COLLOQUIUM MATHEMATICUM

VOL. 143 2016 NO. 1

ON PSEUDO-PRIME MULTIPLICATION MODULES OVER PULLBACK RINGS

BY

FATEMEH ESMAEILI KHALIL SARAEI (Fouman)

Abstract. The purpose of this paper is to present a new approach to the classification of indecomposable pseudo-prime multiplication modules over pullback of two local Dedekind domains. We extend the definitions and the results given by Ebrahimi Atani and Farzalipour (2009) to more general cases.

1. Introduction. One of the aims of modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. The reader is referred to [1], [25], [26, Chapters 1 and 6] and [27] for a detailed discussion of classification problems, representation types (finite, tame, or wild), and useful computational reduction procedures. Unfortunately, for the vast majority of rings, the classification of all modules is unfeasible. For example, the classification of all indecomposable pure-injective modules with infinite-dimensional top over $R/\mathrm{rad}(R)$ (for any module M over a ring R we define its top as $M/\mathrm{rad}(R)M$) over the pullback ring formed by mapping two local Dedekind domains R_1 and R_2 onto a field $\bar{R}$ is at least as difficult as that problem.

Modules over pullback rings have been studied by several authors (see for example [24], [2], [16], [12], [17] and [29]). Notably, there is the monumental work of Levy [19], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings. Common to all these classification is the reduction to a "matrix problem" over a division ring (see [26, Section 17.9] for background on matrix problems and their applications).

In the present paper we introduce a new class of R-modules, called pseudo-prime multiplication modules (see Definition 3.4), and we study them in detail from the classification point of view. We are mainly interested in the case where R is either a Dedekind domain or a pullback ring of two local Dedekind domains. The purpose of this paper is to give

2010 *Mathematics Subject Classification*: Primary 13C05; Secondary 16D70.
Key words and phrases: pseudo-prime multiplication modules, separated modules, pullback ring.
Received 11 January 2015; revised 24 April 2015.
Published online 3 December 2015.

a complete description of the indecomposable pseudo-prime multiplication modules over R. The classification is divided into two stages: we give a list of all separated pseudo-prime multiplication R-modules, and then, using this list, we show that nonseparated indecomposable pseudo-prime multiplication R-modules are factor modules of finite direct sums of separated pseudo-prime multiplication R-modules. Then we use the classification of separated pseudo-prime multiplication R-modules from Section 4, together with results of Levy [19], [18] on the possibility of amalgamating finitely generated separated modules, to classify the nonseparated indecomposable pseudo-prime multiplication modules (see Theorem 5.8). We will see that nonseparated modules may be represented by certain amalgamation chains of separated pseudo-prime multiplication modules.

2. Preliminaries. For the sake of completeness, we state some definitions and notation used throughout. In this paper all rings are commutative with identity and all modules are unitary. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinatewise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [20]).

Definition 2.1. An R-module S is defined to be *separated* if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$).

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module, and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [20, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also S is separated if and only if $P_1S \cap P_2S = 0$ [20, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi = (S \xrightarrow{f} S' \to M)$ of R-modules where S is separated, and if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [20, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [20, Proposition 2.3].

Every module M has a separated representation, which is unique up to isomorphism [20, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [18, Theorem 2.6].

Now, we have the following definition containing several parts which we use throughout this paper.

DEFINITION 2.2. (a) If R is a ring and N is a submodule of an R-module M, then the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N :_R M)$. So $(0 :_R M)$ is the annihilator of M.

(b) A proper submodule N of an R-module M is called *pseudo-prime* if $(N :_R M)$ is a prime ideal of R (see [13]). The set of all pseudo-prime submodules of M is denoted by $\mathrm{psSpec}(M)$. Every maximal submodule of an R-module M is prime and every prime submodule of M is a pseudo-prime submodule. Therefore $\mathrm{Max}(M) \subseteq \mathrm{Spec}(M) \subseteq \mathrm{psSpec}(M)$.

(c) A proper submodule N of an R-module M is *semiprime* if for every ideal I of R and every submodule K of M, $I^k K \subseteq N$ for some positive integer k implies that $IK \subseteq N$. The set of all semiprime submodules in an R-module M is denoted by $\mathrm{seSpec}(M)$.

(d) An R-module M is defined to be a *multiplication module* if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N :_R M)$.

(e) An R-module M is defined to be a *semiprime multiplication module* if for every semiprime submodule N of M, $N = IM$ for some ideal I of R (see [8]).

(d) A submodule N of an R-module M is called *pure* if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called *relatively divisible* (or an *RD-submodule*) in M if $rN = N \cap rM$ for all $r \in R$.

(e) A module M is *pure-injective* if it has the injective property relative to all pure exact sequences.

REMARK 2.3. (i) An R-module M is pure-injective if and only if it is algebraically compact (see [15] and [28]).

(ii) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M [28].

3. Pseudo-prime multiplication modules. In this section, we collect some basic properties concerning pseudo-prime multiplication modules. We begin with a lemma containing several useful properties of pseudo-prime submodules of R-modules.

LEMMA 3.1. *Let $N \subseteq L$ be submodules of an R-module M. Then the following hold:*

(i) *L is a pseudo-prime submodule of M if and only if L/N is a pseudo-prime submodule of M/N.*
(ii) *If L is a pseudo-prime R-submodule of M, and I an ideal of R with $I \subseteq (0 :_R M)$, then L is a pseudo-prime submodule of M as an R/I-module.*
(iii) *L is a prime submodule of M if and only if L is a primary and pseudo-prime submodule of M.*

Proof. (i) This is straightforward, since $(L :_R M) = (L/N :_R M/N)$.

(ii) Assume that $(L :_R M) = P$ for some prime ideal P of R. Then $IM = 0$ implies that $I \subseteq P$. An inspection will show that $(L :_{R/I} M) = P/I$, so L is a pseudo-prime submodule of M as an R/I-module.

(iii) Assume that L is a prime submodule of M. Clearly, L is primary and $(L :_R M) = P$ where P is a prime ideal of R. So L is a pseudo-prime submodule of M. Conversely, assume that L is a primary and pseudo-prime submodule of M, and let $rm \in L$ for some $r \in R$ and $m \in M \setminus L$. Then $r^n \in (L :_R M)$ for some positive integer n since L is a primary submodule of M. So $r \in (L :_R M)$ since L is a pseudo-prime submodule of M. Thus L is a prime submodule of M. ∎

We obtain some results concerning the relationship of semiprime and pseudo-prime submodules of modules over a local Dedekind domain.

LEMMA 3.2. *Let R be a local Dedekind domain with maximal ideal $P = Rp$ and let M be an R-module. Then:*

(i) *Every semiprime submodule of M is a pseudo-prime submodule.*
(ii) *If N is a pseudo-prime submodule of M with $(N :_R M) \neq 0$, then N is a semiprime submodule of M.*

Proof. (i) Let N be a semiprime submodule of M. If $(N :_R M) = 0$, then N is a pseudo-prime submodule of M. Now, suppose that $(N :_R M) = P^n$ for some positive integer n. Then $P^n M \subseteq N$. So $PM \subseteq N$, since N is a semiprime submodule, thus $(N :_R M) = P$. Hence N is a pseudo-prime submodule of M.

(ii) Let K be a submodule of M. Then $(N :_R M) \subseteq (N :_R K)$. So $(N :_R M) = P$ since $(N :_R M) \neq 0$ and N is a pseudo-prime submodule. Thus $(N :_R K) = P$ for every submodule K of M. Now, let $I^m K \subseteq N$ for some positive integer m. So $I^m \subseteq P$ and thus $I \subseteq P = (N :_R K)$. Therefore $IK \subseteq N$ and N is a semiprime submodule of M. ∎

The following example shows that a pseudo-prime submodule of M does not need to be a primary, prime or semiprime submodule of M.

EXAMPLE 3.3. (i) Let $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and let $N = (2,0)\mathbb{Z}$ be the submodule of M generated by $(2,0) \in M$. Then $(N :_{\mathbb{Z}} M) = 0$. Hence N is a pseudo-prime submodule, but N is not a prime submodule (see [13]).

(ii) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}(p^\infty)$ where p is a prime integer. Then M has no prime submodule but every proper submodule of M is a pseudo-prime submodule (see [13]).

(iii) Let $R = \mathbb{Z}$. If $N = 6\mathbb{Z}$, then N is a semiprime submodule of M that is not primary and is not pseudo-prime. If $N = 4\mathbb{Z}$, then N is a primary submodule of M that is not semiprime and is not a pseudo-prime submodule (see [8]).

(iv) If (R, P) is a local Dedekind domain and $M = E(R/P)$, then $E(R/P)$ has no primary submodule and no semiprime submodule by [9, Remark 2.7] and [8, Proposition 3.6]. It is clear that $(L :_R E) = 0$ for every proper submodule L of E, since E is a divisible R-module. Hence every proper submodule of E is a pseudo-prime submodule. So the converse of Lemma 3.2(i) is not true in general.

Now we define pseudo-prime multiplication modules.

DEFINITION 3.4. Let R be a commutative ring. An R-module M is defined to be a *pseudo-prime multiplication module* if for every pseudo-prime submodule N of M, $N = IM$ for some ideal I of R.

One can easily show that if M is a pseudo-prime multiplication module, then $N = (N :_R M)M$ for every pseudo-prime submodule N of M. It is easy to see by Lemma 3.1 that the class of pseudo-prime multiplication modules contains the class of weak multiplication modules defined in [11].

LEMMA 3.5. *Let M be a pseudo-prime multiplication module over a commutative ring R. Then:*

(i) *If I is an ideal of R, and N a nonzero R-submodule of M with $I \subseteq (N :_R M)$, then M/N is a pseudo-prime multiplication R/I-module.*
(ii) *If N is a submodule of M, then M/N is a pseudo-prime multiplication R-module.*
(iii) *Every direct summand of M is a pseudo-prime multiplication R-module.*

Proof. (i) Let L/N be a pseudo-prime submodule of M/N. Then L is a pseudo-prime submodule of M by Lemma 3.1(i), so $L = (L :_R M)M$. An inspection will show that $L/N = (L/N :_{R/I} M/N)M/N$.

(ii) Take $I = 0$ in case (i).

(iii) This is clear by case (ii). ∎

Lemma 3.6. *Let M be a divisible module over an integral domain R. If M is a pseudo-prime multiplication module, then M is a simple module.*

Proof. Let M be a pseudo-prime multiplication module and let L be a proper submodule of M, so $(L :_R M) = 0$ since M is a divisible R-module. Thus $L = (L :_R M)M = 0M = 0$. Hence M has no nonzero proper submodule. ∎

Corollary 3.7. *Let R be a local Dedekind domain with maximal ideal $P = Rp$. Then $E(R/P)$, the injective hull of R/P, and $Q(R)$, the field of fractions of R, are not pseudo-prime multiplication R-modules.*

Proof. It is clear that these modules are divisible. By [6, Lemma 2.6], $E(R/P)$ has nonzero proper submodules and $L = \{r/1 : r \in R\}$ is a nonzero proper submodule of $Q(R)$. Thus $E(R/P)$ and $Q(R)$ are not simple, and the conclusion is clear by Lemma 3.6. ∎

Proposition 3.8. *Let M be a pseudo-prime multiplication module over an integral domain R which is not a field. Then M is either torsion or torsion-free.*

Proof. Let $T(M)$ be the set of all torsion elements of M and suppose $T(M) \neq M$. Then $T(M)$ is a prime submodule of M and $(T(M) :_R M) = 0$ by [23, Lemma 3.8]. So $T(M)$ is a pseudo-prime submodule of M. It follows that $T(M) = (T(M) : M)M = 0M = 0$. Thus M is a torsion-free R-module. ∎

We have the following result containing a complete list of indecomposable pseudo-prime multiplication modules over local Dedekind domains.

Theorem 3.9. *Let R be a local Dedekind domain with maximal ideal $P = Rp$. Then the following is a complete list, up to isomorphism, of indecomposable pseudo-prime multiplication modules:*

(i) *R;*
(ii) *R/P^n, $n \geq 1$, the indecomposable torsion modules.*

Proof. By [5, Proposition 1.3] these modules are indecomposable. Clearly, R and R/P^n $(n \geq 1)$ are multiplication modules, so they are pseudo-prime multiplication modules.

Now, we show that there are no more indecomposable pseudo-prime multiplication R-modules. So let M be an indecomposable pseudo-prime multiplication module, and choose any nonzero element $a \in M$. Let $h(a) = \sup\{n : a \in P^n M\}$ (so $h(a)$ is a nonnegative integer or ∞). Also let $(0 :_R a) = \{r \in R : ra = 0\}$. Then $(0 :_R a)$ is an ideal of the form P^m or 0. Because $(0 :_R a) = P^{m+1}$ implies that $p^m a \neq 0$ and $pp^m a = 0$, we can choose a such that $(0 :_R a) = P$ or 0.

Now, we consider the various possibilities for $h(a)$ and $(0 :_R a)$:

If $\mathrm{psSpec}(M) = \emptyset$, then $\mathrm{Spec}(M) = \emptyset$ since $\mathrm{Spec}(M) \subseteq \mathrm{psSpec}(M)$. It follows from [22, Lemma 1.3, Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and M is not finitely generated. We may assume that $(0 :_R a) = P$ since M is a torsion module. By an argument like that in [4, Proposition 2.7], we have $M \cong E(R/P)$, which is a contradiction by Corollary 3.7. So we may assume that $\mathrm{psSpec}(M) \neq \emptyset$.

Case 1: $h(a) = n$, $(0 :_R a) = P$. Say $a = p^n b$ for some positive integer n and $b \in M$. Then $Rb \cong R/P^{n+1}$ is a pseudo-prime multiplication R-module. By an argument like that in [3, Theorem 2.12, Case 1], Rb is a pure submodule of M. Since Rb is a pure submodule of bounded order of M, we find that Rb is a direct summand of M by [14, Theorem 5]; hence $M = Rb \cong R/P^{n+1}$.

Case 2: $h(a) = n$, $(0 :_R a) = 0$. Assume that $a = p^n b$ for some positive integer n and $b \in M$. Then $(0 :_R b) = 0$. Thus $Rb \cong R$. By an argument as in Case 1, Rb is a pure submodule of M. Since $(0 :_R a) = 0$, M is a torsion-free module by Proposition 3.8. So Rb is a prime submodule of M by [21, Result 2]. Therefore Rb is a pseudo-prime submodule of M, and so $R \cong Rb = P^s M$ for some s. Then there is an element $m \in M$ such that $b = p^s m$; so $a = p^n b = p^{n+s} m$, which is a contradiction, therefore $s = 0$ and we have $R \cong Rb = P^0 M = RM = M$.

Case 3: $h(a) = \infty$. First suppose that $(0 :_R a) = P$. By an argument as in [3, Theorem 2.12, Case 4] we get $M \cong E(R/P)$, which is a contradiction. Now, suppose that $(0 :_R a) = 0$. By [14, Theorem 10], M is a torsion-free module and $(0 :_R M) = 0$ is faithful. So by [4, Lemma 2.3], $P^n M = M$ $(n \geq 1)$. Hence by an argument as in [3, Theorem 2.12, Case 3], $M \cong Q(R)$, which is a contradiction. ∎

In view of Theorem 3.9, we have the following result.

Corollary 3.10. *Let R be a local Dedekind domain with maximal ideal $P = Rp$, and M be a pseudo-prime multiplication R-module. Then M is a direct sum of copies of R/P^n $(n \geq 1)$. In particular, every pseudo-prime multiplication R-module not isomorphic to R is pure-injective.*

Proof. Let N_i denote the indecomposable summand of M. By Lemma 3.5, N_i is an indecomposable pseudo-prime multiplication R-module. Then N_i is a torsion or torsion-free module by Proposition 3.8. If N_i is torsion, then $N_i \cong R/P^n$ for some n by Theorem 3.9. Now, suppose that N_i is torsion-free. So N_i is a prime faithful module and hence $N_i \not\cong R$ by [4, Lemma 2.5]. Then M is a direct sum of copies of R/P^n. Now, the assertion follows from Theorem 3.9 and [5, Proposition 1.3]. ∎

4. The separated case. We devote this section to separated pseudo-prime multiplication modules over a pullback of two local Dedekind domains. Throughout we shall assume, unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{1}$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2; P denotes $P_1 \oplus P_2$; and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is, $P_1 \oplus 0$) and P_2 (that is, $0 \oplus P_2$).

Remark 4.1. Let R be a pullback ring as in (1), and let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ with projection maps $\pi_i : S \twoheadrightarrow S_i$. Set

$$T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\},$$
$$T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}.$$

Then for each $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\pi_1' = \pi_1|T : T \twoheadrightarrow T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/((0 \oplus \mathrm{Ker}(f_2)) \cap T) \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $\mathrm{Ker}(f_1) = P_1S_1$ and $\mathrm{Ker}(f_2) = P_2S_2$).

Proposition 4.2. *Let S be any separated module over a pullback ring as in* (1) *and T be a nonzero proper submodule of S. Then:*

(i) *If $(T :_R S) = P_1 \oplus 0$, then either $(0 :_R S) = 0$, or $(0 :_R S) = P_1^n \oplus 0$ for some positive integer n.*
(ii) *If $(T :_R S) = 0 \oplus P_2$, then either $(0 :_R S) = 0$, or $(0 :_R S) = 0 \oplus P_2^m$ for some positive integer m.*

Proof. (i) Assume $(T :_R S) = P_1 \oplus 0$. If $(0 :_R S) \neq 0$, then $(0 :_R S) = P_1^n \oplus P_2^m$ for some positive integers m and n. So $(P_1^n \oplus P_2^m)S = 0 \subseteq T$. Thus $P_1^n \oplus P_2^m \subseteq (T :_R S) = P_1 \oplus 0$, which is a contradiction. It follows that $m = 0$.

The proof of (ii) is similar. ∎

Proposition 4.3. *Let $T = (T_1 \to \bar{T} \leftarrow T_2)$ be a nonzero proper submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ over a pullback ring as in* (1)*. Then T is a pseudo-prime submodule of S if and only if T_i is a pseudo-prime submodule of S_i for every $i = 1, 2$.*

Proof. This is clear by [8, Proposition 4.2]. ∎

Proposition 4.4. *Let* $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ *be any separated module over a pullback ring as in* (1). *Then* $\mathrm{psSpec}(S) = \emptyset$ *if and only if* $\mathrm{psSpec}(S_i) = \emptyset$ *for every* $i = 1, 2$.

Proof. Assume that $\mathrm{psSpec}(S) = \emptyset$ and let π_i be the projection of R onto R_i for every $i = 1, 2$. Suppose that $\mathrm{psSpec}(S_1) \neq \emptyset$ and let T_1 be a pseudo-prime submodule of S_1. Then T_1 is a pseudo-prime submodule of $S_1 = S/(0 \oplus P_2)S$. Thus $\mathrm{psSpec}(S) \neq \emptyset$ by Lemma 3.1, which is a contradiction. Similarly, $\mathrm{psSpec}(S_2) = \emptyset$.

Now, suppose $\mathrm{psSpec}(S_i) = \emptyset$ for every $i = 1, 2$. If T is a pseudo-prime submodule of S, then $(T :_R S) = I$ where $I \in \{P_1 \oplus P_2, P_1 \oplus 0, 0 \oplus P_2\}$. So $(T_1 :_{R_1} S_1) \in \{0, P_1\}$ and $(T_2 :_{R_2} S_2) \in \{0, P_2\}$ by [8, Proposition 4.2]. So $\mathrm{psSpec}(S_1) \neq \emptyset$ or $\mathrm{psSpec}(S_2) \neq \emptyset$, which is a contradiction. ■

Theorem 4.5. *Let* $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ *be any separated module over a pullback ring as in* (1). *Then S is a pseudo-prime multiplication R-module if and only if each S_i is a pseudo-prime multiplication R_i-module for every $i = 1, 2$.*

Proof. We may assume that $\mathrm{psSpec}(S) \neq \emptyset$ by Proposition 4.4. Assume that S is a separated pseudo-prime multiplication R-module. If $\bar{S} = 0$, then $S = S_1 \oplus S_2$ by [5, Lemma 2.7]; so S_i is a pseudo-prime multiplication R_i-module by Lemma 3.5 for every $i = 1, 2$. Now, we may assume that $\bar{S} \neq 0$. So $S_1 \cong S/(0 \oplus P_2)S$ is a pseudo-prime multiplication R-module by Lemma 3.5, and since $0 \oplus P_2 \subseteq (0 :_R S/(0 \oplus P_2)S)$, it follows that $S_1 \cong S/(0 \oplus P_2)S$ is a pseudo-prime multiplication $R/(0 \oplus P_2) \cong R_1$-module. Similarly, S_2 is a pseudo-prime multiplication R_2-module.

Conversely, suppose that S_i is a pseudo-prime multiplication R_i-module for each i, and let $T = (T_1 \to \bar{T} \leftarrow T_2)$ be a pseudo-prime submodule of S. So $(T :_R S) = I$ where $I \in \{P_1 \oplus P_2, P_1 \oplus 0, 0 \oplus P_2\}$. Now we split the proof into two cases:

Case 1: $(T :_R S) = P_1 \oplus P_2$. Then T_i is a pseudo-prime submodule of S_i for each i, and $(T_1 :_{R_1} S_1) = P_1$ and $(T_2 :_{R_2} S_2) = P_2$ by [8, Proposition 4.2]. So $T_1 = P_1S_1$ and $T_2 = P_2S_2$, since S_1 and S_2 are pseudo-prime multiplication. We will show that $T = (P_1 \oplus P_2)S$.

Since $(T :_R S) = P_1 \oplus P_2$, we have $(P_1 \oplus P_2)S \subseteq T$. Now, suppose that $(t_1, t_2) \in T$. Then $t_1 \in T_1 = P_1S_1$ and $t_2 \in T_2 = P_2S_2$. So $t_1 = p_1s_1$ and $t_2 = p_2s_2$ for some $s_1 \in S_1$ and $s_2 \in S_2$. Thus $(s_1, s_2'), (s_1', s_2) \in S$ for some $s_1' \in S_1$ and $s_2' \in S_2$. Therefore $(t_1, t_2) = (p_1s_1, p_2s_2) = (p_1, 0)(s_1, s_2')$ $+ (0, p_2)(s_1', s_2) \in (P_1 \oplus P_2)S$. Hence $T = (P_1 \oplus P_2)S$.

Case 2: $(T :_R S) = P_1 \oplus 0$. Then T_i is a pseudo-prime submodule of S_i for each i, and $(T_1 :_{R_1} S_1) = P_1$ and $(T_2 :_{R_2} S_2) = 0$ by [8, Proposition 4.2].

So $T_1 = P_1S_1$ and $T_2 = 0$, since S_1 and S_2 are pseudo-prime multiplication. It is clear that $(P_1 \oplus 0)S \subseteq T$. Now, suppose that $(t_1, t_2) \in T$. Then $t_1 = p_1s_1$ and $t_2 = 0$ for some $s_1 \in S_1$. There is an $s_2 \in S_2$ such that $(s_1, s_2) \in S$. Thus $(t_1, t_2) = (t_1, 0) = (p_1s_1, 0) = (p_1, 0)(s_1, s_2) \in (P_1 \oplus 0)S$. Hence $T = (P_1 \oplus 0)S$.

Similarly, if $(T :_R S) = 0 \oplus P_2$, then S is a pseudo-prime multiplication R-module. ∎

Proposition 4.6. *Let $S \neq 0$ be a separated pseudo-prime multiplication module over a pullback ring as in* (1). *Then $\bar{S} \neq 0$ and* $\operatorname{psSpec}(S) \neq \emptyset$.

Proof. First suppose that $\operatorname{psSpec}(S) = \emptyset$. So $\operatorname{Spec}(S) = \emptyset$, since $\operatorname{Spec}(S) \subseteq \operatorname{psSpec}(S)$. Hence $S = PS$, $S_1 = P_1S_1$, $S_2 = P_2S_2$ and $\operatorname{psSpec}(S_i) = \emptyset$ for each i by Proposition 4.4. Thus $(0 :_{R_1} S_1) \neq 0$ and $(0 :_{R_2} S_2) \neq 0$. Otherwise $0 \in \operatorname{psSpec}(S_1)$ and $0 \in \operatorname{psSpec}(S_2)$, which is a contradiction.

Now, assume $(0 :_{R_1} S_1) = P_1^n$ for some positive integer n. If $n \geq 2$, then $P_1^{n-1}S_1 = P_1^{n-1}(P_1S_1) = P_1^nS_1 = 0$. This implies that $P_1^{n-1} \subseteq (0 :_{R_1} S_1) = P_1^n$ which is a contradiction. So $n = 1$, thus $(0 :_{R_1} S_1) = P_1$ and so $S_1 = P_1S_1 = 0$. Similarly, $S_2 = 0$. Then $S = 0$, which is a contradiction. Hence $\operatorname{psSpec}(S) \neq \emptyset$.

Now, we show that $\bar{S} \neq 0$. Let T be a pseudo-prime submodule of M. Then $T \neq S$ and T_1 and T_2 are pseudo-prime submodules by Proposition 4.3. So $T_i \neq S_i$ for each i. If $(T :_R S) = P_1 \oplus P_2$, then $(P_1 \oplus P_2)S \subseteq T \neq S$, so $PS \neq S$ and $\bar{S} \neq 0$.

If $(T :_R S) = P_1 \oplus 0$, then $T = (P_1 \oplus 0)S$ since S is a pseudo-prime multiplication R-module. Thus $T_1 = P_1S_1$ and $T_2 = 0$ by [8, Proposition 4.2] and Theorem 4.5. So $T_1 = P_1S_1 \neq S_1$ and thus $\bar{S} \neq 0$. Similarly, if $(T :_R S) = 0 \oplus P_2$, then $\bar{S} \neq 0$. ∎

In view of Theorems 4.5 and 3.9, we have the following result.

Lemma 4.7. *Let R be a pullback ring as in* (1). *The following separated R-modules are indecomposable and pseudo-prime multiplication modules:*

(1) $R = (R_1 \to \bar{R} \leftarrow R_2)$;
(2) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$.

Proof. By [5, Lemma 2.8], these modules are indecomposable, and by Theorems 3.9 and 4.5, they are pseudo-prime multiplication modules. ∎

Theorem 4.8. *Let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a nonzero indecomposable separated pseudo-prime multiplication module over a pullback ring as in* (1). *Then S is isomorphic to one of the modules listed in Lemma* 4.7. *In particular, every indecomposable separated pseudo-prime multiplication R-module not isomorphic to R is pure-injective.*

Proof. Let $S \neq R$ be an indecomposable separated pseudo-prime multiplication R-module. Then $\bar{S} \neq 0$ and $\mathrm{psSpec}(S) \neq \emptyset$ by Proposition 4.6. By Theorem 4.5, S_i is a pseudo-prime multiplication R_i-module for each $i = 1, 2$. Note that for each i, S_i is torsion and is not a divisible R_i-module by [7, Lemma 4.3] and Corollary 3.10. There exist positive integers m, n and k such that $P_1^m S_1 = 0$, $P_2^n S_2 = 0$ and $P^k S = 0$. For $s \in S$, let $o(s)$ denote the least positive integer l such that $p^l s = 0$. Now, choose $s \in S_1 \cup S_2$ with $\bar{s} \neq 0$ and such that $o(s)$ is maximal. There exists $s = (s_1, s_2)$ such that $o(s_1) = n_1$, $o(s_2) = m_1$ and $o(s) = k_1$. Then $R_i s_i$ is pure in S_i for $i = 1, 2$ by [5, Theorem 2.9]. Therefore, $R_1 s_1 \cong R_1/P_1^{n_1}$ (resp. $R_2 s_2 \cong R_2/P_2^{m_1}$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i s_i$ is pure-injective (see [5]). Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{s}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 s_1 = M_1 \to \bar{M} \leftarrow M_2 = R_2 s_2)$. Then M is an R-submodule of S which is a pseudo-prime multiplication module by Lemma 4.7 and is a direct summand of S; this implies that $S = M$ and S is as in (2) of Lemma 4.7 (see [5, Theorem 2.9]). ∎

In view of Theorem 4.8, we have the following result.

Corollary 4.9. *Let R be a pullback ring as in* (1), *and let $S \neq R$ be a separated pseudo-prime multiplication R-module. Then S is a direct sum of copies of the modules described in* (2) *of Lemma* 4.7. *In particular, every separated pseudo-prime multiplication R-module not isomorphic to R is pure-injective.*

Proof. Apply Theorem 4.8, Corollary 3.10 and [5, Theorem 2.9]. ∎

Theorem 4.10. *Let R be a pullback ring as in* (1), *and let S be a separated pseudo-prime multiplication R-module. Then S has finite-dimensional top.*

Proof. Apply Corollary 4.9 and [10, Theorem 3.14]. ∎

5. The nonseparated case. In this section, we find the indecomposable nonseparated pseudo-prime multiplication modules. We begin with the following proposition.

Proposition 5.1. *Let R be a pullback ring as in* (1). *Then $E(R/P)$, the injective hull of R/P, is a nonseparated pseudo-prime multiplication R-module.*

Proof. It suffices to show that $\mathrm{psSpec}(E(R/P)) = \emptyset$. Let L be a proper submodule of $E(R/P)$. Then $(L :_R E(R/P)) = 0$ since $E(R/P)$ is divisible. Since R is not an integral domain, L is not a pseudo-prime submodule of M. Thus $\mathrm{psSpec}(E(R/P)) = \emptyset$ as required. ∎

Proposition 5.2. *Let R be a pullback ring as in* (1) *and let M be any nonseparated R-module. Let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. Then S is a faithful R-module if and only if M is a faithful R-module.*

Proof. Let S be a faithful R-module. Then $(0 :_R S) = 0$. Since $M \cong S/K$, it suffices to show that $(K :_R S) = 0$. Assume that $rS \subseteq K$ for some $r = (r_1, r_2) \in R$. So $rPS \subseteq PK = 0$. Hence $rp \in rP \subseteq (0 :_R S)$. Thus $(r_1p_1, r_2p_2) = rp = (0, 0)$, so $r_1 = 0$ and $r_2 = 0$ since R_1 and R_2 are integral domains.

Conversely, assume that $(0 :_R M) = 0$. Then $(0 :_R S) \subseteq (K :_R S) = (0 :_R M) = 0$. Thus S is a faithful R-module. ∎

The following proposition shows that if M is any nonseparated pseudo-prime multiplication R-module, then S need not be a separated pseudo-prime multiplication R-module.

Proposition 5.3. *Let R be a pullback ring as in* (1) *and let M be any nonseparated pseudo-prime multiplication R-module. Let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. Then:*

- (i) *If $(T :_R S) = P_1 \oplus 0$ is a pseudo-prime submodule of S, then $T = (P_1 \oplus 0)S \oplus (K \cap T)$.*
- (ii) *If $(T :_R S) = 0 \oplus P_2$ is a pseudo-prime submodule of S, then $T = (0 \oplus P_2)S \oplus (K \cap T)$.*

Proof. (i) Suppose that $(T :_R S) = P_1 \oplus 0$. If $T + K = S$, then $PS = PT + PK = PT \subseteq T$ since $PK = 0$. So $P \subseteq (T :_R S) = P_1 \oplus 0$, which is a contradiction. One can show that $(T + K :_R S) = (\varphi(T) :_R M)$. First we show $(T + K :_R S) = P_1 \oplus 0$. It is clear that $(P_1 \oplus 0)S \subseteq T \subseteq T + K$. Now, let $rS \subseteq T+K$ for some $r = (r_1, r_2) \in R$. Thus $rPS \subseteq PT+PK = PT \subseteq T$. So $(r_1p_1, r_2p_2) = rp \in rP \subseteq P_1 \oplus 0$ and $p_2r_2 = 0$. Hence $r_2 = 0$ since R_2 is an integral domain and so $r \in P_1 \oplus 0$.

Now, we show that $T + K = (P_1 \oplus 0)S \oplus K$. Since $T + K \neq S$, it follows that $\varphi(T) \neq M$ and so $\varphi(T) = (P_1 \oplus 0)M$ since M is a pseudo-prime multiplication R-module. Let $t \in T$. Then $\varphi(t) = (p_1, 0)m$ for some $m \in M$. Since $m = \varphi(s)$ for some $s \in S$, we have $\varphi(t - (p_1, 0)s) = 0$. Thus $t = (p_1, 0)s + k$ for some $k \in K$. Therefore $T + K \subseteq (P_1 \oplus 0)S \oplus K$. The converse is clear.

Now, we show that $T = (P_1 \oplus 0)S \oplus (K \cap T)$. Since $PK = 0$, K is a vector space over $\bar{R}$. Then $K = (T \cap K) \oplus L$ for some R-submodule L of K. Thus $T \cap L = 0$. So $T + (T \cap K) + L = T + K = (P_1 \oplus 0)S \oplus ((T \cap K) + L)$. Let $t \in T$, so $t = a+b+l$ for some $a \in (P_1 \oplus 0)S$, $b \in T \cap K$ and $l \in L$. Thus $t - a - b = l \in T \cap L = 0$, so $t = a + b$. Therefore $T \subseteq (P_1 \oplus 0)S \oplus (K \cap T)$. The reverse inclusion is clear. The proof of (ii) is similar. ∎

Our next goal is to show that M is a pseudo-prime multiplication R-module if and only if S is one, when M is not a faithful module. Proposition 5.3 shows that $(0 :_R M) \neq 0$ is necessary.

THEOREM 5.4. *Let R be a pullback ring as in* (1) *and let M be any nonseparated R-module with* $(0 :_R M) \neq 0$. *Let* $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ *be a separated representation of M. Then S is a pseudo-prime multiplication R-module if and only if M is a pseudo-prime multiplication R-module.*

Proof. Let S be a pseudo-prime multiplication R-module. Then $\bar{S} \neq 0$ and $\mathrm{psSpec}(S) \neq \emptyset$ by Proposition 4.6. So $\bar{M} \neq 0$ ($\bar{M} \cong \bar{S}$). Therefore $(PM :_R M) = P$ and thus $PM \in \mathrm{psSpec}(M)$. Hence $\mathrm{psSpec}(M) \neq \emptyset$. Since $M \cong S/K$ and S is a pseudo-prime multiplication R-module, M is a pseudo-prime multiplication R-module by Lemma 3.5.

Conversely, suppose that M is a pseudo-prime multiplication R-module. If $\mathrm{psSpec}(S) = \emptyset$, then clearly S is a pseudo-prime multiplication R-module. Now, suppose that $\mathrm{psSpec}(S) \neq \emptyset$. Let T be a pseudo-prime submodule of S. We split the proof into two cases:

CASE 1: $(T :_R S) = P = P_1 \oplus P_2$. Then $K \subseteq PS \subseteq T$. So T/K is a pseudo-prime submodule of S/K by Lemma 3.1. Since $M \cong S/K$ is a pseudo-prime multiplication module, we must have $T/K = P(S/K) = PS/K$, hence $T = PS$.

CASE 2: $(T :_R S) = P_1 \oplus 0$. Then $(0 :_R S) = P_1^n \oplus 0$ for some positive integer n by Propositions 4.2 and 5.2. Since $(P_1^n \oplus 0)S = 0 \subseteq (0 \oplus P_2)S$, we have $P_1^n \oplus 0 \subseteq ((0 \oplus P_2)S :_R S)$. On the other hand, it is clear that $0 \oplus P_2 \subseteq ((0 \oplus P_2)S :_R S)$. Therefore $((0 \oplus P_2)S :_R S) = P_1^m \oplus P_2$ for some positive integer m. Then $K \subseteq P^m S \subseteq (P_1^m \oplus P_2)S \subseteq (0 \oplus P_2)S$ by [9, Lemma 4.3]. Hence $K = 0$, since $K \cap (0 \oplus P_2)S = 0$. So $S \cong M$ is a pseudo-prime multiplication R-module.

CASE 3: $(T :_R S) = 0 \oplus P_2$. The proof is similar to that in Case 2. ■

PROPOSITION 5.5. *Let R be a pullback ring as in* (1) *and let M be an indecomposable pseudo-prime multiplication nonseparated R-module with* $(0 :_R M) \neq 0$. *Let* $0 \to K \xrightarrow{i} S \to \varphi M \to 0$ *be a separated representation of M. Then S is pure-injective.*

Proof. Apply Theorem 5.4 and Corollary 4.9. ■

LEMMA 5.6. *Let R be a pullback ring as in* (1) *and let M be an indecomposable pseudo-prime multiplication nonseparated R-module with* $(0 :_R M)$ $\neq 0$. *Let* $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ *be a separated representation of M. Then R does not occur among the direct summands of S.*

Proof. Suppose $S = R \oplus L$ for some submodule L of S. Then $K \subseteq L$, since $\mathrm{Soc}(R) = 0$. Therefore $M \cong L/K \oplus R$, which is a contradiction, since M is indecomposable and nonseparated. ■

Let R be a pullback ring as in (1) and let M be an indecomposable pseudo-prime multiplication nonseparated R-module. Consider the separated representation $0 \to K \to S \to M \to 0$. Then by Proposition 5.5, S is a pure-injective R-module. So in the proofs of [5, Lemma 3.1, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [5, Proposition 2.6(ii)]), we can replace the statement "M is an indecomposable pure-injective nonseparated R-module" by "M is an indecomposable nonseparated pseudo-prime multiplication R-module", because the key properties in those results are the pure-injectivity of S, the indecomposability and the nonseparability of M. So we have the following result.

Corollary 5.7. *Let R be a pullback ring as in* (1)*, let M be an indecomposable nonseparated pseudo-prime multiplication R-module with* $(0 :_R M) \neq 0$ *and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is a direct sum of finitely many indecomposable pseudo-prime multiplication modules.*

Now, we are in a position to state the main theorem of this section.

Theorem 5.8. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the indecomposable nonseparated pseudo-prime multiplication modules with nonzero annihilator are the indecomposable modules of finite length* (*apart from R/P which is separated*).

Proof. We already know that every indecomposable nonseparated pseudo-prime multiplication module has this form by Corollary 5.7, so it remains to show that the modules obtained by this amalgamation are, indeed, indecomposable pseudo-prime multiplication modules.

Note that every indecomposable R-module of finite length is a pseudo-prime multiplication module since it is a quotient of a pseudo-prime multiplication R-module by Corollary 5.7. The indecomposability follows from [18, 1.9]. ■

Corollary 5.9. *Let R be a pullback ring as in Theorem* 5.8*. Then every indecomposable pseudo-prime multiplication R-module is pure-injective.*

Proof. Apply [5, Theorem 3.5] and Theorem 5.8. ■

Acknowledgements. The author is deeply grateful to the referee for a careful reading of the manuscript and helpful suggestions.

References

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory, of Associative Algebras, Vol. 1. Techniques of Representation Theory*, London Math. Soc. Student Texts 65, Cambridge Univ. Press, Cambridge, 2007.

[2] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. 82 (1963), 8–29.

[3] S. Ebrahimi Atani, *Indecomposable weak multiplication modules over Dedekind domains*, Demonstratio Math. 41 (2008), 33–43.

[4] S. Ebrahimi Atani, *On prime modules over pullback rings*, Czechoslovak Math. J. 54 (2004), 781–789.

[5] S. Ebrahimi Atani, *On pure-injective modules over pullback rings*, Comm. Algebra 28 (2000), 4037–4069.

[6] S. Ebrahimi Atani, *On secondary modules over pullback rings*, Comm. Algebra 30 (2002), 2675–2685.

[7] R. Ebrahimi Atani and S. Ebrahimi Atani, *On primarily multiplication modules over pullback rings*, Algebra Discrete Math. 11 (2011), 1–17.

[8] R. Ebrahimi Atani and S. Ebrahimi Atani, *On semiprime multiplication modules over pullback rings*, Comm. Algebra 41 (2013), 776–791.

[9] S. Ebrahimi Atani and F. Esmaeili Khalil Saraei, *Indecomposable primarily comultiplication modules over a pullback of two Dedekind domains*, Colloq. Math. 120 (2010), 23–42.

[10] S. Ebrahimi Atani and F. Esmaeili Khalil Saraei, *On P-torsion modules over a pullback of two Dedekind domains*, Georgian Math. J. 19 (2012), 473–488.

[11] S. Ebrahimi Atani and F. Farzalipour, *Weak multiplication modules over a pullback of Dedekind domains*, Colloq. Math. 114 (2009), 99–112.

[12] A. Facchini and P. Vamos, *Injective modules over pullback rings*, J. London Math. Soc. 31 (1985), 125–138.

[13] D. Hassanzadeh Lelekami and H. Roshan Shekalgourabi, *Pseudo-prime submodules of modules*, submitted.

[14] I. Kaplansky, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc. 72 (1952), 327–340.

[15] R. Kiełpiński, *On Γ-pure injective modules*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 15 (1967), 127–131.

[16] V. V. Kirichenko, *Classification of the pairs of mutually annihilating operators in a graded space and representations of a dyad of generalized uniserial algebra*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 75 (1978), 91–109, 196–197 (in Russian).

[17] L. Klingler, *Integral representation of groups of square-free order*, J. Algebra 129 (1990), 26–74.

[18] L. Levy, *Mixed modules over ZG, G cyclic of prime order, and over related Dedekind pullbacks*, J. Algebra 71 (1981), 62–114.

[19] L. Levy, *Modules over Dedekind-like rings*, J. Algebra 93 (1985), 1–116.

[20] L. Levy, *Modules over pullbacks and subdirect sums*, J. Algebra 71 (1981), 50–61.

[21] C. P. Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Pauli 33 (1984), 61–69.

[22] R. L. MacCasland, M. E. Moore and P. F. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra 25 (1997), 79–103.

[23] M. Moore and S. J. Smith, *Prime and radical submodules of modules over commutative rings*, Comm. Algebra 10 (2002), 5037–5064.

[24] L. A. Nazarova and A. V. Roĭter, *Finitely generated modules over a dyad of local Dedekind rings and finite groups having an abelian normal subgroup of index p*, Izv. Akad. Nauk SSSR 33 (1969), 65–69 (in Russian).

[25] M. Prest, *Model Theory and Modules*, London Math. Soc. Lecture Note Ser. 130, Cambridge Univ. Press, Cambridge, 1988.

[26] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra Logic Appl. 4, Gordon and Breach, 1992.

[27] D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, Vol. 3. Representation-Infinite Tilted Algebras*, London Math. Soc. Student Texts 72, Cambridge Univ. Press, Cambridge, 2007.

[28] R. B. Warfield, *Purity and algebraic compactness for modules*, Pacific J. Math. 28 (1969), 699–719.

[29] A. N. Wiseman, *Projective modules over pullback rings*, Math. Proc. Cambridge Philos. Soc. 97 (1985), 399–406.

Fatemeh Esmaeili Khalil Saraei
College of Engineering
Faculty of Fouman
University of Tehran
P.O. Box 43515-1155
Fouman 43516-66456, Guilan, Iran
E-mail: f.esmaeili.kh@ut.ac.ir