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SEPARABLE FUNCTORS FOR THE CATEGORY OF DOI HOM-HOPF MODULES

BY

SHUANGJIAN GUO (Guiyang) and XIAOHUI ZHANG (Qufu)

Abstract. Let $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ be the category of Doi Hom-Hopf modules, $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ be the category of A-Hom-modules, and F be the forgetful functor from $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ to $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$. The aim of this paper is to give a necessary and suffcient condition for F to be separable. This leads to a generalized notion of integral. Finally, applications of our results are given. In particular, we prove a Maschke type theorem for Doi Hom-Hopf modules.

1. Introduction. The paper investigates variations on the theme of Hom-algebras, a topic which has recently received much attention from various researchers. The study of Hom-associative algebras originates with work by Hartwig, Larsson and Silvestrov in the Lie case [13], where a notion of Hom-Lie algebra was introduced in the context of deformations of Witt and Virasoro algebras. Later, it was extended to the associative case by Makhlouf and Silvestrov [17], where associativity is replaced by Hom-associativity $\alpha(a)(bc) = (ab)\alpha(c)$. Hom-coassociativity for a Hom-coalgebra can be considered in a similar way (see [17], [18]). Caenepeel and Goyvaerts [1] studied Hom-structures from the point of view of monoidal categories. This leads to the natural definition of monoidal Hom-algebras, Hom-coalgebras etc. They constructed a symmetric monoidal category, and then introduced monoidal Hom-algebras, Hom-coalgebras etc. as algebras, coalgebras etc. in this monoidal category. Recently, many more properties and structures of Hom-Hopf algebras have been developed: see [4]–[6], [10]–[12], [14], [16], [24], [25] and references cited therein.

Let H be a Hopf algebra with bijective antipode over a commutative ring k. As is well-known, a Hopf module is a k-module that is at once an H-module and an H-comodule, with a certain compatibility relation; see [22] for details. Doi [9] generalized this concept in the following way: if A is an H-comodule algebra and C is an H-module coalgebra, then he introduced

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a so-called unified Hopf module, that is, a k-module that is at once an A-module and a C-comodule, satisfying a compatibility relation which is an immediate generalization of the one that may be found in Sweedler's book [22]. One of the nice features here is that Doi's Hopf modules (we will call them *Doi–Hopf modules*) really unify a lot of module structures that have been studied by several authors; let us mention Sweedler's Hopf modules [22], Takeuchi's relative Hopf modules [23], graded modules, and modules graded by a G-set. In [2], induction functors between categories of Doi–Hopf modules and their adjoints are studied, and it turns out that many pairs of adjoint functors studied in the literature (the forgetful functor and its adjoint, extension and restriction of scalars, ...) are special cases. Caenepeel et al. [3] proved a Maschke type theorem for the category of Doi–Hopf modules. In fact, they gave necessary and sufficient conditions for the functor that forgets the H-coaction to be separable. This leads to a generalized notion of integrals of Doi [8].

The following questions arise naturally:

1. How do we introduce the notion of a Doi Hom-Hopf module?
2. There is an evident functor F from the category of Doi Hom-Hopf modules to the category of modules over the Hom algebra (A, β); does it possess a right adjoint? What is a sufficient and necessary condition for its separability?
3. How do we give a Maschke type theorem for Doi Hom-Hopf modules?

The aim of this article is to answer these questions.

In this paper we study the generalization of the previous results to monoidal Hom-Hopf algebras. In Sec. 3, we introduce the notion of a Doi Hom-Hopf module and prove that the functor F from the category of Doi Hom-Hopf modules to the category of right (A, β)-Hom-modules has a right adjoint (see Proposition 3.3). In Sec. 4, we obtain the main result of this paper: we give necessary and sufficient conditions for the functor that forgets the (C, γ)-coaction to be separable (see Theorem 4.2). Applications of our results are considered in Sec. 5.

Throughout this paper we freely use the Hopf algebra and coalgebra terminology introduced in [7], [19], [21] and [22].

2. Preliminaries. Throughout this paper we work over a commutative ring k. We recall from [1] some information about Hom-structures which are needed in what follows.

Let $\mathcal{C}$ be a category. We introduce a new category $\widetilde{\mathscr{H}}(\mathcal{C})$ as follows: objects are couples (M, μ), with $M \in \mathcal{C}$ and $\mu \in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $f : (M, \mu) \to (N, \nu)$ is a morphism $f : M \to N$ in $\mathcal{C}$ such that $\nu \circ f = f \circ \mu$.

Let $\mathscr{M}_k$ denote the category of k-modules. Then $\mathscr{H}(\mathscr{M}_k)$ will be the Hom-category associated to $\mathscr{M}_k$. If $(M,\mu) \in \mathscr{M}_k$, then $\mu : M \to M$ is obviously a morphism in $\mathscr{H}(\mathscr{M}_k)$. It is easy to show that $\widetilde{\mathscr{H}}(\mathscr{M}_k) = (\mathscr{H}(\mathscr{M}_k), \otimes, (I,I), \widetilde{a}, \widetilde{l}, \widetilde{r}))$ is a monoidal category by [1, Proposition 1.1]: the tensor product of (M,μ) and (N,ν) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is given by the formula $(M,\mu)\otimes(N,\nu) = (M \otimes N, \mu \otimes \nu)$.

Assume that $(M,\mu),(N,\nu),(P,\pi) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$. The associativity and unit constraints are given by the formulas

$$\begin{aligned} \widetilde{a}_{M,N,P}((m \otimes n) \otimes p) &= \mu(m) \otimes (n \otimes \pi^{-1}(p)), \\ \widetilde{l}_M(x \otimes m) = \widetilde{r}_M(m \otimes x) &= x\mu(m). \end{aligned}$$

An algebra in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ will be called a monoidal Hom-algebra.

Definition 2.1. A *monoidal Hom-algebra* is a couple $(A,\alpha) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $m_A : A \otimes A \to A$ and an element $1_A \in A$ such that

$$\begin{aligned} \alpha(ab) &= \alpha(a)\alpha(b), \quad && \alpha(1_A) = 1_A, \\ \alpha(a)(bc) &= (ab)\alpha(c), && a1_A = 1_A a = \alpha(a), \end{aligned}$$

for all $a,b,c \in A$. Here we use the notation $m_A(a \otimes b) = ab$.

Definition 2.2. A *monoidal Hom-coalgebra* is an object $(C,\gamma) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with k-linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation implicitly understood) and $\gamma : C \to C$ such that

$$\Delta(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}), \qquad \varepsilon(\gamma(c)) = \varepsilon(c)$$

and

$$\begin{aligned} \gamma^{-1}(c_{(1)}) \otimes c_{(2)(1)} \otimes c_{(2)(2)} &= c_{(1)(1)} \otimes c_{(1)(2)} \otimes \gamma^{-1}(c_{(2)}), \\ \varepsilon(c_{(1)})c_{(2)} &= \varepsilon(c_{(2)})c_{(1)} = \gamma^{-1}(c), \end{aligned}$$

for all $c \in C$.

Definition 2.3. A *monoidal Hom-bialgebra* $H = (H,\alpha,m,\eta,\Delta,\varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathscr{H}}(\mathscr{M}_k)$. This means that (H,α,m,η) is a Hom-algebra, (H,Δ,α) is a Hom-coalgebra, and Δ and ε are morphisms of Hom-algebras, that is,

$$\begin{aligned} \Delta(ab) &= a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \quad && \Delta(1_H) = 1_H \otimes 1_H, \\ \varepsilon(ab) &= \varepsilon(a)\varepsilon(b), && \varepsilon(1_H) = 1_H. \end{aligned}$$

Definition 2.4. A *monoidal Hom-Hopf algebra* is a monoidal Hom-bialgebra (H,α) together with a linear map $S : H \to H$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$S * I = I * S = \eta\varepsilon, \qquad S\alpha = \alpha S.$$

DEFINITION 2.5. Let (A, α) be a monoidal Hom-algebra. A *right (A, α)-Hom-module* is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ consisting of a k-module and a linear map $\mu : M \to M$ together with a morphism $\psi : M \otimes A \to M$, $\psi(m \cdot a) = m \cdot a$, in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab), \qquad m \cdot 1_A = \mu(m),$$

for all $a \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f : (M, \mu) \to (N, \nu)$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is called *right A-linear* if it preserves the A-action, that is, $f(m \cdot a) = f(m) \cdot a$. The category of right (A, α)-Hom-modules and A-linear morphisms will be denoted $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$.

DEFINITION 2.6. Let (C, γ) be a monoidal Hom-coalgebra. A *right (C, γ)-Hom-comodule* is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$ written $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}),$$
$$m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m),$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right (C, γ)-Hom-comodules are defined in the obvious way. The category of right (C, γ)-Hom-comodules will be denoted by $\widetilde{\mathscr{H}}(\mathscr{M}_k)^C$.

THEOREM 2.7 (Rafael theorem [20]). *Let $L : \mathcal{C} \to \mathcal{D}$ be the left adjoint functor of $R : \mathcal{D} \to \mathcal{C}$. Then L is a separable functor if and only if the unit η of the adjunction (L, R) has a natural retraction, i.e., there is a natural transformation $\nu : RL \to \mathrm{id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{id}$.*

3. Adjoint functor

DEFINITION 3.1. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a *right (H, α)-Hom-comodule algebra* if (A, β) is a right (H, α)-Hom-comodule with coaction $\rho_A : A \to A \otimes H$, $\rho_A(a) = a_{[0]} \otimes a_{[1]}$, such that

$$\rho_A(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}, \qquad \rho_A(1_A) = 1_A \otimes 1_H,$$

for all $a, b \in A$.

DEFINITION 3.2. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-coalgebra (C, γ) is called a *right (H, α)-Hom-module coalgebra* if (C, γ) is a right (H, α)-Hom-module with action $\phi : C \otimes H \to C$, $\phi(c \otimes h) =$

$c \cdot h$, such that

$$\Delta(c \cdot h) = c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)}, \qquad \varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h),$$

for all $c \in C$ and $g, h \in H$.

A *Doi Hom-Hopf datum* is a triple (H, A, C), where H is a monoidal Hom-Hopf algebra, A a right (H, α)-Hom-comodule algebra and (C, γ) a right (H, α)-Hom-module coalgebra.

Definition 3.3. Given a Doi Hom-Hopf datum (H, A, C), a *Doi Hom-Hopf module* (M, μ) is a right (A, β)-Hom-module which is also a right (C, γ)-Hom-comodule with the coaction structure $\rho_M : M \to M \otimes C$ defined by $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ such that the following compatibility condition holds: for all $m \in M$ and $a \in A$,

$$\rho_M(m \cdot a) = m_{[0]} \cdot a_{[0]} \otimes m_{[1]} a_{[1]}.$$

A morphism between two right Doi Hom-Hopf modules is a k-linear map which is a morphism in the categories $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ and $\widetilde{\mathscr{C}}(\mathscr{M}_k)^C$ at the same time. $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ will denote the category of right Doi Hom-Hopf modules and morphisms between them.

Proposition 3.4. *The forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ has a right adjoint $G : \widetilde{\mathscr{H}}(\mathscr{M}_k)_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ defined by*

$$G(N) = N \otimes C,$$

with structure maps

$$(n \otimes c) \cdot a = n \cdot a_{[0]} \otimes c \cdot a_{[1]}, \qquad \rho_{G(M)}(n \otimes c) = (\nu^{-1}(n) \otimes c_{(1)}) \otimes \gamma(c_{(2)}),$$

for all $a \in A$, $n \in N$ and $c \in C$.

Proof. Let us first show that $G((N, \nu)) = (N \otimes C, \nu \otimes \gamma)$ is an object of $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$. It is routine to check that $G((N, \nu))$ is a right (C, γ)-Hom-comodule and a right (A, β)-Hom-module. Now we only check the compatibility condition, for all $a \in A$ and $c \in C$. Indeed,

$$\begin{aligned}
\rho_{G((N,\nu))}((n \otimes c) \cdot a) &= \rho_{G((N,\nu))}(n \cdot a_{[0]} \otimes c \cdot a_{[1]}) \\
&= \nu^{-1}(n) \cdot \beta^{-1}(a_{[0]}) \otimes c_{(1)} \cdot a_{[1](1)} \otimes \gamma(c_{(2)} \cdot a_{[1](2)}) \\
&= \nu^{-1}(n) \cdot a_{[0][0]} \otimes c_{(1)} \cdot a_{[0][1]} \otimes \gamma(c_{(2)}) \cdot a_{[1]} \\
&= (n \otimes c)_{[0]} \cdot a_{[0]} \otimes (n \otimes c)_{[1]} \cdot a_{[1]} = \rho_{G(M)}(n \otimes c) \cdot a.
\end{aligned}$$

This is exactly what we have to show.

For an (A, β)-linear map $\varphi : (M, \mu) \to (N, \nu)$, we set

$$G(\varphi) = \varphi \otimes \mathrm{id}_C : M \otimes C \to N \otimes C.$$

Standard computations show that $G(\varphi)$ is a morphism of right (A, β)-Hom-modules and right (C, γ)-Hom-comodules. Let us describe the unit η and the counit δ of the adjunction. The unit is described by the coaction: for $(M,\mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$, we define $\eta_M : M \to M \otimes C$ by setting, for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes m_{[1]}.$$

We can check that $\eta_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$. In fact, for any $m \in M$, we have

$$\begin{aligned}\eta_M(m \cdot a) &= (m \cdot a)_{[0]} \otimes (m \cdot a)_{[1]} = m_{[0]} \cdot a_{[0]} \otimes m_{[1]} a_{[1]} \\ &= (m_{[0]} \otimes m_{[1]}) \cdot a = \eta_M(m) \cdot a\end{aligned}$$

and

$$\begin{aligned}\rho_{M\otimes C} \circ \eta_M(m) &= \rho_{M\otimes C}(m_{[0]} \otimes m_{[1]}) = (\mu^{-1}(m_{[0]}) \otimes m_{[1](1)}) \otimes \gamma(m_{[1](2)}) \\ &= (m_{[0][0]} \otimes m_{[0][1]}) \otimes m_{[1]} = (\eta_M \otimes \mathrm{id}_C)(m_{[0]} \otimes m_{[1]}) \\ &= (\eta_M \otimes \mathrm{id}_C) \circ \rho_M(m).\end{aligned}$$

For any $(N,\nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$, we define $\delta_N : N \otimes C \to N$ by setting, for all $n \in N$ and $c \in C$,

$$\delta_N(n \otimes c) = \varepsilon(c)\nu(n).$$

We can check that δ_N is (A,β)-linear. In fact, for any $n \in N$, we have

$$\begin{aligned}\delta_N((n \otimes c) \cdot a) &= \delta_N(n \cdot a_{[0]} \otimes c \cdot a_{[1]}) = \varepsilon(c \cdot a_{[1]})\nu(n \cdot a_{[0]}) \\ &= \varepsilon(c)\nu(n) \cdot a = \delta_N(n \otimes c) \cdot a.\end{aligned}$$

This is what we need to show. We can check that η and δ defined above are all natural transformations, and satisfy

$$\begin{tikzcd} m \arrow[r, "F(\eta_M)"'] \arrow[rr, bend left, "\mathrm{id}_{FM}"] & m_{[0]} \otimes m_{[1]} \arrow[r, "\delta F_M"'] & \varepsilon_C(m_{[1]})\mu(m_{[0]}) = m, \end{tikzcd}$$

$$\begin{tikzcd} n \otimes c \arrow[r, "\eta_{GN}"] \arrow[rr, bend right, "\mathrm{id}_{GN}"'] & (\nu^{-1}(n) \otimes c_{(1)}) \otimes \gamma(c_{(2)}) \arrow[r, "G(\delta_N)"] & \varepsilon_C(c_{(1)})n \otimes \gamma(c_{(2)}) = n \otimes c, \end{tikzcd}$$

for all $(M,\mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$, $(N,\nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$, $m \in M$, $n \in N$, $c \in C$.

4. Separable functors for the category of Doi Hom-Hopf modules. In this section, we shall give necessary and sufficient conditions for the functor F which forgets the (C,γ)-coaction to be separable.

DEFINITION 4.1. Let (H,A,C) be a Doi Hom-Hopf datum. A k-linear map

$$\theta : (C,\gamma) \otimes (C,\gamma) \to (A,\beta)$$

such that $\theta \circ (\gamma \otimes \gamma) = \beta \circ \theta$ is called a *normalized* (A,β)*-integral* if θ satisfies the following conditions:

(1) For all $c, d \in C$,

$$\theta(\gamma^{-1}(d) \otimes c_{(1)}) \otimes \gamma(c_{(2)}) = \beta\big(\theta(d_{(2)} \otimes \gamma^{-1}(c))_{[0]}\big) \otimes d_{(1)} \cdot \theta(d_{(2)} \otimes \gamma^{-1}(c))_{[1]}. \tag{4.1}$$

(2) For all $c \in C$,

$$\theta(c_{(1)} \otimes c_{(2)}) = 1_A \varepsilon(c). \tag{4.2}$$

(3) For all $a \in A$ and $c, d \in C$,

$$\beta^2(a_{[0][0]})\theta\big(\gamma^{-1}(d) \cdot a_{[0][1]} \otimes \gamma^{-1}(c) \cdot \alpha^{-1}(a_{[1]})\big) = \theta(d \otimes c)a. \tag{4.3}$$

Theorem 4.2. *For any Doi Hom-Hopf datum (H, A, C), the following assertions are equivalent:*

(1) *The left adjoint functor F in Proposition* 3.4 *is separable.*
(2) *There exists a normalized (A, β)-integral $\theta : (C, \gamma) \otimes (C, \gamma) \to (A, \beta)$.*

Proof. (2)⇒(1). For any Doi Hom-Hopf module (M, μ), we define

$$\nu_M : M \otimes C \to M, \quad m \otimes c \mapsto \mu(m_{[0]})\theta(m_{[1]} \otimes \gamma^{-1}(c)),$$

for all $m \in M$ and $c \in C$. Now, we shall check that $\nu_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$. In fact, for all $m \in M$, $c \in C$ and $a \in A$, it is easy to see that

$$\nu_M(\mu(m) \otimes \gamma(c)) = \mu(\nu_M(m \otimes c)).$$

We also have

$$\begin{aligned}
\nu_M((m \otimes c) \cdot a) &= \nu_M(m \cdot a_{[0]} \otimes c \cdot a_{[1]}) \\
&= (\mu(m_{[0]}) \cdot \beta(a_{[0][0]}))\theta\big(m_{[1]}a_{[0][1]} \otimes \gamma^{-1}(c) \cdot \alpha^{-1}(a_{[1]})\big) \\
&= \mu^2(m_{[0]}) \cdot (\beta(a_{[0][0]})\beta^{-1}\big(\theta(m_{[1]}a_{[0][1]} \otimes \gamma^{-1}(c) \cdot \alpha^{-1}(a_{[1]}))\big) \\
&= \mu^2(m_{[0]}) \cdot \big(\beta(a_{[0][0]})\theta\big(\gamma^{-1}(m_{[1]})\gamma^{-1}(a_{[0][1]}) \otimes \gamma^{-2}(c) \cdot \alpha^{-2}(a_{[1]})\big)\big) \\
&\overset{(4.3)}{=} \mu^2(m_{[0]}) \cdot \big(\theta(m_{[1]} \otimes \gamma^{-1}(c)) \cdot \beta^{-1}(a)\big) \\
&= \big(\mu(m_{[0]}) \cdot \theta(m_{[1]} \otimes \gamma^{-1}(c))\big) \cdot a = (\nu_M(m \otimes c)) \cdot a.
\end{aligned}$$

Hence it is a morphism of (A, β)-Hom-modules. Next, we shall check that ν_M is a morphism of comodules over (C, γ). It is sufficient to check that

$$\rho_M \circ \nu_M = (\nu_M \otimes \mathrm{id}_C) \circ \rho_M.$$

Indeed, for all $m \in M$ and $c \in C$, we have

$$\begin{aligned}
\rho_M \circ \nu_M(m \otimes c) &= \rho_M\big(\mu(m_{[0]})\theta(m_{[1]} \otimes \gamma^{-1}(c))\big) \\
&= (\mu(m_{[0]}) \cdot \theta(m_{[1]} \otimes \gamma^{-1}(c))_{[0]} \otimes \big(\mu(m_{[0]}) \cdot \theta(m_{[1]} \otimes \gamma^{-1}(c))\big)_{[1]} \\
&= \mu(m_{[0][0]}) \cdot \theta(m_{[1]} \otimes \gamma^{-1}(c))_{[0]} \otimes \gamma(m_{[0][1]}) \cdot \theta(m_{(1)} \otimes \gamma^{-1}(c))_{[1]}
\end{aligned}$$

$$= m_{[0]} \cdot \theta(\gamma(m_{[1](2)}) \otimes \gamma^{-1}(c))_{[0]} \otimes \gamma(m_{[1](1)}) \cdot \theta(\gamma(m_{[1](2)}) \otimes \gamma^{-1}(c))_{[1]}$$
$$\overset{(4.1)}{=} m_{[0]} \cdot \beta^{-1}(\theta(m_{[1]} \otimes c_{(1)})) \otimes \gamma(c_{(2)})$$
$$= m_{[0]} \cdot \theta(\gamma^{-1}(m_{[1]}) \otimes \gamma^{-1}(c_{(1)})) \otimes \gamma(c_{(2)}) = (\nu_M \otimes \mathrm{id}_C) \circ \rho_M(m \otimes c).$$

For all $m \in M$,

$$\nu_M \circ \eta_M(m) = \nu_M(m_{[0]} \otimes m_{[1]}) = \mu(m_{[0][0]}) \cdot \theta(m_{[0][1]} \otimes \gamma^{-1}(m_{[1]}))$$
$$= m_{[0]} \cdot \theta(m_{[1](1)} \otimes m_{[1](2)}) \overset{(4.2)}{=} m.$$

So the left adjoint F in Proposition 3.4 is separable by the Rafael theorem (see [20] for details).

(1)⇒(2). We consider the Doi Hom-Hopf module $(A \otimes C, \beta \otimes \gamma)$, with the (A, β)-action and (C, γ)-coaction defined as follows:

$$\begin{cases} (a \otimes c) \cdot b = ab_{[0]} \otimes c \cdot b_{[1]}, \\ \rho_{A\otimes C}(a \otimes c) = (\beta^{-1}(a) \otimes c_{(1)}) \otimes \gamma(c_{(2)}), \end{cases}$$

for any $a, b \in A$ and $c \in C$.

By evaluating at this object, the retraction ν of the unit of the adjunction in Proposition 3.4 yields a morphism

$$\nu_{A\otimes C} : (A \otimes C) \otimes C \to A \otimes C$$

such that, for all $a \in A$ and $c \in C$,

$$\nu_{A\otimes C}\big((\beta^{-1}(a) \otimes c_{(1)}) \otimes \gamma(c_{(2)})\big) = a \otimes c.$$

It can be used to construct $\theta : C \otimes C \to A$ as follows:

$$\theta(c \otimes d) = \widetilde{r}_A(\mathrm{id}_A \otimes \varepsilon_C)\nu_{A\otimes C}((1_A \otimes c) \otimes \gamma(d))$$

for all $c \in C$, where r means the right unit constraint, since

$$\theta(c_{(1)} \otimes c_{(2)}) = \widetilde{r}_A(\mathrm{id}_A \otimes \varepsilon_C)\nu_{A\otimes C}((1_A \otimes c_{(1)}) \otimes \gamma(c_{(2)}))$$
$$= \widetilde{r}_A(\mathrm{id}_A \otimes \varepsilon_C)(1_A \otimes c) = 1_A\varepsilon_C(c).$$

Hence condition (4.2) follows.

In order to prove (4.3), for any right (A, β)-Hom-module (M, μ), we need to give $(M \otimes C, \mu \otimes \gamma)$ the new right (A, β)-action and new right (C, γ)-coaction by

$$(m \otimes c) \leftharpoonup a := m \cdot a_{[0]} \otimes c \cdot \alpha(a_{[1]}),$$
$$\varrho^{M\otimes C}(m \otimes c) = (m \otimes c)_{\{0\}} \otimes (m \otimes c)_{\{1\}} := \mu^{-1}(m) \otimes c_{(1)} \otimes c_{(2)},$$

for any $m \in M$, $a \in A$, $c \in C$.

We define a left (H, α)-Hom-module structure on $((A\otimes C)\otimes C, (\beta\otimes\gamma)\otimes\gamma)$ by

$$h \rightharpoonup ((a\otimes c)\otimes d) := (\beta^{-2}(h)a\otimes\gamma(c))\otimes\gamma(d) \quad \text{for } h \in H,\ a \in A \text{ and } c, d \in C,$$

and a left (H,α)-Hom-module structure on $(A\otimes C,\beta\otimes\gamma)$ by

$$h \rightharpoondown (a\otimes c) := \beta^{-1}(h)a\otimes\gamma(c).$$

Then it is easy to check that

$$\eta_{A\otimes C}: A\otimes C\to (A\otimes C)\otimes C, \qquad a\otimes c\mapsto (\beta^{-1}(a)\otimes c_1)\otimes\gamma(c_2),$$

is an (A,β)-Hom-module morphism. Thus $\nu_{A\otimes C}$ is also left (A,β)-linear.

Furthermore, obviously

$$\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C): A\otimes C\to A, \qquad a\otimes c\mapsto\beta(a)\varepsilon(c),$$

is both a morphism of left (A,β)-Hom-modules and right (A,β)-Hom-modules.

Then we obtain

$$\begin{aligned}
&\beta^2(a_{[0][0]})\theta\big(\gamma^{-1}(d)\cdot a_{[0][1]}\otimes\gamma^{-1}(c)\cdot\alpha^{-1}(a_{[1]})\big)\\
&\quad=\beta^2(a_{[0][0]})\big((\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C))\big(\nu_{A\otimes C}((1_A\otimes\gamma^{-1}(d)\cdot a_{[0][1]})\otimes c\cdot a_{[1]})\big)\big)\\
&\quad=(\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C))\big(\beta^2(a_{[0][0]})\rightharpoondown\nu_{A\otimes C}((1_A\otimes\gamma^{-1}(d)\cdot a_{[0][1]})\otimes c\cdot a_{[1]})\big)\\
&\quad=(\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C))\big(\nu_{A\otimes C}\big(\beta^2(a_{[0][0]})\rightharpoonup((1_A\otimes\gamma^{-1}(d)\cdot a_{[0][1]})\otimes c\cdot a_{[1]})\big)\big)\\
&\quad=(\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C))\big(\nu_{A\otimes C}\big((\beta(a_{[0][0]})\otimes d\cdot\alpha(a_{[0][1]}))\otimes\gamma(c)\cdot\alpha(a_{[1]})\big)\big)\\
&\quad=(\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C))\big(\nu_{A\otimes C}(((1_A\otimes d)\otimes\gamma(c))\leftharpoondown a)\big)\\
&\quad=(\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C))\big((\nu_{A\otimes C}((1_A\otimes d)\otimes\gamma(c)))\leftharpoondown a\big)\\
&\quad=\big((\widetilde{r}_A\circ(\mathrm{id}_A\otimes\varepsilon_C))(\nu_{A\otimes C}((1_A\otimes d)\otimes\gamma(c)))\big)a=\theta(d\otimes c)a.
\end{aligned}$$

The verification of (4.1) is more involved. For any right (H,α)-comodule (M,μ), we consider the Doi Hom-Hopf module $(M\otimes A,\mu\otimes\beta)$ with the (A,β)-action and (C,γ)-coaction defined as follows: for all $m\in M$ and $a,b\in A$,

$$\begin{cases}(m\otimes a)\cdot b=\mu^{-1}(m)\otimes a\beta(b),\\ \rho_{M\otimes A}(m\otimes a)=(m_{[0]}\otimes a_{[0]})\otimes m_{[1]}\cdot a_{[1]}.\end{cases}$$

In particular, there is a Doi Hom-Hopf module $(C\otimes A,\gamma\otimes\beta)$ and the map $\xi: C\otimes A\to A\otimes C$ defined by

$$\xi(c\otimes a)=\beta(a_{[0]})\otimes\gamma^{-1}(c)\cdot a_{[1]}.$$

Since ξ is both right (A,β)-linear and right (C,γ)-colinear, we have

$$\begin{aligned}\xi(\nu_{C\otimes A}((c\otimes a)\otimes d))&=\nu_{A\otimes C}((\xi\otimes\mathrm{id}_C)((c\otimes a)\otimes d))\\&=\nu_{A\otimes C}\big((\beta(a_{[0]})\otimes\gamma^{-1}(c)\cdot a_{[1]})\otimes d\big).\end{aligned}\tag{4.4}$$

It is not hard to check that $GF(C\otimes A,\gamma\otimes\beta)=((C\otimes A)\otimes C,(\gamma\otimes\beta)\otimes\gamma)\in{}^C\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$, and its left (C,γ)-Hom-comodule structure is given by

$$(c\otimes a)\otimes d\mapsto\gamma^2(c_{(1)})\otimes\big((c_{(2)}\otimes\beta^{-1}(a))\otimes\gamma^{-1}(d)\big).$$

Also $(C\otimes A,\gamma\otimes\beta)\in {}^{C}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$, and the left (C,γ)-coaction of $(C\otimes A,$ $\gamma\otimes\beta)$ is given by

$$c\otimes a\mapsto \gamma(c_{(1)})\otimes(c_{(2)}\otimes\beta^{-1}(a)).$$

We also deduce that $\nu_{C\otimes A}:(C\otimes A)\otimes C\to C\otimes A$ is a morphism in ${}^{C}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$, which means

$$\begin{aligned}&\nu_{C\otimes A}((c\otimes a)\otimes d)_{[-1]}\otimes\nu_{C\otimes A}((c\otimes a)\otimes d)_{[0]}\\&\qquad=\gamma^2(c_{(1)})\otimes\nu_{C\otimes A}\big((c_{(2)}\otimes\beta^{-1}(a))\otimes\gamma^{-1}(d)\big).\end{aligned}$$

Thus we conclude that $\nu_{C\otimes A}$ is left and right (C,γ)-colinear. Take $c,d\in C$, and set

$$\nu_{A\otimes C}((1_A\otimes c)\otimes\gamma(d))=\sum_i a_i\otimes q_i\in A\otimes C,$$

$$\nu_{C\otimes A}((c\otimes 1_A)\otimes\gamma(d))=\sum_i p_i\otimes b_i\in C\otimes A;\tag{4.5}$$

we obtain

$$\begin{aligned}&\beta(\theta(c_{(2)}\otimes\gamma^{-1}(d))_{[0]})\otimes c_{(1)}\cdot\theta\big(c_{(2)}\otimes\gamma^{-1}(d)\big)_{[1]}\\
&=\beta\big(\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)\nu_{A\otimes C}((1_A\otimes c_{(2)})\otimes d)_{[0]}\big)\otimes c_{(1)}\\
&\quad\cdot\big(\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)\nu_{A\otimes C}((1_A\otimes c_{(2)})\otimes d)\big)_{[1]}\\
&\overset{(4.4)}{=}\beta\big(\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)\xi\nu_{C\otimes A}((c_{(2)}\otimes 1_A)\otimes d)_{[0]}\big)\otimes c_{(1)}\\
&\quad\cdot\big(\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)\xi\nu_{C\otimes A}((c_{(2)}\otimes 1_A)\otimes d)_{[1]}\big)\\
&\overset{(4.5)}{=}\sum_i\beta\big(r_A(\mathrm{id}_A\otimes\varepsilon_C)\xi(p_{i(2)}\otimes\beta^{-1}(b_i))_{[0]}\big)\\
&\quad\otimes\gamma^{-1}(p_{i(1)})\big(\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)\xi(p_{i(2)}\otimes\beta^{-1}(b_i))_{[1]}\big)\\
&=\sum_i\beta\big(\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)(b_{i[0]}\otimes\gamma^{-1}(p_{i(2)}\cdot b_{i[1]}))_{[0]}\big)\\
&\quad\otimes\gamma^{-1}(p_{i(1)})\big(\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)\\
&\quad\cdot(b_{i[0]}\otimes\gamma^{-1}(p_{i(2)}\cdot b_{i[1]})_{[1]})\big)\\
&=\sum_i\beta(b_{i[0]})\otimes\gamma^{-1}(p_{i(1)})\varepsilon_C(p_{i(2)})\cdot b_{i[1]}\\
&=\sum_i\xi(\gamma^{-1}(p_i)\otimes b_i)=\xi\big(\nu_{C\otimes A}((\gamma^{-1}(c)\otimes 1_A)\otimes\gamma(d))\big).\end{aligned}$$

Using the fact that $\nu_{A\otimes C}$ is a morphism of right (C,γ)-Hom-comodules, we also have

$$\begin{aligned}&\theta(\gamma^{-1}(c)\otimes d_{(1)})\otimes\gamma(d_{(2)})\\
&\qquad=\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)\nu_{A\otimes C}\big((1_A\otimes\gamma^{-1}(c))\otimes\gamma(d_{(1)})\big)\otimes\gamma(d_{(2)})\\
&\qquad=\sum_i\widetilde{r}_A(\mathrm{id}_A\otimes\varepsilon_C)(\beta^{-1}(a_i)\otimes q_{i(1)})\otimes q_{i(2)}\end{aligned}$$

$$= \sum_i a_i \otimes \gamma^{-1}(q_i) = \nu_{A\otimes C}\big((1_A \otimes \gamma^{-1}(c)) \otimes \gamma(d)\big)$$

$$\overset{(4.4)}{=} \xi\big(\nu_{C\otimes A}((\gamma^{-1}(c) \otimes 1_A) \otimes \gamma(d))\big).$$

Hence, we can get condition (4.1).

5. Applications

5.1. A Maschke type theorem for Doi Hom-Hopf modules. Since separable functors reflect well the semisimplicity of the objects of a category, by Theorem 4.2, we will prove a Maschke type theorem for Doi Hom-Hopf modules as an application, where Maschke type theorems were proved in a similar context to that of [12].

Corollary 5.1. *Let (H, A, C) be a Doi Hom-Hopf datum, and (M, μ), $(N, \nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$. Suppose that there exists a normalized (A, β)-integral $\theta : (C, \gamma) \otimes (C, \gamma) \to (A, \beta)$. Then a monomorphism (resp. epimorphism) $f : (M, \mu) \to (N, \nu)$ splits in $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^C_A$ if the monomorphism (resp. epimorphism) f splits as an (A, β)-linear morphism.*

5.2. Relative Hom-Hopf modules. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α)-Hom-comodule algebra. Then the triple (H, A, H) is a Doi Hom-Hopf datum. The category $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)^H_A$ is called an (H, A)-*Hom-Hopf module category* and denoted by $\widetilde{\mathscr{H}}(\mathscr{M}_k)^H_A$.

Corollary 5.2. *Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α)-Hom-comodule algebra. Then the following statements are equivalent:*

(1) *The forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)^H_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ is separable.*
(2) *There exists a normalized (A, β)-integral $\theta : (H, \alpha) \otimes (H, \alpha) \to (A, \beta)$.*

5.3. Hom-Yetter–Drinfeld modules. First, we give the definition of Yetter–Drinfeld modules over a monoidal Hom-Hopf algebra, which were also introduced by Liu and Shen [15] similarly.

Definition 5.3. Let (H, α) be a monoidal Hom-Hopf algebra. A *right-right (H, α)-Hom-Yetter–Drinfeld module* is an object (M, β) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that (M, β) a right (H, α)-Hom-module and a right (H, α)-Hom-comodule with the following compatibility condition:

$$m_{[0]} \cdot h_{(1)} \otimes m_{[1]} \cdot h_{(2)} = \mu((\mu^{-1}(m) \cdot h_{(2)})_{[0]}) \otimes h_{(1)}(\mu^{-1}(m) \cdot h_{(2)})_{[1]} \tag{5.1}$$

for all $h \in H$ and $m \in M$. We denote by $\mathscr{H}\mathscr{Y}\mathscr{D}^H_H$ the category of right-right (H, α)-Hom-Yetter–Drinfeld modules, morphisms being right (H, α)-linear right (H, α)-colinear maps.

PROPOSITION 5.4. *The equality* (5.1) *holds if and only if*

$$\rho(m\cdot h)=m_{[0]}\cdot\alpha(h_{(2)(1)})\otimes S(h_{(1)})(\alpha^{-1}(m_{[1]})h_{(2)(2)}) \tag{5.2}$$

for all $h\in H$ *and* $m\in M$.

Proof. (5.1)⇒(5.2). For $h\in H$ and $m\in M$, we have

$$\begin{aligned}
&m_{[0]}\cdot\alpha(h_{(2)(1)})\otimes S(h_{(1)})(\alpha^{-1}(m_{[1]})h_{(2)(2)})\\
&\quad=\mu(\mu^{-1}(m_{[0]})\cdot h_{(2)(1)})\otimes S(h_{(1)})(\alpha^{-1}(m_{[1]})h_{(2)(2)})\\
&\quad\overset{(5.1)}{=}\mu^2((\mu^{-2}(m)\cdot h_{(2)(2)})_{[0]})\otimes S(h_{(1)})\big(h_{(2)(1)}((\mu^{-2}(m)\cdot h_{(2)(2)})_{[1]})\big)\\
&\quad=\mu^2\big((\mu^{-2}(m)\cdot\alpha^{-1}(h_{(2)}))_{[0]}\big)\\
&\qquad\otimes(S(h_{(1)(1)})h_{(1)(2)})\alpha\big((\mu^{-2}(m)\cdot\alpha^{-1}(h_{(2)}))_{[1]}\big)\\
&\quad=\mu^2((\mu^{-2}(m)\cdot\alpha^{-2}(h))_{[0]})\otimes\alpha^2((\mu^{-2}(m)\cdot\alpha^{-2}(h))_{[1]})\\
&\quad=(m\cdot h)_{[0]}\otimes(m\cdot h)_{[1]}.
\end{aligned}$$

(5.2)⇒(5.1). We compute as follows:

$$\begin{aligned}
&\mu((\mu^{-1}(m)\cdot h_{(2)})_{[0]})\otimes h_{(1)}(\mu^{-1}(m)\cdot h_{(2)})_{[1]}\\
&\quad\overset{(5.2)}{=}\mu(\mu^{-1}(m_{[0]})\cdot\alpha(h_{(2)(2)(1)}))\otimes h_{(1)}(S(h_{(2)(1)})(\alpha^{-2}(m_{[1]})h_{(2)(2)(2)}))\\
&\quad=m_{[0]}\cdot\alpha^2(h_{(2)(2)(1)})\otimes(\alpha^{-1}(h_{(1)})S(h_{(2)(1)}))(\alpha^{-1}(m_{[1]})\alpha(h_{(2)(2)(2)}))\\
&\quad=m_{[0]}\cdot\alpha(h_{(2)(1)})\otimes(h_{(1)(1)}S(h_{(1)(2)}))(\alpha^{-1}(m_{[1]})h_{(2)(2)})\\
&\quad=m_{[0]}\cdot h_{(1)}\otimes m_{[1]}\cdot h_{(2)}.
\end{aligned}$$

Thus the conclusion holds.

THEOREM 5.5. *Let* (H,α) *be a monoidal Hom-Hopf algebra with bijective antipode.*

(1) (H,α) *can be made into a right* $H^{\mathrm{op}}\otimes H$*-Hom-comodule algebra. The coaction* $H\to H\otimes(H^{\mathrm{op}}\otimes H)$ *is given by the formula*

$$h\mapsto\alpha(h_{(2)(1)})\otimes\big(h_{(2)(2)}\otimes S(\alpha^{-1}(h_{(1)}))\big).$$

(2) (H,α) *can be made into a right* $(H^{\mathrm{op}}\otimes H,\alpha\otimes\alpha)$*-Hom-module coalgebra. The action of* $(H^{\mathrm{op}}\otimes H,\alpha\otimes\alpha)$ *on* (H,α) *is given by the formula*

$$c\triangleleft(h\otimes k)=\alpha(k)(\alpha^{-1}(c)h).$$

(3) *The category* $\mathscr{HYD}_H^H$ *of right-right Hom-Yetter–Drinfeld modules is isomorphic to a category of Doi Hom-Hopf modules, namely* $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ $(H^{\mathrm{op}}\otimes H)_H^H$.

Proof. (1) Let us first prove that (H,α) is a right $H^{\mathrm{op}}\otimes H$-Hom-comodule. For all $h\in H$,

$$
\begin{aligned}
&(\alpha^{-1}\otimes\Delta_{H^{\rm op}\otimes H})\rho_H(h) = h_{(2)(1)}\otimes\Delta_{H^{\rm op}\otimes H}\big(h_{(2)(2)}\otimes S(\alpha^{-1}(h_{(1)}))\big)\\
&\qquad = h_{(2)(1)}\otimes h_{(2)(2)(1)}\otimes S(\alpha^{-1}(h_{(1)(2)}))\otimes h_{(2)(2)(2)}\otimes S^{-1}(\alpha^{-1}(h_{(1)(1)}))\\
&= \alpha(h_{(2)(1)(1)})\otimes h_{(2)(1)(2)}\otimes S(\alpha^{-1}(h_{(1)(2)}))\otimes\alpha^{-1}(h_{(2)(2)})\otimes S(\alpha^{-1}(h_{(1)(1)}))\\
&= \alpha^2(h_{(2)(2)(1)(1)})\otimes\alpha(h_{(2)(2)(1)(2)})\otimes S(\alpha^{-1}(h_{(2)(1)}))\otimes h_{(2)(2)(2)}\\
&\quad\otimes S(\alpha^{-2}(h_{(1)}))\\
&= \alpha^2(h_{(2)(1)(2)(1)})\otimes\alpha(h_{(2)(1)(2)(2)})\otimes S(h_{(2)(1)(1)})\otimes\alpha^{-1}(h_{(2)(2)})\\
&\quad\otimes S^{-1}(\alpha^{-2}(h_{(1)}))\\
&= \rho(\alpha(h_{(2)(1)}))\otimes\alpha^{-1}(h_{(2)(2)})\otimes S(\alpha^{-2}(h_{(1)})) = (\rho_H\otimes\alpha^{-1})\rho_H(h),
\end{aligned}
$$

and therefore (H,α) is a right $(H^{\rm op}\otimes H,\alpha\otimes\alpha)$-Hom-comodule.

We also have

$$
\begin{aligned}
\rho(hg) &= \alpha(h_{(2)(1)}g_{(2)(1)})\otimes\big(h_{(2)(2)}g_{(2)(2)}\otimes S(\alpha^{-1}(h_{(1)}g_{(1)}))\big)\\
&= a(h_{(2)(1)})\alpha(g_{(2)(1)})\otimes(g_{(2)(2)}h_{(2)(2)}\otimes S(\alpha^{-1}(g_{(1)}))S(\alpha^{-1}(h_{(1)})))\\
&= \big(\alpha(h_{(2)(1)})\otimes(h_{(2)(2)}\otimes S(\alpha^{-1}(h_{(1)})))\big)\\
&\quad\big(\alpha(g_{(2)(1)})\otimes(g_{(2)(2)}\otimes S(\alpha^{-1}(g_{(1)})))\big)\\
&= \rho_H(h)\rho_H(g).
\end{aligned}
$$

(2) We will first prove that (H,α) is a $(H^{\rm op}\otimes H,\alpha\otimes\alpha)$-Hom-comodule. For all $h,l,k,m,c\in H$, we have

$$
\begin{aligned}
&[c\triangleleft(h\otimes k)]\triangleleft(\alpha(l)\otimes\alpha(m))\\
&\qquad= [\alpha(k)(\alpha^{-1}(c)h)]\triangleleft(\alpha(l)\otimes\alpha(m)) = \alpha^2(m)[[k(\alpha^{-2}(c)\alpha^{-1}(h))]\alpha(l)]\\
&\qquad= \alpha^2(m)[[(\alpha^{-1}(k)\alpha^{-2}(c))h]\alpha(l)] = \alpha^2(m)[(k\alpha^{-1}(c))(hl)]\\
&\qquad= [c(mk)](\alpha(h)\alpha(l)) = \alpha(c)\triangleleft(hl\otimes mk) = \alpha(c)\triangleleft[(h\otimes k)(l\otimes m)],
\end{aligned}
$$

and this implies that (H,α) is a $(H^{\rm op}\otimes H,\alpha\otimes\alpha)$-Hom-comodule.

Using the fact that (H,α) is an (H,α)-Hom-bimodule algebra, we can infer that (H,α) is a left $(H^{\rm op}\otimes H,\alpha\otimes\alpha)$-Hom-module coalgebra.

(3) Let $((M,\mu),\mu,\cdot,\rho_M)$ be such that $((M,\mu),\cdot)$ is a right (H,α)-module and $((M,\mu),\rho_M)$ is a right (H,α)-comodule. Then $(M,\mu)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)(H^{\rm op}\otimes H)_H^H$ if and only if

$$
\begin{aligned}
\rho_M(m\cdot h) &= m_{[0]}\cdot\alpha(h_{(2)(1)})\otimes m_{[1]}\triangleleft(h_{(2)(2)}\otimes S(\alpha^{-1}(h_{(1)})))\\
&= m_{[0]}\cdot\alpha(h_{(2)(1)})\otimes S(h_{(1)})(\alpha^{-1}(m_{[1]})h_{(2)(2)})
\end{aligned}
$$

for all $h\in H$ and $m\in M$. This shows that $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H^{\rm op}\otimes H)_H^H$ is isomorphic to $\mathscr{H}\mathscr{Y}\mathscr{D}_H^H$.

From Theorem 4.2 we obtain immediately the following Maschke type theorem for Hom-Yetter–Drinfeld modules:

Corollary 5.6. *Let (H, α) be a monoidal Hom-Hopf algebra with bijective antipode. Then the following statements are equivalent:*

1. *The forgetful functor $F : \mathscr{H}\mathscr{Y}\mathscr{D}_H^H \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_H$ is separable.*
2. *There exists a normalized (H, α)-integral $\theta : (H, \alpha) \otimes (H, \alpha) \to (H, \alpha)$.*

5.4. Doi Hom-Hopf datum (k, k, H). Let (H, α) be a finite-dimensional monoidal Hom-Hopf algebra. Recall from [4] that $\varphi \in H^*$ is called a *right integral* on H^* if $\varphi * h^* = \langle h^*, 1_H\rangle\varphi$ and $\alpha^*\varphi = \varphi$ for all $h^* \in H^*$, or, equivalently, if $\varphi : H \to k$ is right (H, α)-colinear. Now suppose that there exists a normalized k-integral $\theta : (H, \alpha) \otimes (H, \alpha) \to k$. The map φ defined by $\langle \varphi, h\rangle = \theta(1_H \otimes h)$ is a right integral. Conversely, if $\varphi \in \int_{H^*}^{l}$, the k-module consisting of classical integrals on (H, α), then $\theta(h \otimes g) = \varphi(gS^{-1}(h))$ is a k-integral. This can be proved directly.

Corollary 5.7. *With the notation as above, the following statements are equivalent:*

1. *The forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)^H \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_k$ (the category of all vector spaces) is separable.*
2. *There exists a normalized k-integral $\theta : (H, \alpha) \otimes (H, \alpha) \to k$ such that*

$$\alpha(h_{(2)})\theta(\alpha^{-1}(g) \otimes h_{(1)}) = \alpha(g_{(1)})\theta(g_{(2)} \otimes \alpha^{-1}(h)),$$
$$\theta(h_{(1)} \otimes h_{(2)}) = \varepsilon_H(h).$$

Corollary 5.8. *If (H, α) is a finite-dimensional cosemisimple monoidal Hom-Hopf algebra, then the forgetful functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)^H \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_k$ is separable.*

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Shuangjian Guo
School of Mathematics and Statistics
Guizhou University of Finance and Economics
Guiyang, 550025, P.R. China
E-mail: shuangjguo@gmail.com

Xiaohui Zhang
School of Mathematical Sciences
Qufu Normal University
Qufu, Shandong 273165, P.R. China
E-mail: zxhui-000@126.com