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FINITE SPEED OF PROPAGATION FOR A NON-LOCAL POROUS MEDIUM EQUATION

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Abstract. This note is concerned with proving the finite speed of propagation for some non-local porous medium equation by adapting arguments developed by Caffarelli and Vázquez (2010).

1. Introduction. Caffarelli and Vázquez [CV] proved finite speed of propagation for non-negative weak solutions of

$$\partial_t u = \nabla \cdot (u \nabla^{\alpha-1} u), \qquad t > 0,\, x \in \mathbb{R}^d, \tag{1.1}$$

where $\alpha \in (0,2)$ and $\nabla^{\alpha-1}$ stands for $\nabla(-\Delta)^{\alpha/2-1}$. We adapt here their proof in order to treat the more general case

$$\partial_t u = \nabla \cdot (u \nabla^{\alpha-1} u^{m-1}), \qquad t > 0,\, x \in \mathbb{R}^d, \tag{1.2}$$

for $m > m_\alpha := 1 + d^{-1}(1-\alpha)_+ + 2(1-\alpha^{-1})_+$. Equation (1.2) is supplemented with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \tag{1.3}$$

for some $u_0 \in L^1(\mathbb{R}^d)$. Our result gives a positive answer to a question posed in [STV1] (see also [STV2]) where finite or infinite speed of propagation is studied for another generalization of (1.1). We recall that weak solutions of (1.2)–(1.3) are constructed in [BIK2] for $m > m_\alpha$ (see also [BIK1]).

In the following statement (and the remainder of the note), B_R denotes the ball of radius $R > 0$ centered at the origin.

THEOREM 1.1 (Finite speed of propagation). *Let $m > m_\alpha$ and assume that $u_0 \ge 0$ is integrable and supported in B_{R_0}. Then every non-negative weak solution u of (1.2)–(1.3) is supported in $B_{R(t)}$ where*

$$R(t) = R_0 + Ct^{1/\alpha} \quad \textit{with} \quad C = C_0 \|u_0\|_\infty^{(m-1)/\alpha}$$

for some constant $C_0 > 0$ only depending on dimension, α and m.

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REMARK 1.2. The technical assumption $m > m_\alpha$ is imposed to ensure the existence of weak solutions; see [BIK2].

REMARK 1.3. In view of the Barenblatt solutions constructed in [BIK2], the previous estimate of the speed of propagation is optimal.

The remainder of the note is organized as follows. In Section 2, the equation is written in non-divergence form, non-local operators appearing in it are written as singular integrals, invariant scalings are exhibited and an approximation procedure is recalled. Section 3 is devoted to the contact analysis. A first lemma for a general barrier is derived in Subsection 3.1. The barrier to be used in the proof of the theorem is constructed in Subsection 3.2. The main error estimate is obtained in Subsection 3.3. Theorem 1.1 is finally proved in Section 4.

NOTATION. For $a \in \mathbb{R}$, a_+ denotes $\max(0, a)$. An inequality written as $A \lesssim B$ means that there exists a constant C only depending on dimension, α and m such that $A \le CB$. If $\alpha \in (0,1)$, a function u being in $\mathcal{C}^\alpha$ means that it is α-Hölder continuous. If $\alpha \in (1,2)$, it means that ∇u is $(\alpha-1)$-Hölder continuous. For $\alpha \in (0,2)$, a function u is in $\mathcal{C}^{\alpha+0}$ if it is in $\mathcal{C}^{\alpha+\varepsilon}$ for some $\varepsilon > 0$ and $\alpha + \varepsilon \neq 1$.

2. Preliminaries. The contact analysis relies on writing (1.2) in the following non-divergence form:

$$\partial_t u = \nabla u \cdot \nabla p + u \Delta p \tag{2.1}$$

where p stands for the pressure term and is defined as

$$p = (-\Delta)^{\alpha/2-1} u^{m-1}.$$

It is also convenient to write $v = u^{m-1} = G(u)$.

We recall that for a smooth and bounded function v, the non-local operators appearing in (2.1) have the following singular integral representations:

$$\nabla(-\Delta)^{\alpha/2-1} v = c_\alpha \int (v(x+z) - v(x)) z \frac{dz}{|z|^{d+\alpha}},$$

$$-(-\Delta)^{\alpha/2} v = \bar{c}_\alpha \int \big(v(x+z) + v(x-z) - 2v(x)\big) \frac{dz}{|z|^{d+\alpha}}.$$

The following elementary lemma makes the scaling of the equation precise.

LEMMA 2.1 (Scaling). *If u satisfies* (1.2) *then $U(t,x) = Au(Tt, Bx)$ satisfies* (1.2) *as soon as*

$$T = A^{m-1} B^\alpha.$$

Consider non-negative solutions of the viscous approximation of (1.2), i.e.

$$\partial_t u = \nabla \cdot (u \nabla^{\alpha-1} G(u)) + \delta \Delta u, \quad t > 0,\, x \in \mathbb{R}^d. \tag{2.2}$$

For sufficiently smooth initial data u_0, solutions are at least $\mathcal{C}^2$ with respect to x and $\mathcal{C}^1$ with respect to t.

3. Contact analysis

3.1. The contact analysis lemma. In the following lemma, we analyse what happens when a sufficiently regular barrier U touches a solution u of (2.2) from above. The monotone terms such as $\partial_t u$, Δu or $-(-\Delta)^{\alpha/2} u$ are naturally ordered. But this is not the case for the non-local drift term $\nabla u \cdot \nabla p$. The idea is to split it is a "good" part (i.e. with the same monotony as Δu for instance) and a "bad" part. It turns out that the bad part can be controlled by a fraction of the "good" part; see (3.6) in the proof of the lemma.

LEMMA 3.1 (Contact analysis). *Let u be a solution of the approximate equation* (2.2) *and $U(t,x)$ be $\mathcal{C}^2((0,\infty)\times(\mathbb{R}^d \setminus B_1))$, radially symmetric with respect to x, and non-increasing with respect to $|x|$. If*

$$u \le U \quad \textit{for } (t,x) \textit{ in } [0,t_c]\times\mathbb{R}^d, \qquad u(t_c,x_c) = U(t_c,x_c),$$

then

$$\partial_t U \le \nabla U \cdot \nabla P + U \Delta P + \delta \Delta U + e \tag{3.1}$$

at $(t_c,x_c) \in (0,\infty)\times(\mathbb{R}^d \setminus B_1))$ where

$$\begin{aligned} V &= G(U),\\ P &= (-\Delta)^{\alpha/2-1} V,\\ e &= |\nabla U|(I_{\mathrm{out},+}(V) - I_{\mathrm{out},+}(v)) \ge 0 \end{aligned}$$

with

$$I_{\mathrm{out},+}(w) = \begin{cases} \displaystyle\int_{\substack{|y|\ge\gamma\\ y\cdot\hat{x}_c \ge 0}} (w(x_c+y) - w(x_c))(y\cdot\hat{x}_c)\,\frac{dy}{|y|^{d+\alpha}} & \textit{if } \alpha \ge 1,\\ \displaystyle\int_{\substack{|y|\ge\gamma\\ y\cdot\hat{x}_c \ge 0}} w(x_c+y)(y\cdot\hat{x}_c)\,\frac{dy}{|y|^{d+\alpha}} & \textit{if } \alpha \in (0,1) \end{cases}$$

(where $\hat{x}_C = x_C/|x_C|$) for γ such that

$$c_\alpha \gamma |\nabla U(x_c)| \le \bar{c}_\alpha U(x_c)$$

where c_α and $\bar{c}_\alpha$ are the constants appearing in the definitions of the two non-local operators.

Proof. At the contact point (t_c, x_c), we have

$$\partial_t u \geq \partial_t U, \qquad \nabla u = \nabla U = -|\nabla U|\hat{x}_c, \qquad \Delta u \leq \Delta U.$$

This implies that

$$\partial_t U \leq \nabla U \cdot \nabla p + U\Delta p + \delta \Delta U. \tag{3.2}$$

We next turn our attention to ∇p and Δp. We drop the time dependence of functions since it plays no role in the remaining analysis.

The fact that U is radially symmetric and non-decreasing implies in particular that $\nabla U(x) = -|\nabla U(x)|x/|x|$, which in turn implies

$$\nabla U \cdot \nabla p = -|\nabla U| I(v) \tag{3.3}$$

where

$$I(v) = \begin{cases} c_\alpha \int (v(x_c+y) - v(x_c))(y \cdot \hat{x}_c) \dfrac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in [1,2), \\ c_\alpha \int v(x_c+y)(y \cdot \hat{x}_c) \dfrac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in (0,1). \end{cases}$$

We now split I into several pieces by splitting the domain of integration $\mathbb{R}^d$ into $B_\gamma^{\text{in},\pm} = \{y \in B_\gamma : \pm y \cdot \hat{x}_c \geq 0\}$ and $B_\gamma^{\text{out},\pm} = \{y \notin B_\gamma : \pm y \cdot \hat{x}_c \geq 0\}$ for some parameter $\gamma > 0$ to be fixed later. We can thus write

$$I(v) = I_{\text{in},+}(v) + I_{\text{in},-}(v) + I_{\text{out},+}(v) + I_{\text{out},-}(v)$$

where

$$I_{\text{in/out},\pm}(v) = \begin{cases} c_\alpha \displaystyle\int_{B_\gamma^{\text{in/out},\pm}} (v(x_c+y) - v(x_c))(y \cdot \hat{x}_c) \dfrac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in [1,2), \\ c_\alpha \displaystyle\int_{B_\gamma^{\text{in/out},\pm}} v(x_c+y)(y \cdot \hat{x}_c) \dfrac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in (0,1). \end{cases}$$

We can proceed similarly for Δp. Note that

$$\Delta p = J(v) \tag{3.4}$$

where

$$J(v) = \bar{c}_\alpha \int \big(v(x+y) + v(x-y) - 2v(x)\big) \frac{dy}{|y|^{d+\alpha}}.$$

We can introduce $J_{\text{in/out},\pm}(v)$ analogously.

We first remark that

$$-I_{\text{in/out},-}(v) \leq -I_{\text{in/out},-}(V), \qquad J_{\text{in/out},\pm}(v) \leq J_{\text{in/out},\pm}(V) \tag{3.5}$$

at x_c, where $V = G(U)$.

We next remark that since G is non-decreasing and vanishes at 0, and $w = v - V$ reaches a zero maximum at $x = x_c$, we have

$$-I_{\text{in},+}(v - V) \leq -\tilde{c}_\alpha \gamma J_{\text{in},+}(v - V) \tag{3.6}$$

at x_c. Indeed, for $\alpha \in (1,2)$ (the proof is the same in the other case),

$$\begin{aligned}\gamma J_{\mathrm{in},+}(w)(x_c) &= \bar{c}_\alpha \gamma \int\limits_{B_\gamma^{\mathrm{in},+}} \big(w(x_c+y)+w(x_c-y)-2w(x_c)\big)\,\frac{dy}{|y|^{d+\alpha}}\\ &= 2\bar{c}_\alpha\gamma \int\limits_{B_\gamma^{\mathrm{in},+}} (w(x_c+y)-w(x_c))\,\frac{dy}{|y|^{d+\alpha}}\\ &\ge 2\bar{c}_\alpha \int\limits_{B_\gamma^{\mathrm{in},+}} (w(x_c+y)-w(x_c))(y\cdot \hat{x}_c)\,\frac{dy}{|y|^{d+\alpha}}.\end{aligned}$$

Combining (3.2)–(3.6), we get (at x_c)

$$\begin{aligned}\partial_t U \le{}& |\nabla U|\big(-I_{\mathrm{in},+}(V)-I_{\mathrm{in},-}(V)-I_{\mathrm{out},+}(v)-I_{\mathrm{out},-}(V)+\tilde{c}_\alpha\gamma J_{\mathrm{in},+}(V)\big)\\ &+(U-\tilde{c}_\alpha\gamma|\nabla U|)J_{\mathrm{in},+}(v)\\ &+U\big(J_{\mathrm{in},-}(V)+J_{\mathrm{out},+}(V)+J_{\mathrm{out},-}(V)\big)+\delta\Delta U.\end{aligned}$$

In view of the choice of γ, we get

$$\partial_t U \le -|\nabla U| I(V) + UJ(V) + |\nabla U|(-I_{\mathrm{out},+}(v)+I_{\mathrm{out},+}(V)) + \delta\Delta U.$$

We now remark that $-|\nabla U|I(V) = \nabla U\cdot\nabla P$ and $J(V)=\Delta P$ to get the desired inequality. ∎

3.2. Construction of the barrier. The previous lemma holds true for general barriers U. In this subsection, we specify the barrier we are going to use. We would like to use $(R(t)-|x|)^2$, but this does not work. First the power 2 is changed to β large enough such that $V=U^{m-1}$ is regular enough. Second, a small ω^β is added in order to ensure that the contact does not happen at infinity. Third, a small slope in time of the form $\omega^\beta t/T$ is added to control some error terms.

LEMMA 3.2 (Construction of a barrier). *Assume that*

$$\|u\|_\infty \le 1 \quad \text{and} \quad 0\le u_0(x)\le (R_0-|x|)_+^\beta \quad \text{with } R_0\ge 2$$

for some $\beta > \max(2,\alpha(m-1)^{-1})$. Then there exist $C,T>0$ (only depending on d,m,α,β) and $U\in\mathcal{C}^2((0,\infty)\times(\mathbb{R}^d\setminus B_1))$ defined by

$$U(t,x)=\omega^\beta+(R(t)-|x|)_+^\beta+\omega^\beta t/T, \tag{3.7}$$

where $R(t)=R_0+Ct$ and $\omega=\omega(\delta)$ is small enough, such that:

(i) *the functions*

$$\nabla P, \Delta P, J_{\mathrm{in},+}(V), I_{\mathrm{out},+}(V), \Delta U \ \text{ are bounded}; \tag{3.8}$$

(ii) *u and U cannot touch at a time $t<T$ and a point $x_c\in B_1$ or $x_c\notin \bar{B}_{R(t)}$;*

(iii) *if* U *touches* u *from above at* (t_c, x_c) *with* $t_c < T$ *and* $x_c \in B_{R(t)}$, *then*

$$C \lesssim 1 - I_{\text{out},+}(v) + \delta/\omega. \tag{3.9}$$

Proof. We first remark that the condition $R_0 \geq 2$ ensures that the contact point is out of B_1 since $\|u\|_\infty \leq 1$.

The fact that U is $\mathcal{C}^2$ in $(0,\infty) \times (\mathbb{R}^d \setminus B_1)$ and $V = U^{m-1}$ is $\mathcal{C}^{\alpha+0}$ in $\mathbb{R}^d \setminus B_1$ ensures that (3.8) holds true. Notice that the condition $\beta(m-1) > \alpha$ is used here.

We should now justify that the contact point cannot be outside $B_{R(t)}$ at a time $t \in (0,T)$ for some small time T under control. If $|x_c| > R(t)$ and $t_c < T$ then

$$0 \leq U \leq 2\omega^\beta, \qquad \partial_t U = \omega^\beta/T, \qquad |\nabla U| = 0, \qquad \Delta U = 0.$$

Then (3.1) of the contact analysis Lemma 3.1 (with $\gamma = 1$, say) and (3.8) imply that

$$\omega^\beta/T \leq |\Delta P| U \lesssim \omega^\beta,$$

and choosing T small enough (but under control) yields a contradiction.

It remains to study what happens if $t_c < T$ and $x_c \in B_{R(t)} \setminus B_1$. To do so, we first define h and H as follows:

$$U = h^\beta + H^\beta \leq 1$$

with $H^\beta = \omega^\beta t T^{-1} \leq \omega^\beta$ for $t \in (0,T)$. Note that $h \geq \omega \geq H$. In the contact analysis Lemma 3.1, we choose γ such that

$$\beta c_\alpha \gamma \leq \bar{c}_\alpha h.$$

If $x_c \in B_{R(t)} \setminus B_1$ then

$$\partial_t U = \beta C h^{\beta-1} + \omega^\beta/T \geq \beta C h^{\beta-1}, \qquad |\nabla U| = \beta h^{\beta-1}. \tag{3.10}$$

Combining Lemma 3.1 with (3.8)–(3.10), we get (3.9). ∎

3.3. Estimate of the error term

Lemma 3.3. *At* x_c,

$$-I_{\text{out},+}(v) \lesssim \begin{cases} G(2h^\beta)h^{1-\alpha} & \text{if } \alpha > 1, \\ R_0^{1-\alpha+\varepsilon} & \text{if } \alpha \leq 1, \end{cases} \tag{3.11}$$

for all $\varepsilon > 0$.

Proof. We begin with the easy case $\alpha > 1$. In this case, we simply write

$$
\begin{aligned}
-I_{\text{out},+}(v) &= \int_{\substack{|y|\geq\gamma \\ y\cdot\hat{x}_c\geq 0}} (v(x_c) - v(x_c+y))(y\cdot\hat{x}_c)\,\frac{dy}{|y|^{d+\alpha}} \\
&\leq v(x_c) \int_{\substack{|y|\geq\gamma \\ y\cdot\hat{x}_c\geq 0}} (y\cdot\hat{x}_c)\,\frac{dy}{|y|^{d+\alpha}} \leq v(x_c)\int_{|y|\geq\gamma} \frac{dy}{|y|^{d+\alpha-1}}
\end{aligned}
$$

where we have used the fact that $v \geq 0$. By remarking that

$$v = G(u) = G(h^\beta + H^\beta) \leq G(2h^\beta)$$

at the contact point and through an easy and standard computation, we get the desired estimate in the case $\alpha > 1$.

We now turn to the more subtle case $\alpha \in (0,1]$. In this case,

$$I_{\text{out},+}(v) = I^*[v] + K \star v$$

where

$$-I^*[v] = \int_{\substack{\gamma\leq|y|\leq 1 \\ y\cdot\hat{x}_c\geq 0}} -v(x_c+y)(y\cdot\hat{x}_c)\,\frac{dy}{|y|^{d+\alpha}}, \qquad K = \frac{y\cdot\hat{x}_c}{|y|^{d+\alpha}}\mathbf{1}_{|y|\geq 1,\, y\cdot\hat{x}_c\geq 0}.$$

We first remark that

$$|I^*[v]| \leq \|v\|_\infty \int_{B_1} \frac{dy}{|y|^{d+\alpha-1}} \lesssim 1.$$

We next observe that $K \in L^p(\mathbb{R}^d)$ for all $p > \frac{d}{d-(1-\alpha)} \geq 1$. Hence,

$$|K\star v| \leq \|K\|_p \|v\|_q = \|K\|_p \|u\|_{(m-1)q}^{m-1}$$

with p as above and $q^{-1} = 1 - p^{-1}$.

We next estimate $\|u\|_{(m-1)q}$. Interpolation leads to

$$\|u\|_{(m-1)q}^{m-1} \leq \|u\|_1^{1/q}\|u\|_\infty^{(m-1)-1/q} \leq \|u\|_1^{1/q}$$

since $\|u\|_\infty \leq 1$. Finally, we use mass conservation in order to get

$$\|u\|_1 = \|u_0\|_1 \leq \int \min(1, (R_0 - |x|)_+^\beta)\,dx \leq \omega_d R_0^d.$$

Finally, we have

$$|I_{\text{out},+}(v)| \lesssim R_0^{d/q}$$

for all $q < d/(1-\alpha)$, which yields the desired result. ∎

Combining now Lemmas 3.2 and 3.3, we get the following one.

Lemma 3.4 (Estimate of the speed of propagation). *Assume that*

$$\|u\|_\infty \leq 1 \quad \textit{and} \quad 0 \leq u_0(x) \leq (R_0 - |x|)_+^\beta \quad \textit{with } R_0 \geq 2$$

for some $\beta > \max(2, \alpha(m-1)^{-1})$. *Then there exist* $T, C_0 > 0$ *only depending on dimension,* m*,* α *and* β *(and* ε *for* $\alpha \le 1$*) such that, for* $t \in (0,T)$*,* u *is supported in* B_{R_0+Ct} *with*

$$C = \begin{cases} C_0 & \text{if } \alpha > 1, \\ C_0 R_0^{1-\alpha-\varepsilon} & \text{if } \alpha \le 1 \end{cases} \tag{3.12}$$

(for $\varepsilon > 0$ *arbitrarily small).*

Proof. In view of Lemma 3.2, the parameter ω is chosen so that $\omega \gg \delta$, say $\omega = \sqrt{\delta}$. Now Lemmas 3.2 and 3.3 imply that if C is chosen as indicated in (3.12), then u remains below U at least up to time T. Letting (ω, δ) go to 0 yields the desired result. ∎

4. Proof of Theorem 1.1. We treat successively the cases $\alpha > 1$ and $\alpha \le 1$.

First case. In the case $\alpha > 1$, if $\|u\|_\infty = \|u_0\|_\infty \le 1$ and

$$u_0(x) \le (R_0 - |x|)_+^\beta,$$

then Lemma 3.4 implies that the support of u is contained in $B_{R(t)}$ with

$$R(t) = R_0 + C_0 t$$

for some constant C_0 only depending on dimension, m and α. Rescaling the solution (see Lemma 2.1), we get

$$R(t) = R_0 + C_0 L^{m-1-(\alpha-1)/\beta} a^{(\alpha-1)/\beta} t$$

as soon as

$$u_0(x) \le a(R_0 - |x|)_+^\beta \quad \text{and} \quad L = \|u\|_\infty = \|u_0\|_\infty.$$

If we simply know that u_0 is supported in B_{R_0} and $\|u\|_\infty = \|u_0\|_\infty = L$, then we can pick any $a, r_1 > 0$ such that $a r_1^\beta = L$ and get

$$u_0(x) \le a(r_1 + R_0 - |x|)_+^\beta.$$

By the previous reasoning, we get

$$R(t) \le R_0 + r_1 + C_0 L^{m-1-(\alpha-1)/\beta} a^{(\alpha-1)/\beta} t = R_0 + r_1 + C_0 L^{m-1} r_1^{1-\alpha} t.$$

Minimizing with respect to r_1 yields the desired result in the case $\alpha > 1$.

Second case. We now turn to the case $\alpha \in (0,1]$. Lemma 3.4 yields, for $t \in [0, T_1]$ with $T_1 = R_0/C_1$,

$$C_1 \lesssim R_0^{1-\alpha+\varepsilon}$$

(recall that $R_0 \ge 2$).

We now start with $R_1 = R_0 + C_1 T_1 = 2R_0$, and for $t \in [T_1, T_2]$ we get

$$C_2 \lesssim (3R_0)^{1-\alpha+\varepsilon} \quad \text{with} \quad T_2 - T_1 = R_0/C_2.$$

More generally, for $t \in [T_k, T_{k+1}]$,

$$C_k \simeq ((k+1)R_0)^{1-\alpha+\varepsilon} \simeq (kR_0)^{1-\alpha+\varepsilon}$$

with

$$T_{k+1} - T_k = \frac{R_0}{C_k} \simeq \frac{R_0^{\alpha-\varepsilon}}{(k+1)^{1-\alpha+\varepsilon}}.$$

We readily see that the series $\sum_k (T_{k+1} - T_k)$ diverges. More precisely, $T_k \simeq (kR_0)^{\alpha-\varepsilon}$. Moreover, the function u is supported in $B_{R(t)}$ with

$$R(t) - R_0 \lesssim kR_0 + C_k(t - T_k) \lesssim (T_k)^{1/(\alpha-\varepsilon)} + (T_k)^{\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}} t \lesssim t^{1/(\alpha-\varepsilon)}$$

for $t \in [T_k, T_{k+1}]$. Hence, we get the result but not with the right power. More precisely, for $L = 1$ and

$$0 \leq u_0(x) \leq (R_0 - |x|)_+^\beta$$

we get $R(t) = R_0 + C_0 t^\beta$ with $\beta > 1/\alpha$. Rescaling and playing again with r_1 and a such that $ar_1^\beta = L$ yields the desired result in the case $\alpha < 1$. The proof is now complete. ■

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