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A NOTE ON REPRESENTATION FUNCTIONS WITH DIFFERENT WEIGHTS

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Abstract. For any positive integer k and any set A of nonnegative integers, let $r_{1,k}(A,n)$ denote the number of solutions (a_1,a_2) of the equation $n = a_1 + ka_2$ with $a_1, a_2 \in A$. Let $k, l \geq 2$ be two distinct integers. We prove that there exists a set $A \subseteq \mathbb{N}$ such that both $r_{1,k}(A,n) = r_{1,k}(\mathbb{N}\setminus A,n)$ and $r_{1,l}(A,n) = r_{1,l}(\mathbb{N}\setminus A,n)$ hold for all $n \geq n_0$ if and only if $\log k/\log l = a/b$ for some odd positive integers a, b, disproving a conjecture of Yang. We also show that for any set $A \subseteq \mathbb{N}$ satisfying $r_{1,k}(A,n) = r_{1,k}(\mathbb{N}\setminus A,n)$ for all $n \geq n_0$, we have $r_{1,k}(A,n) \to \infty$ as $n \to \infty$.

1. Introduction. We use $\mathbb{N}$ to denote the set of nonnegative integers. For a set $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_1(A,n)$, $R_2(A,n)$ and $R_3(A,n)$ be the number of solutions (a_1,a_2) of $n = a_1 + a_2$ with $a_1, a_2 \in A$; with $a_1, a_2 \in A$, $a_1 < a_2$; and with $a_1, a_2 \in A$, $a_1 \leq a_2$, respectively. These representation functions have been studied by many authors. The reader may refer to the excellent survey paper [SS] for many results concerning representation functions.

For $i = 1, 2, 3$, Sárközy asked whether there exist sets $A, B \subseteq \mathbb{N}$ with infinite symmetric difference such that $R_i(A,n) = R_i(B,n)$ for all sufficiently large integers n. Dombi [D] observed that the answer is negative for $i = 1$, and affirmative for $i = 2$. Chen and Wang [CW] constructed a set $A \subseteq \mathbb{N}$ with $R_3(A,n) = R_3(\mathbb{N}\setminus A,n)$ for all $n \geq 1$. Later Lev [L], Sándor [S] and Tang [T] characterized all sets $A \subseteq \mathbb{N}$ such that $R_i(A,n) = R_i(\mathbb{N}\setminus A,n)$ for $n \geq N$ and $i = 2, 3$.

One may extend these problems by considering the representation functions in a more general form. Let k_1, k_2 be positive integers. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, denote by $r_{k_1,k_2}(A,n)$ the number of solutions (a_1,a_2) of $k_1a_1 + k_2a_2 = n$ with $a_1, a_2 \in A$. Yang and Chen [YC] determined all pairs (k_1,k_2) of positive integers for which there exists a set $A \subseteq \mathbb{N}$ such that $r_{k_1,k_2}(A,n) = r_{k_1,k_2}(\mathbb{N}\setminus A,n)$ for all $n \geq n_0$. Let $1 \leq k_1 < k_2$, and $(k_1,k_2) = 1$. They

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proved that there exists $A \subseteq \mathbb{N}$ such that $r_{k_1,k_2}(A,n) = r_{k_1,k_2}(\mathbb{N} \setminus A, n)$ for all $n \geq n_0$ if and only if $k_1 = 1$.

From now on, we denote by Ψ_k the set of all $A \subseteq \mathbb{N}$ such that $r_{1,k}(A,n) = r_{1,k}(\mathbb{N} \setminus A, n)$ for all sufficiently large integers n. Yang [Y] studied the problem of when $\Psi_k \cap \Psi_l$ is nonempty, where $k, l \geq 2$ are distinct integers.

THEOREM A ([Y]). *Let $k, l \geq 2$ be two distinct integers. If k, l are multiplicatively independent* (*equivalently,* $\log k/\log l$ *is irrational*), *then* $\Psi_k \cap \Psi_l = \emptyset$.

The proof in [Y] also works for $\log k/\log l = a/b$ with a, b positive integers of different parities. It is conjectured in [Y] that $\Psi_k \cap \Psi_l = \emptyset$ also for a, b both odd. However, this is not the case. In this paper we will prove the following theorem.

THEOREM 1.1. *Let $k, l \geq 2$ be two distinct integers. Then $\Psi_k \cap \Psi_l \neq \emptyset$ if and only if $\log k/\log l = a/b$ for some odd positive integers a, b.*

Theorem A proves one direction of Theorem 1.1. We provide a new proof here since an ingredient in the proof is also needed for the other direction. Motivated by [C, CT], Yang and Chen asked about the asymptotic behavior of $r_{1,k}(A,n)$ for sets $A \in \Psi_k$.

PROBLEM 1.2 ([YC]). *For any set $A \in \Psi_k$, is it true that $r_{1,k}(A,n) \geq 1$ for all sufficiently large integers n? Is it true that $r_{1,k}(A,n) \to \infty$ as $n \to \infty$?*

We give an affirmative answer to this problem.

THEOREM 1.3. *Let $k \geq 2$ be an integer, and $A \in \Psi_k$. Then*

$$\lim_{n\to\infty} r_{1,k}(A,n) = \infty.$$

2. Proofs. For the proof of Theorem 1.1, we first obtain a criterion for $A \in \Psi_k$ in terms of generating functions. We use $[x, y)$ to denote the set of all integers n satisfying $x \leq n < y$. Noting that both A and $\mathbb{N} \setminus A$ are infinite sets for $A \in \Psi_k$, it is convenient for us to write A in "blocks", that is,

$$A = \bigcup_{i=0}^{\infty} [t_{2i}, t_{2i+1}), \tag{2.1}$$

where $0 \leq t_0 < t_1 < t_2 < \cdots$ is an increasing sequence of integers. Let

$$f_A(x) = \sum_{a \in A} x^a, \qquad |x| < 1.$$

LEMMA 2.1. *Let $k > 1$ be a given integer. With the notation above, $A \in \Psi_k$ if and only if there exists an odd positive integer a such that $t_{i+a} = kt_i$ for all $i \geq i_0$, and the polynomial*

$$-1+\sum_{i=0}^{i_0+a-1}(-1)^i x^{t_i}+\sum_{j=0}^{i_0-1}(-1)^j x^{kt_j}$$

is divisible by $(1-x)(1-x^k)$.

Proof. Let $B=\mathbb{N}\setminus A$. First note that

$$f_A(x)f_A(x^k)=\sum_{a_1,a_2\in A}x^{a_1+ka_2}=\sum_{n\ge 0}r_{1,k}(A,n)x^n.$$

Thus $A\in\Psi_k$ if and only if

$$P(x):=f_A(x)f_A(x^k)-f_B(x)f_B(x^k) \tag{2.2}$$

is a polynomial. Substituting $f_B(x)=1/(1-x)-f_A(x)$ in (2.2), we get

$$P(x)=-\frac{1}{(1-x)(1-x^k)}+\frac{f_A(x)}{1-x^k}+\frac{f_A(x^k)}{1-x},$$

hence

$$(1-x)(1-x^k)P(x)=-1+f_A(x)(1-x)+f_A(x^k)(1-x^k). \tag{2.3}$$

Writing A in the form of (2.1) yields

$$f_A(x)(1-x)=\sum_{i=0}^{\infty}(-1)^i x^{t_i}. \tag{2.4}$$

Substituting (2.4) in (2.3), we obtain

$$(1-x)(1-x^k)P(x)=-1+\sum_{i=0}^{\infty}(-1)^i x^{t_i}+\sum_{j=0}^{\infty}(-1)^j x^{kt_j}. \tag{2.5}$$

Since the right hand side of (2.5) is a polynomial, there exist positive integers i_0, j_0 such that

$$(-1)^{j_0+m}x^{t_{j_0+m}}+(-1)^{i_0+m}x^{kt_{i_0+m}}=0$$

for all $m\ge 0$. This means that $t_{j_0+m}=kt_{i_0+m}$ and j_0-i_0 is odd. Set $a=j_0-i_0$. Clearly $j_0>i_0$, thus a is an odd positive integer, and $t_{i+a}=kt_i$ for all $i\ge i_0$. Consequently,

$$(1-x)(1-x^k)P(x)=-1+\sum_{i=0}^{i_0+a-1}(-1)^i x^{t_i}+\sum_{j=0}^{i_0-1}(-1)^j x^{kt_j}$$

is a polynomial divisible by $(1-x)(1-x^k)$.

The other half of the statement of the lemma is now trivial. ∎

Proof of Theorem 1.1. Suppose $A\in\Psi_k\cap\Psi_l$. By Lemma 2.1, there exist odd positive integers a, b such that $t_{i+a}=kt_i$ and $t_{i+b}=lt_i$ for all $i\ge i_0$. It follows that

$$k^b t_i=t_{i+ab}=l^a t_i$$

for all $i\ge i_0$, hence $\log k/\log l=a/b$ with a, b odd positive integers.

Assume now that $\log k/\log l = a/b$ with a, b odd and $(a, b) = 1$; then $k = m^a$ and $l = m^b$ for some positive integer m. Without loss of generality, we may assume that $a > b$. Let $t_0 = 0$, $t_1 = m^a$, $t_2 = (m+1)t_1$, and $t_{i+1} = mt_i$ for all $i \geq 2$. We prove that $A \in \Psi_k \cap \Psi_l$. In view of Lemma 2.1 (with $i_0 = 2$), it remains to show that

$$-x^{kt_1} + \sum_{i=0}^{a+1} (-1)^i x^{t_i} \tag{2.6}$$

is divisible by $(1-x)(1-x^k)$, and

$$-x^{lt_1} + \sum_{i=0}^{b+1} (-1)^i x^{t_i}$$

is divisible by $(1-x)(1-x^l)$. We prove the case for k, and the case for l is similar. Since

$$x^n \equiv 1 \pmod{1 - x^k}$$

for $k \mid n$, and $k \mid t_i$ for all $i \geq 0$, it follows that

$$-x^{kt_1} + \sum_{i=0}^{a+1} (-1)^i x^{t_i} \equiv -1 + \sum_{i=0}^{a+1} (-1)^i = 0 \pmod{1 - x^k},$$

thus $1 - x^k$ divides (2.6). Taking derivative of (2.6) and setting $x = 1$, we get

$$-kt_1 + \sum_{i=0}^{a+1} (-1)^i t_i = -(k+1)t_1 + t_2 \frac{1 - (-m)^a}{1 - (-m)} = 0.$$

Thus $x = 1$ is a double root, hence $(1-x)(1-x^k)$ divides (2.6).

This completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.3. Let $A \in \Psi_k$. It follows from Lemma 2.1 that A can be written in the form of (2.1) such that $t_{i+a} = kt_i$ for some odd positive integer a and all $i \geq i_0$. All we need is this condition, thus Theorem 1.3 is actually valid for a larger class of sets $A \subseteq \mathbb{N}$.

For $i \geq i_0 + a$, we have

$$t_{i+1} - t_i = k(t_{i+1-a} - t_{i-a}) \geq k.$$

By eliminating the first several blocks of A, we may assume without loss of generality that $t_{i+a} = kt_i$ and $t_{i+1} - t_i \geq k$ for all $i \geq 0$.

Let s be an arbitrary positive integer. Fix $\alpha \in (1/2, 1)$. It is clear that the sequence $\{t_{i+1}/t_i\}_{i \geq 0}$ is periodic with period a, hence

$$\liminf_{i\to\infty} \frac{t_{i+1}}{t_i} = \min_{0 \leq i < a} \frac{t_{i+1}}{t_i} > 1 = \lim_{i\to\infty} 1 + \frac{t_i^\alpha}{t_i}.$$

It follows that

$$\frac{t_{i+1}}{t_i} > 1 + \frac{t_i^\alpha}{t_i}$$

for $i \geq i_1$, that is,

$$t_{i+1} - t_i > t_i^\alpha \tag{2.7}$$

for $i \geq i_1$. Since

$$\frac{t_i^\alpha}{\sqrt{t_{i+1}} + k} = \frac{t_i^\alpha}{\sqrt{kt_{i+1-a}} + k} \geq \frac{t_i^\alpha}{\sqrt{kt_i} + k} \to \infty$$

as $i \to \infty$, we have

$$t_i^\alpha > k^{2s+1}(\sqrt{t_{i+1}} + k) \tag{2.8}$$

for $i \geq i_2$. Finally,

$$t_i > k^{4s+2} t_0^2 \tag{2.9}$$

for $i \geq i_3$. Let $m = \max\{i_1, i_2, i_3\} + 1$. We show that $r_{1,k}(A, n) \geq s$ for all $n \geq t_m$, which would then imply our result.

Let $I_j = [t_j, t_{j+1})$; then $I_j \subset A$ if j is even. For a set $I \subseteq \mathbb{N}$, write

$$k * I = \{kx : x \in I\}.$$

Since $t_{i+1} - t_i \geq k$, it follows that

$$I_i + k * I_j = \bigcup_{u=t_j}^{t_{j+1}-1} [t_i + ku, t_{i+1} + ku) = [t_i + kt_j, t_{i+1} + kt_{j+1} - k).$$

Let $n \geq t_m$. Assume that $n \in I_i$ for some $i \geq m$. We shall distinguish four cases.

Case 1: i is even and $n - t_i \leq \sqrt{t_i}$. Since $\{t_{ai}\}_{i \geq 0}$ is a geometric progression with common ratio k, and

$$t_0 < \frac{\sqrt{t_i}}{k^{2s+1}} < \frac{t_{i-1}^\alpha}{k^{2s+1}}$$

by (2.9), at least $2s$ of the t_j's satisfy

$$t_j \in (t_{i-1}^\alpha / k^{2s+1}, t_{i-1}^\alpha). \tag{2.10}$$

Indeed, let j_1 be the largest with $t_{j_1} \leq t_{i-1}^\alpha / k^{2s+1}$ and j_2 be the smallest with $t_{j_2} \geq t_{i-1}^\alpha$. Then

$$\frac{t_{j_2}}{t_{j_1}} \geq \frac{t_{i-1}^\alpha}{t_{i-1}^\alpha / k^{2s+1}} = k^{2s+1},$$

thus $j_2 \geq j_1 + (2s+1)a \geq j_1 + 2s + 1$. Hence

$$t_{j_1+1}, \ldots, t_{j_1+2s} \in (t_{i-1}^\alpha / k^{2s+1}, t_{i-1}^\alpha).$$

For each t_j satisfying (2.10) with j even (there are at least s of them), we claim that

$$n \in I_j + k * I_{i-1-a} = [t_j + t_{i-1}, t_{j+1} + t_i - k).$$

By (2.10) and (2.7), we have

$$t_j + t_{i-1} < t_{i-1}^{\alpha} + t_{i-1} < t_i - t_{i-1} + t_{i-1} = t_i \le n.$$

On the other hand, by (2.8), (2.10) and the assumption on n, we have

$$t_{j+1} + t_i - k \ge t_{j+1} + n - \sqrt{t_i} - k > t_j + n - \frac{t_{i-1}^{\alpha}}{k^{2s+1}} > n,$$

hence the claim follows.

For each t_j satisfying (2.10) with j even, the equation $x + ky = n$ has a solution with $x \in I_j$ and $y \in I_{i-1-a}$. Noting that j and $i - 1 - a$ are both even, we have $x, y \in A$, thus $r_{1,k}(A, n) \ge s$.

Case 2: i is even and $n - t_i > \sqrt{t_i}$. Since $\sqrt{t_i}/k > k^{2s} t_0$ by (2.9), it follows that at least $2s$ of the t_j's satisfy

$$t_j \in [t_0, \sqrt{t_i}/k). \tag{2.11}$$

For each such t_j with j even (there are at least s of them), we claim that

$$n \in I_i + k * I_j = [t_i + kt_j, t_{i+1} + kt_{j+1} - k).$$

It is clear that

$$t_{i+1} + kt_{j+1} - k \ge t_{i+1} > n.$$

On the other hand, by (2.11) and the assumption on n,

$$t_i + kt_j < t_i + \sqrt{t_i} < n,$$

hence the claim follows.

For each t_j satisfying (2.11) with j even, the equation $x + ky = n$ has a solution with $x \in I_i$ and $j \in I_j$. Noting that i and j are both even, we have $x, y \in A$, thus $r_{1,k}(A, n) \ge s$.

Case 3: i is odd and $n - t_i \le \sqrt{t_i}$. By (2.7)–(2.9), we have

$$t_i - t_{i-1} > t_{i-1}^{\alpha} > k^{2s+1}(\sqrt{t_i} + k) > kt_0,$$

hence at least $2s$ of the t_j's satisfy

$$t_j \in \left(\frac{\sqrt{t_i} + k}{k}, \frac{t_i - t_{i-1}}{k} \right). \tag{2.12}$$

For each such t_j with j odd (there are at least s of them), we claim that

$$n \in I_{i-1} + k * I_{j-1} = [t_{i-1} + kt_{j-1}, t_i + kt_j - k).$$

It is clear, by (2.12), that

$$t_{i-1} + kt_{j-1} < t_{i-1} + kt_j < t_{i-1} + (t_i - t_{i-1}) = t_i \le n.$$

On the other hand, by (2.12) and the assumption on n,

$$t_i + kt_j - k > t_i + (\sqrt{t_i} + k) - k = t_i + \sqrt{t_i} \geq n,$$

hence the claim follows.

For each t_j satisfying (2.12) with j odd, the equation $x + ky = n$ has a solution with $x \in I_{i-1}$ and $j \in I_{j-1}$. Noting that $i - 1$ and $j - 1$ are both even, we have $x, y \in A$, thus $r_{1,k}(A, n) \geq s$.

Case 4: i is odd and $n - t_i > \sqrt{t_i}$. Since $\sqrt{t_i} > k^{2s+1} t_0$ by (2.9), at least $2s$ of the t_j's satisfy

$$t_j \in [t_0, \sqrt{t_i}). \tag{2.13}$$

For each such t_j with j even (there are at least s of them), we claim that

$$n \in I_j + k * I_{i-a} = [t_j + t_i, t_{j+1} + t_{i+1} - k).$$

It is clear that

$$t_{j+1} + t_{i+1} - k \geq t_{i+1} > n.$$

On the other hand, by (2.13) and the assumption on n,

$$t_j + t_i < t_i + \sqrt{t_i} < n,$$

hence the claim follows.

For each t_j satisfying (2.13) with j even, the equation $x + ky = n$ has a solution with $x \in I_j$ and $y \in I_{i-a}$. Noting that j and $i - a$ are both even, we have $x, y \in A$, thus $r_{1,k}(A, n) \geq s$.

This completes the proof of Theorem 1.3. ∎

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