f = (j, k)$. From the congruence condition in the inner sum, we have $f^2 \mid \ell$. So we write

$$j_0 = j/f, \qquad k_0 = k/f, \qquad \ell_0 = \ell/f^2.$$

The congruence then becomes

$$j_0^2 ru \equiv -\ell_0 \ (\mathrm{mod}\ k_0^2).$$

Now, for $g = (k_0^2, r)$ as above we have $g \mid \ell_0$. We write

$$k_1 = k_0^2/g, \qquad s = r/g, \qquad t = \ell_0/g.$$

That transforms the congruence into

$$j_0^2 su \equiv -t \ (\mathrm{mod}\ k_1).$$

Finally, let $h = (k_1, t)$. From the considerations above, we must have $h \mid u$. We write

$$k' = k_1/h, \qquad t' = t/h, \qquad u' = u/h.$$

So the congruence becomes

$$j_0^2 su' \equiv -t' \ (\mathrm{mod}\ k'),$$

and since $(t', k') = 1$, it has at most $2 \cdot 2^{\omega(k')} \le 2d(k_0)$ solutions in j_0 (mod k'). Therefore

$$\begin{aligned}
\mathcal{N}(J,K) &\le \sum_{g|r} \sum_{f^2h|\ell} \sum_{\substack{k_0\sim K/f \\ gh|k_0^2}} \sum_{u' \le XJ^{-2}h^{-1}} \sum_{\substack{j_0\sim J/f \\ j_0^2 su' \equiv -t' \,(\mathrm{mod}\, k_0^2/gh)}} 1 \\
&\le 2 \sum_{g|r} \sum_{f^2h|\ell} \sum_{k_0\sim K/f} XJ^{-2}h^{-1} \left\{ \frac{Jgh}{fk_0^2} + 1 \right\} d(k_0) \\
&\ll_r \sum_{f^2h|\ell} \sum_{k_0\sim K/f} XJ^{-2} \left\{ \frac{J}{fk_0^2} + 1 \right\} d(k_0) \\
&\ll \sum_{f^2h|\ell} XJ^{-2} \left\{ \frac{J}{K^2} + \frac{1}{f} \right\} K \log K \\
&\ll d_3(\ell) XJ^{-2} \left\{ \frac{J}{K^2} + 1 \right\} K \log X.
\end{aligned}$$

Hence

$$\mathcal{N}(J,K) \ll_r d_3(\ell)\{Xy^{-1} + XJ^{-2}K\}\log X.$$

A similar inequality with the roles of J and K interchanged on the right-hand side can be obtained in an analogous way. Combining the two formulas, we deduce

$$\begin{aligned}\mathcal{N}(J,K) &\ll_r d_3(\ell)\{Xy^{-1} + X(JK)^{-1/2}\}\log X \\ &\ll d_3(\ell)Xy^{-1/2}\log X.\end{aligned} \tag{4.20}$$

Replacing (4.20) in (4.19) and adding the result to (4.16) gives

$$S(\ell,r) = \prod_p\Big(1 - \frac{u_p(\ell,r)}{p^2}\Big)|I(X,\ell,r)| + O_r\big(y\log y + d_3(\ell)Xy^{-1/2}(\log X)^3\big).$$

We make the choice $y = X^{2/3}(\log X)^{4/3}$, obtaining

$$S(\ell,r) = \prod_p\Big(1 - \frac{u_p(\ell,r)}{p^2}\Big)|I(X,\ell,r)| + O_r\big(d_3(\ell)X^{2/3}(\log X)^{7/3}\big). \tag{4.21}$$

We finish by a study of $u_p(\ell,r)$. We distinguish five cases (recall that r is squarefree):

- If $p\,|\,r$ and $p^2\,|\,\ell$ then $u_p(\ell,r) = p$.
- If $p\,|\,r$ and $p\,|\,\ell$ but $p^2\nmid\ell$ then $u_p(\ell,r) = p+1$.
- If $p\,|\,r$ and $p\nmid\ell$ then $u_p(\ell,r) = 1$.
- If $p\nmid r$ and $p^2\,|\,\ell$ then $u_p(\ell,r) = 1$.
- If $p\nmid r$ and $p^2\nmid\ell$ then $u_p(\ell,r) = 2$.

The lemma is now a consequence of (4.21) and of the different values of $u_p(\ell,r)$. ■

4.1. Sums involving the B_2 function. In the following we study certain sums involving the Bernoulli polynomials $B_2(x)$. In the next lemma, we deal with the simplest case

$$A(Y;q,a) = \sum_{\substack{n\ge1\\(n,q)=1}}\Big\{B_2\Big(\frac{Y^2}{n^2} + \frac{a\bar{n}^2}{q}\Big) - B_2\Big(\frac{a\bar{n}^2}{q}\Big)\Big\}, \tag{4.22}$$

where Y is a positive real number and a,q are coprime integers. The sum above will serve as an archetype for more complicated sums appearing in the proof of Proposition 4.10, which in turn will be central to estimating $C[\gamma](X,q)$.

One elementary bound for $A(Y;q,a)$ can be given by noticing that we have both

$$B_2\Big(\frac{Y^2}{n^2}+\frac{a\bar{n}^2}{q}\Big)-B_2\Big(\frac{a\bar{n}^2}{q}\Big)\ll 1, \tag{4.23}$$

since B_2 is bounded, and

$$B_2\Big(\frac{Y^2}{n^2}+\frac{a\bar{n}^2}{q}\Big)-B_2\Big(\frac{a\bar{n}^2}{q}\Big)=\int\limits_{a\bar{n}^2/q}^{Y^2/n^2+a\bar{n}^2/q} B_1(v)\,dv\ll\frac{Y^2}{n^2}, \tag{4.24}$$

since B_1 is also a bounded function. Gathering (4.23) and (4.24), we obtain

$$A(Y;q,a)\ll\sum_{n\le Y}1+\sum_{n>Y}\frac{Y^2}{n^2}\ll Y. \tag{4.25}$$

In the following lemma we give a non-trivial bound for the sum above by means of Bourgain's bound, via Corollary 4.3. What we obtain is better than trivial by just a small power of $\log q$, but it is sufficient to obtain Theorem 1.2.

Lemma 4.7. *There exists $\delta>0$ such that for every $\epsilon>0$,*

$$A(Y;q,a)\ll_\epsilon Y(\log q)^{-\delta} \tag{4.26}$$

uniformly for integers a and q such that $q\ge 2$, $(a,q)=1$ and for real numbers $Y>q^\epsilon$.

Proof. By Corollary 4.3, there exists $\delta_1>0$ such that

$$\sum_{\substack{n\le Y\\(n,q)=1}} e\Big(\frac{a\bar{n}^2}{q}\Big)\ll_\epsilon Y(\log q)^{-\delta_1} \tag{4.27}$$

uniformly for $(a,q)=1$ and $Y>q^{\epsilon/10}$. For simplification, we write

$$\Delta_Y(n;q,a)=B_2\Big(\frac{Y^2}{n^2}+\frac{a\bar{n}^2}{q}\Big)-B_2\Big(\frac{a\bar{n}^2}{q}\Big). \tag{4.28}$$

The sum in (4.27) appears naturally once we use the Fourier series development

$$B_2(x)=\sum_{h\ne0}\frac{1}{4\pi^2h^2}e(hx) \tag{4.29}$$

in formula (4.22). Let

$$\theta(q)=(\log q)^{\delta_1/2}. \tag{4.30}$$

By (4.23) and (4.29), we have

$$\sum_{\substack{n\le Y\theta(q)\\(n,q)=1}} \Delta_Y(n;q,a) = \sum_{\substack{Y\theta(q)^{-1}\le n\le Y\theta(q)\\(n,q)=1}} \Delta_Y(n;q,a) + O(Y\theta(q)^{-1}) \tag{4.31}$$

$$= \sum_{h\neq 0} \frac{1}{4\pi^2h^2} \sum_{\substack{Y\theta(q)^{-1}\le n\le Y\theta(q)\\(n,q)=1}} \left(e\left(\frac{hY^2}{n^2}\right)-1\right)e\left(\frac{ah\bar{n}^2}{q}\right) + O(Y\theta(q)^{-1})$$

$$= \sum_{1\le|h|\le\theta(q)^3} \frac{1}{4\pi^2h^2} \sum_{\substack{Y\theta(q)^{-1}\le n\le Y\theta(q)\\(n,q)=1}} \left(e\left(\frac{hY^2}{n^2}\right)-1\right)e\left(\frac{ah\bar{n}^2}{q}\right) + O(Y\theta(q)^{-1}).$$

Summing by parts, we see that the inner sum on the right-hand side is

$$\ll \sum_{Y\theta(q)^{-1}\le m\le Y\theta(q)} \frac{|h|Y^2}{m^3}\Bigg|\sum_{\substack{Y\theta(q)^{-1}\le n\le m\\(n,q)=1}} e\left(\frac{ah\bar{n}^2}{q}\right)\Bigg| + \Bigg|\sum_{\substack{Y\theta(q)^{-1}\le n\le Y\theta(q)\\(n,q)=1}} e\left(\frac{ah\bar{n}^2}{q}\right)\Bigg|.$$

Now, if q is prime and sufficiently large, then any integer h satisfying $1\le |h|\le\theta(q)^3$ is coprime to q. Then, by (4.27), the above expression is

$$\ll \sum_{Y\theta(q)^{-1}\le m\le Y\theta(q)} \frac{|h|Y^2}{m^2}(\log q)^{-\delta_1} + Y\theta(q)^{-1} \ll |h|Y\theta(q)^{-1}. \tag{4.32}$$

If we insert this upper bound in (4.31), we obtain

$$\sum_{\substack{n\le Y\theta(q)\\(n,q)=1}} \Delta_Y(n;q,a) \ll Y\theta(q)^{-1}\log\log q \ll Y(\log q)^{-\delta_1/4}. \tag{4.33}$$

For the remainder terms we use the trivial upper bound (4.24) to deduce

$$\sum_{\substack{n> Y\theta(q)\\(n,q)=1}} \Delta_Y(n;q,a) \ll \sum_{n>Y\theta(q)} \frac{Y^2}{n^2} \ll Y\theta(q)^{-1}. \tag{4.34}$$

Combining (4.33) and (4.34), we obtain

$$\sum_{\substack{n\ge 1\\(n,q)=1}} \Delta_Y(n;q,a) \ll Y(\log q)^{-\delta_1/4}$$

uniformly for $(a,q)=1$ and $Y>q^{\epsilon}$. The proof of Lemma 4.7 is now complete. ∎

Remark 4.8. Among the hypotheses of Lemma 4.7, it is essential that $(a,q)=1$. In the case where $q\,|\,a$, one cannot improve on (4.25). Indeed, it is possible to show that (see [5, Lemma 4.3])

$$A(Y;q,0) = -\frac{\varphi(q)}{q}\frac{\zeta(3/2)}{2\pi}Y + O(d(q)Y^{2/3}) \quad (Y\ge 1).$$

4.2. A consequence of Lemma 4.7. In order to evaluate $S[\gamma](X,q)$ (see (3.4)), it is important to consider the sum below.

Definition 4.9. For integers q,r,s such that $q\geq 1$ and $q\nmid rs$, let

$$\mathfrak{S}[\gamma_{r,s}](X,q) := \sum_{\ell\equiv s \,(\mathrm{mod}\, q)} f(\ell,r)|I(X,\ell,r)|.$$

Note that this sum is actually finite since whenever $|\ell|>2|r|X$, we have $I(X,\ell,r)=\emptyset$.

The purpose of this subsection is to prove the following.

Proposition 4.10. *There exists $\delta>0$ such that for every $\epsilon>0$ and every $r\neq 0$ such that r is squarefree, one has*

$$\mathfrak{S}[\gamma_{r,s}](X,q) = \left(\frac{6}{\pi^2}\right)^2\left(1+\frac{1}{q^2(q^2-2)}\right)^{-1}X^2/q + O_{\epsilon,r}\big(q^{1+\epsilon}+X^{1/2}q^{1/2}(\log q)^{-\delta}\big) \tag{4.35}$$

uniformly for $X\geq 2$, for integers s and prime numbers q such that $q\nmid r$.

The special case $r=1$ simplifies many of the calculations in the proof below. For instance, the sums over ρ, σ and τ disappear. This simpler result is, however, equally deep, and it might be helpful on a first reading to think of $r=1$, in order to see more clearly the connection between the upper bound (4.26) and the error term in (4.35).

Proof of Proposition 4.10. We start by recalling (4.9):

$$f(\ell,r)=C_2\bigg(\prod_{p|r}\frac{p^2-1}{p^2-2}\bigg)\bigg(\prod_{\substack{p^2|\ell\\ p\nmid r}}\frac{p^2-1}{p^2-2}\bigg)\kappa((\ell,r^2)),$$

where C_2 is as in (2.4). We notice that the term

$$C_2\prod_{p|r}\frac{p^2-1}{p^2-2}$$

is independent of ℓ. We consider the sum

$$\begin{aligned}\mathfrak{S}'[\gamma_{r,s}](X,q) &= C_2^{-1}\bigg(\prod_{p|r}\frac{p^2-1}{p^2-2}\bigg)^{-1}\mathfrak{S}[\gamma_{r,s}](X,q)\\ &= \sum_{\ell\equiv s\,(\mathrm{mod}\, q)}|I(X,\ell,r)|\bigg(\prod_{\substack{p^2|\ell\\ p\nmid r}}\frac{p^2-1}{p^2-2}\bigg)\kappa((\ell,r^2)).\end{aligned} \tag{4.36}$$

We expand the last product as follows:

$$\prod_{\substack{p^2|\ell \\ p\nmid r}} \frac{p^2-1}{p^2-2} = \sum_{\substack{d^2|\ell \\ (d,r)=1}} \frac{h(d)}{d^2},$$

from which we deduce

$$\begin{aligned}(4.37)\qquad \mathfrak{S}'[\gamma_{r,s}](X,q) &:= \sum_{\rho|r^2} \kappa(\rho) \sum_{\substack{\ell\equiv s \,(\mathrm{mod}\, q) \\ (\ell,r^2)=\rho}} |I(X,\ell,r)| \sum_{\substack{d^2|\ell \\ (d,r)=1}} \frac{h(d)}{d^2} \\ &= \sum_{\rho\sigma|r^2} \kappa(\rho)\mu(\sigma) \sum_{\ell_0\equiv\overline{\rho\sigma}s \,(\mathrm{mod}\, q)} |I(X,\rho\sigma\ell_0,r)| \sum_{\substack{d^2|\ell_0 \\ (d,r)=1}} \frac{h(d)}{d^2} \\ &= \sum_{\rho\sigma|r^2} \kappa(\rho)\mu(\sigma) \sum_{(d,qr)=1} \frac{h(d)}{d^2} \sum_{\ell_1\equiv\overline{(\rho\sigma d^2)}s \,(\mathrm{mod}\, q)} |I(X,\rho\sigma d^2\ell_1,r)|,\end{aligned}$$

where in the second line we used the Möbius inversion formula for detecting the condition $(\ell,r^2)=\rho$, and we noticed that the congruence satisfied by ℓ_0 implies $(d,q)=1$.

We write the inner sum as an integral:

$$\begin{aligned}(4.38)\qquad \sum_{\ell_1\equiv\overline{(\rho\sigma d^2)}s \,(\mathrm{mod}\, q)} &|I(X,\rho\sigma d^2\ell_1,r)| \\ &= \int_0^X \sum_{\ell_1\equiv\overline{(\rho\sigma d^2)}s \,(\mathrm{mod}\, q)} \mathbf{1}_{(0,X)}(ru+\rho\sigma d^2\ell_1)\,du,\end{aligned}$$

where $\mathbf{1}_{(0,X)}$ is the characteristic function of the interval $(0,X)$. Hence the inner sum above equals

$$\begin{aligned}&\left\lfloor \frac{X-ru}{\rho\sigma d^2 q} - \frac{\overline{(\rho\sigma d^2)}s}{q} \right\rfloor - \left\lfloor \frac{-ru}{\rho\sigma d^2 q} - \frac{\overline{(\rho\sigma d^2)}s}{q} \right\rfloor \\ &\quad = \frac{X}{\rho\sigma d^2 q} - B_1\Big(\frac{X-ru}{\rho\sigma d^2 q} - \frac{\overline{(\rho\sigma d^2)}s}{q}\Big) + B_1\Big(\frac{-ru}{\rho\sigma d^2 q} - \frac{\overline{(\rho\sigma d^2)}s}{q}\Big)\end{aligned}$$

for almost all $u\in(0,X)$ in the sense of Lebesgue measure. If we apply this formula in (4.38), we get

$$\begin{aligned}(4.39)\qquad \sum_{\ell_1\equiv\overline{(\rho\sigma d^2)}s \,(\mathrm{mod}\, q)} &|I(X,\rho\sigma d^2\ell_1,r)| \\ = \frac{X^2}{\rho\sigma d^2 q} - \frac{\rho\sigma d^2 q}{r}\bigg\{ &B_2\Big(\frac{X^2}{\rho\sigma d^2 q} - \frac{\overline{(\rho\sigma d^2)}s}{q}\Big) - B_2\Big(-\frac{\overline{(\rho\sigma d^2)}s}{q}\Big) \\ &- B_2\Big(\frac{(1-r)X}{\rho\sigma d^2 q} - \frac{\overline{(\rho\sigma d^2)}s}{q}\Big) + B_2\Big(\frac{-rX}{\rho\sigma d^2 q} - \frac{\overline{(\rho\sigma d^2)}s}{q}\Big)\bigg\}.\end{aligned}$$

From this point on, we suppose that $r < 0$. The case $r > 0$ requires only minor modifications. With this hypothesis, both

$$\frac{(1-r)X}{\rho\sigma d^2 q} \quad \text{and} \quad \frac{-rX}{\rho\sigma d^2 q}$$

are positive for every $\rho, \sigma \geq 1$.

We insert (4.39) in (4.37) and define

$$B(D;q,a;r) := \sum_{(d,qr)=1} h(d) \varDelta_D(d,q;a),$$

where $\varDelta_D(d,q;a)$ is as in (4.28). From (4.37) and (4.39) we deduce that

$$\begin{aligned}(4.40)\qquad \mathfrak{S}'[\gamma_{r,s}](X,q) = \lambda(q,r)\frac{X^2}{q} \\ - \frac{q}{r}\Big\{G\Big(\frac{X}{q};q,-s;r\Big) - G\Big(\frac{(1-r)X}{q};q,-s;r\Big) \\ + G\Big(\frac{-rX}{q};q,-s;r\Big)\Big\},\end{aligned}$$

where

$$G(Y;q,s;r) = \sum\sum_{\rho\sigma\mid r^2} \kappa(\rho)\mu(\sigma)\rho\sigma B\Big(\sqrt{\frac{Y}{\rho\sigma}}, q, \overline{\rho\sigma}s; r\Big),$$

$$\lambda(q,r) = \sum_{\rho\sigma\mid r^2} \frac{\kappa(\rho)\mu(\sigma)}{\rho\sigma} \times \sum_{(d,qr)=1} \frac{h(d)}{d^4}.$$

Let $\beta(m)$ be the function defined in Lemma 4.1. We observe that for all $D > 0$,

$$\begin{aligned}B(D;q,a;r) &= \sum_{(m,qr)=1} \beta(m) \sum_{(n,qr)=1} \varDelta_D(mn;q,a) \\ &= \sum_{(m,qr)=1} \beta(m) \sum_{(n,qr)=1} \varDelta_{D/m}(n;q,\overline{m}^2 a) \\ &= \sum_{(m,qr)=1} \beta(m) \sum_{\tau\mid r} \mu(\tau) \sum_{(n,q)=1} \varDelta_{D/(\tau m)}(n;q,\overline{\tau}^2\overline{m}^2 a) \\ &= \sum_{(m,qr)=1} \beta(m) \sum_{\tau\mid r} \mu(\tau) A(D/(\tau m), q; \overline{\tau}^2\overline{m}^2 a).\end{aligned}$$

We apply the equality above with $D = \sqrt{Y/(\rho\sigma)}$ and $a = \overline{\rho\sigma}s$, multiply by $\kappa(\rho)\mu(\sigma)\rho\sigma$ and sum over ρ, σ such that $\rho\sigma \mid r^2$. We obtain

$$\begin{aligned}(4.41)\qquad & G(Y;q,s;r) \\ &= \sum\sum_{\rho\sigma\mid r^2}\sum_{\tau\mid r}\sum_{(m,qr)=1} \kappa(\rho)\mu(\sigma)\mu(\tau)\rho\sigma\beta(m) A\Big(\sqrt{\frac{Y}{\rho\sigma\tau^2 m^2}}; q, \overline{\rho\sigma\tau^2 m^2} s\Big).\end{aligned}$$

Our discussion depends on the size of Y:

- If $Y \leq q^\epsilon$, we have the trivial bound (see (4.25))

$$A\Big(\sqrt{\frac{Y}{\rho\sigma\tau^2m^2}}; q, \overline{\rho\sigma\tau^2m^2s}\Big) \ll \sqrt{\frac{Y}{\rho\sigma\tau^2m^2}} \leq \frac{Y^{1/2}}{m}$$

for every $\rho, \sigma, \tau \geq 1$. Summing over ρ, σ, τ and m gives

$$G(Y; q, s; r) \ll_r Y^{1/2} \sum_{m\geq 1} \frac{\beta(m)}{m} \ll q^{\epsilon/2}, \tag{4.42}$$

as a consequence of the upper bound (4.1).

- If $Y > q^\epsilon$, we decompose the quadruple sum of (4.41) as

$$\sum_{\substack{m\leq q^{\epsilon/2} \\ \rho\sigma\tau^2m^2>Y/q^\epsilon}}\sum\sum\sum + \sum_{\substack{m\leq q^{\epsilon/2} \\ \rho\sigma\tau^2m^2\leq Y/q^\epsilon}}\sum\sum\sum + \sum_{m>q^{\epsilon/2}}\sum\sum\sum.$$

For the first sum we use again the trivial bound

$$A\Big(\sqrt{\frac{Y}{\rho\sigma\tau^2m^2}}; q, \overline{\rho\sigma\tau^2m^2s}\Big) \ll \sqrt{\frac{Y}{\rho\sigma\tau^2m^2}} \leq q^{\epsilon/2}. \tag{4.43}$$

The most delicate is the second sum, for which we appeal to (4.26). This gives

$$A\Big(\sqrt{\frac{Y}{\rho\sigma\tau^2m^2}}; q, \overline{\rho\sigma\tau^2m^2s}\Big) \ll_\epsilon \sqrt{\frac{Y}{\rho\sigma\tau^2m^2}}\,(\log q)^{-\delta}. \tag{4.44}$$

For the third sum, we use the trivial bound

$$A\Big(\sqrt{\frac{Y}{\rho\sigma\tau^2m^2}}; q, \overline{\rho\sigma\tau^2m^2s}\Big) \ll \sqrt{\frac{Y}{\rho\sigma\tau^2m^2}}. \tag{4.45}$$

Applying inequalities (4.43)–(4.45) in (4.41), we obtain

$$G(Y; q, s; r) \ll_{\epsilon,r} q^{\epsilon/2} \sum_{m\leq q^{\epsilon/2}} |\beta(m)| + \sqrt{Y}(\log q)^{-\delta} \sum_{m\leq q^{\epsilon/2}} \frac{|\beta(m)|}{m} + \sqrt{Y} \sum_{m>q^{\epsilon/2}} \frac{|\beta(m)|}{m},$$

and finally, by Lemma 4.1,

$$G(Y; q, s; r) \ll_{\epsilon,r} q^\epsilon + \sqrt{Y}(\log q)^{-\delta} \qquad (Y > q^\epsilon). \tag{4.46}$$

Comparing with (4.42), we see that (4.46) is true for any $Y \geq 1$.

Combining (4.46) and (4.40), one has

$$\mathfrak{S}'[\gamma_{r,s}](X, q) = \lambda(q, r)\frac{X^2}{q} + O_{\epsilon,r}\big(q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta}\big). \tag{4.47}$$

If we multiply the above formula by $C_2 \prod_{p\mid r} \frac{p^2-1}{p^2-2}$ (recall (4.36)), we deduce

$$\mathfrak{S}[\gamma_{r,s}](X,q) = \varLambda(q,r)\frac{X^2}{q} + O_{\epsilon,r}\big(q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta}\big), \tag{4.48}$$

where

$$\varLambda(q,r) = C_2\Big(\prod_{p\mid r}\frac{p^2-1}{p^2-2}\Big)\sum_{\rho\sigma\mid r^2}\frac{\kappa(\rho)\mu(\sigma)}{\rho\sigma} \times \sum_{(d,qr)=1}\frac{h(d)}{d^4}.$$

Since for r squarefree, we have

$$\sum_{\rho\sigma\mid r^2}\frac{\kappa(\rho)\mu(\sigma)}{\rho\sigma} = \prod_{p\mid r}\frac{p^2-1}{p^2},$$

standard calculations show that $\varLambda(q,r)$ does not depend on r. More precisely, since q is prime and $(q,r)=1$, we have

$$\varLambda(q,r) = \Big(\frac{6}{\pi^2}\Big)^2\Big(1+\frac{1}{q^2(q^2-2)}\Big)^{-1}.$$

Now, formula (4.48) completes the proof of Proposition 4.10. ∎

5. Study of $S[\gamma_{r,s}](X,q)$. We rewrite $S[\gamma_{r,s}](X,q)$ (see (3.4)) as

$$S[\gamma_{r,s}](X,q) = \sum_{\ell\equiv s \,(\mathrm{mod}\, q)}\ \sum_{n\in I(X,\ell,r)} \mu(n)^2\mu(rn+\ell)^2. \tag{5.1}$$

Recall that for $|\ell| > 2|r|X$ we have defined $I(X,\ell,r)$. Hence, by (4.8) (recall Definition 4.9),

$$\begin{aligned} S[\gamma_{r,s}](X,q) &= \sum_{\substack{\ell\equiv s \,(\mathrm{mod}\, q)\\ |\ell|\le 2|r|X}} f(\ell,r)|I(X,\ell,r)| + O_r\Big(\frac{X}{q}X^{2/3+\epsilon}\Big)\\ &= \mathfrak{S}[\gamma_{r,s}](X,q) + O_r\Big(\frac{X^{5/3+\epsilon}}{q}\Big). \end{aligned}$$

From Proposition 4.10, we deduce that

$$\begin{aligned} S[\gamma_{r,s}](X,q) = {}& \Big(\frac{6}{\pi^2}\Big)^2\Big(1+\frac{1}{q^2(q^2-2)}\Big)^{-1}\frac{X^2}{q}\\ &+ O_{\epsilon,r}\Big(q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta} + \frac{X^{5/3+\epsilon}}{q}\Big). \end{aligned} \tag{5.2}$$

By the definition (3.2) for $C(q)$, one can easily see that

$$\Big(\frac{6}{\pi^2}\Big)^2\Big(1+\frac{1}{q^2(q^2-2)}\Big)^{-1} = C(q)^2 + O\Big(\frac{1}{q^2}\Big).$$

In conclusion, we have proved

Proposition 5.1. *Let $C(q)$ be as in (3.2). There exists $\delta > 0$ such that for every $\epsilon > 0$ and every $r \neq 0$, one has the asymptotic formula*

$$S[\gamma_{r,s}](X,q) = C(q)^2\frac{X^2}{q} + O_{\epsilon,r}\left(q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta} + \frac{X^{5/3+\epsilon}}{q} + \frac{X^2}{q^3}\right) \tag{5.3}$$

uniformly for $X \geq 2$, for integers s and prime numbers q such that $q \nmid rs$ and $q \leq X$.

6. Proof of the main theorem. We start by recalling (3.3):

$$C[\gamma_{r,s}](X,q) = S[\gamma_{r,s}](X,q) - 2C(q)\frac{X}{q}\sum_{n\leq X}\mu(n)^2 + C(q)^2\frac{X^2}{q} + O\left(\frac{X^2}{q^2}\right).$$

By Proposition 5.1 and formula (3.5), we deduce the inequality

$$C[\gamma](X,q) \ll_{\epsilon,r} q^{1+\epsilon} + X^{1/2}q^{1/2}(\log q)^{-\delta} + \frac{X^{5/3+\epsilon}}{q} + \frac{X^2}{q^2}.$$

The proof of Theorem 1.2 is now complete.

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Ramon M. Nunes
Laboratoire de Mathématiques
Université Paris Sud
Campus d'Orsay
91405 Orsay Cedex, France
E-mail: ramon.moreira@math.u-psud.fr