INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

DISSERTATIONES MATHEMATICAE

512

E. M. ROJAS

Boundary value problems and singular integral equations on Banach function spaces

WARSZAWA 2015

E. M. Rojas
Departamento de Matemáticas
Universidad Nacional de Colombia
Carrera 30 No. 45-03
11321, Bogotá, Colombia
E-mail: emrojass@unal.edu.co

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound in Poland by HermanDruK, Warszawa
Nakład 200 egz.

Abstracted/Indexed in: Mathematical Reviews, Zentralblatt MATH, Science Citation Index Expanded, Journal Citation Reports/Science Edition, Google Science, Scopus, EBSCO Discovery Service.

Available online at **http://journals.impan.pl**

DOI: 10.4064/dm742-12-2015

ISSN 0012-3862

Contents

Abstract

We study the solvability and Fredholmness of binomial boundary value problems for analytic functions represented by integrals of Cauchy type with density on abstract nonstandard Banach function spaces, assuming continuous, piecewise continuous and essentially bounded factorizable functions as coefficients. The representation of the solutions of those problems allows us to describe the explicit form of the solutions of the associated singular integral equations in each case. The solvability and explicit representation of the solutions of a class of singular integral equations with Carleman shifts is also considered.

Acknowledgements. The author is grateful to the referee whose comments and suggestions led to an improvement of the paper.

2010 *Mathematics Subject Classification*: Primary 30E25, 46E30; Secondary 42B20, 47B38.

Key words and phrases: Riemann boundary problem, singular integral operator, singular integral equation, Banach function space, Lebesgue space, variable exponent.

Received 25 March 2015; revised 6 July 2015.

Published online 14 December 2015.

1. Introduction

The first formulation of linear boundary value problems (also called Riemann problems or two-term boundary value problems) for analytic functions was due to Riemann [44] in 1853, while the theory of singular integral equations with principal value integrals was originated almost directly after the development of the classical theory of integral equations by E. Fredholm in 1903. Singular integral equations were investigated by D. Hilbert [18] and H. Poincaré [43] while studying two different problems: Hilbert investigated some boundary value problems for analytic functions and Poincaré studied the theory of tides. J. Plemelj [42] applied further the Cauchy singular integral as a mathematical device for solving boundary value problems.

The complete solution of the Riemann problem was first given by F. D. Gakhov [13, 14] and N. I. Muskhelishvili [39, 40]. Subsequently, several authors have extensively studied boundary value problems and singular integral equations on classical spaces of integrable functions. Much of this groundwork is collected in [9, 15, 16, 24, 36, 37, 38, 47, 49] and the references therein.

In the last two decades, several studies have been devoted to singular integral equations and boundary value problems in the general setting of variable exponent Lebesgue spaces $L^{p(\cdot)}$, which is one of the prototypical examples of nonstandard Banach function spaces. The basic properties as boundedness, invertibility and the Fredholm property of singular integral operators over diverse domains and curves in $L^{p(\cdot)}$ (including weighted versions) as well as the solvability theory of singular integral equations were established by various authors: we refer to A. Yu. Karlovich [21], A. Yu. Karlovich and A. K. Lerner [22], A. Yu. Karlovich and I. M. Spitkovsky [23], V. Kokilashvili, N. Samko and S. Samko [31] and V. Kokilashvili and S. Samko [32, 33] and their references.

The Riemann boundary value problem for analytic functions within the framework of $L^{p(\cdot)}$ spaces was first explored by V. Kokilashvili, V. Paatashvili and S. Samko [30]; the Haseman problem, the Riemann–Hilbert problem and the Dirichlet problem were considered by V. Kokilashvili and V. Paatashvili [25–28, 41]. The recent monograph [29] synthesizes numerous developments in this direction.

In the more abstract scheme of Banach function spaces, singular integral operators and their corresponding equations, with coefficients belonging to different classes of functions, have been studied by A. Yu. Karlovich [20] (in the case of reflexive rearragement-invariant spaces), A. Yu. Karlovich and A. K. Lerner [22] and V. Kokilashvili and S. Samko [32, 33]. However, the solvability theory of singular integral equations is far from being complete in this general framework.

The aim of this paper is to extend the study of solvability and Fredholmness of two-term boundary value problems for analytic functions represented by integrals of Cauchy type on Lebesgue spaces to the case of density on Banach function spaces over Lyapunov curves assuming some conditions (see (2.2)–(2.6) below). We will consider continuous, piecewise continuous and essentially bounded functions as coefficients. Since singular integral equations are related to boundary value problems, we will use the representation of their solutions to describe the explicit form of solutions of the associate equations in each case. For the case of essentially bounded functions we are going to introduce the notion of Wiener–Hopf factorization, and we will establish Simonenko's Fredholm criterion for singular integral operators with factorizable functions in this context. Moreover, the solvability and explicit representation of solutions of a class of singular integral equations with shift will be considered.

The paper is organized as follows. Chapter 2 contains the definitions and basic facts about Banach function spaces; here we introduce the notion of factorization in $\mathcal{X}(\Gamma)$. Chapter 3 is devoted to the study of solvability and representations of solutions of Riemann problems with continuous, piecewise continuous and essentially bounded factorizable coefficients.

In Chapter 4 we prove Simonenko's Fredholm criterion for singular integral equations with essentially bounded factorizable coefficients. As a consequence, we establish the effective solution of the corresponding singular integral equations through the lateral inverses of the operator.

In Chapter 5, by using the Fredholmness criteria for singular integral equations with continuous and piecewise continuous functions proved by V. Kokilashvili and S. Samko [32], Simonenko's Fredholmness criterion proved in Chapter 4 and the representations of solutions of boundary value problems considered in Chapter 3, we prove the Fredholm property for the above mentioned boundary value problems, and we will describe the form of solutions of equation (4.2) for each class of essentially bounded functions under study.

Chapter 6 deals with a class of singular integral equations with Carleman shift. For the shift function we assume both behaviors: preserving or reversing the orientation of the curve Γ. The existence and uniqueness of solutions will be established by projection methods, so that we will be able to transform the initial equation into a system of equations which can be solved by means of a Riemann boundary value problem technique. Thus, using the results of Chapter 3 and the Sokhotski–Plemelj formulas we will give an explicit form of the solutions. Furthermore, with the Fredholmness criteria mentioned above and the projection methods, which in this case take the form of a nonexplicit equivalence relation between operators, the Fredholm property for the associated singular integral operator with shift is proved.

In Chapter 7 we show that all the assumptions imposed on the abstract Banach function space $\mathcal{X}(\Gamma)$ are, in fact, well-known results in variable exponent Lebesgue spaces, therefore our results are valid in those spaces.

2. Definitions and preliminary statements

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le \ell\}$ be an oriented rectifiable closed simple Lyapunov curve in the complex plane $\mathbb{C}$ with arc-length s. We denote by D^+ and D^- the bounded and unbounded components of $\mathbb{C} \setminus \Gamma$ respectively. We will assume that $0 \in D^+$ and, as usual, Γ has the natural counterclockwise orientation.

Recall that a simple oriented curve Γ in the complex plane is called a *Lyapunov curve* if the tangent to Γ at each point t exists and forms an angle $\theta(t)$ with the real axis which satisfies the Hölder condition:

$$|\theta(t_1) - \theta(t_2)| \le A|t_1 - t_2|^\mu, \qquad A > 0,\ 0 < \mu < 1.$$

We denote by $\mathcal{R}(\Gamma)$ the Banach algebra of *rational functions without poles* on Γ which, as is well-known, can be decomposed as $\mathcal{R}(\Gamma) = \mathcal{R}_+(\Gamma) \dotplus \mathcal{R}_-(\Gamma)$, where $\mathcal{R}_\pm(\Gamma)$ denote the sets of all *functions on* $\mathcal{R}(\Gamma)$ *with poles outside of* $D^\pm$. The *continuous functions*, the *smooth functions* and the *essentially bounded measurable functions* on Γ, endowed with the essential supremum norm $\|\cdot\|_\infty$, are denoted by $C(\Gamma)$, $C^\infty(\Gamma)$ and $L^\infty(\Gamma)$ respectively. For $p \in [1,\infty)$, $L^p(\Gamma)$ denotes the usual Banach space of all *Lebesgue measurable complex-valued functions* on Γ with absolutely integrable pth power.

The *Cauchy singular integral operator* along the curve Γ of finite length ℓ is defined by

$$(Sf)(t) := \frac{1}{\pi i}\,\text{p.v.}\int_\Gamma \frac{f(\tau)}{\tau - t}\,d\tau, \qquad t = t(s),\ 0 \le s \le \ell,$$

where the integral is understood in the sense of principal value.

Let (Ω, μ) be a nonatomic σ-finite measure space, i.e., a measure space with nonatomic σ-finite measure μ given on a σ-algebra of subsets of Ω. The set of all Lebesgue measurable complex-valued functions on Ω is denoted by $\mathcal{M}$. Let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0,\infty]$. The characteristic function of a measurable set $E \subset \Omega$ is denoted by χ_E, and the Lebesgue measure of E is denoted by $|E|$.

Definition 2.1 ([1, Ch. 1, Definition 1.1]). A mapping $\rho : \mathcal{M}^+ \to [0,\infty]$ is called a *Banach function norm* if, for all functions f, g, f_n ($n \in \mathbb{N}$) in $\mathcal{M}^+$, for all constants $a \ge 0$, and for all measurable subsets E of Ω, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e. $\rho(af) = a\rho(f)$, $\rho(f+g) \le \rho(f) + \rho(g)$,
(A2) $0 \le g \le f$ a.e. $\Rightarrow \rho(g) \le \rho(f)$ (the *lattice property*),
(A3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the *Fatou property*),
(A4) $|E| < \infty \Rightarrow \rho(\chi_E) < \infty$,
(A5) $|E| < \infty \Rightarrow \int_E f\,d\mu \le C_E\rho(f)$

with $C_E \in (0,\infty)$ which may depend on E and ρ but is independent of f.

Here, functions differing only on a set of measure zero are identified. The set $\mathcal{X}(\Omega)$ of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in \mathcal{X}(\Omega)$, the norm of f is defined by

$$\|f\|_{\mathcal{X}(\Omega)} := \rho(|f|).$$

The set $\mathcal{X}(\Omega)$ under the natural linear space operations and with this norm becomes a Banach space (see [1, Ch. 1, Theorems 1.4 and 1.6]).

If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{\int_\Omega f(x)g(x)\,d\mu : f \in \mathcal{M}^+,\ \rho(f) \le 1\right\}, \quad g \in \mathcal{M}^+.$$

It is a Banach function norm itself [1, Ch. 1, Theorem 2.2]. The Banach function space $\mathcal{X}'(\Omega)$ determined by the Banach function norm ρ' is called the *associate space* (*Köthe dual*) of $\mathcal{X}(\Omega)$.

LEMMA 2.1 (Hölder's inequality, see [1, Ch. 1, Theorem 2.4]). *Let $\mathcal{X}(\Omega)$ be a Banach function space with associate space $\mathcal{X}'(\Omega)$. If $f \in \mathcal{X}(\Omega)$ and $g \in \mathcal{X}'(\Omega)$, then fg is summable and*

$$\int_\Omega |fg|\,d\mu \le \|f\|_{\mathcal{X}(\Omega)}\|g\|_{\mathcal{X}'(\Omega)}. \tag{2.1}$$

Let Σ denote the collection of all subsets of Ω of finite measure, where any two such subsets which differ by a set of μ-measure zero are identified. With the distance

$$d(E,F) := \int_\Omega |\chi_E - \chi_F|\,d\mu, \qquad E, F \in \Sigma,$$

(Σ, d) is a complete metric space. A measure μ is said to be separable if the corresponding metric space (Σ, d) is separable.

LEMMA 2.2 ([1, Ch. 1, Corollaries 4.3–4.5]). *Let μ be a separable measure.*

(a) *A Banach function space $\mathcal{X}(\Omega)$ is separable if and only if its associate space $\mathcal{X}'(\Omega)$ is canonically isometrically isomorphic to the dual space $\mathcal{X}^*(\Omega)$ of $\mathcal{X}(\Omega)$.*
(b) *A Banach function space $\mathcal{X}(\Omega)$ is reflexive if and only if both $\mathcal{X}(\Omega)$ and its associate space $\mathcal{X}'(\Omega)$ are separable.*

In this paper we will consider $\mathcal{X}(\Gamma)$ to be a Banach function space over a closed simple Lyapunov curve Γ satisfying the following conditions:

$$C(\Gamma) \subset \mathcal{X}(\Gamma) \subset L^1(\Gamma), \tag{2.2}$$

$$\|af\|_{\mathcal{X}(\Gamma)} \le \sup_{t\in\Gamma}|a(t)| \cdot \|f\|_{\mathcal{X}(\Gamma)} \quad \text{for } a \in L^\infty(\Gamma), \tag{2.3}$$

$$\text{the operator } S \text{ is bounded in } \mathcal{X}(\Gamma), \tag{2.4}$$

$$\mathcal{X}(\Gamma) \text{ is reflexive}, \tag{2.5}$$

$$C^\infty(\Gamma) \text{ is dense in } \mathcal{X}(\Gamma). \tag{2.6}$$

The boundedness of the adjoint operator S^* in the dual space $\mathcal{X}^*(\Gamma)$ is given in the following result.

Lemma 2.3. *Suppose the operator S is bounded in the space $\mathcal{X}(\Gamma)$. Then its adjoint S^* is connected with the operator S in the dual space $\mathcal{X}^*(\Gamma)$ via the equality*

$$S^* = -HSH$$

where H is the operator defined in $\mathcal{X}^*(\Gamma)$ by $(H\varphi)(t) := \overline{h(t)\varphi(t)}$, where $h(t) = \exp(i\Theta(t))$ and $\Theta(t)$ is the angle of inclination of Γ at t to the positive direction of the real axis.

Proof. Since $\mathcal{X}(\Gamma)$ is reflexive, Lemma 2.2 shows that $\mathcal{X}(\Gamma)$ and $\mathcal{X}'(\Gamma)$ are separable. Furthermore, $\mathcal{X}^*(\Gamma)$ can be identified with the associate space $\mathcal{X}'(\Gamma)$ (see also [20, Lemma 1.2]). That is, the general form of a linear functional on $\mathcal{X}(\Gamma)$ is given by

$$f(u) = (u, v) = \int_\Gamma u(t)\overline{v(t)}\,|dt|, \qquad \text{where } u \in \mathcal{X}(\Gamma),\ v \in \mathcal{X}'(\Gamma).$$

Let $\phi, \psi \in \mathcal{R}(\Gamma)$. Then from Cauchy's Theorem it follows that

$$\int_\Gamma \psi(t)(S\phi)(t)\,dt = -\int_\Gamma \phi(t)(S\psi)(t)\,dt.$$

Hence,

$$\begin{aligned}(\phi, S^*\psi) = (S\phi, \psi) &= \int_\Gamma (S\phi)(t)\overline{\psi(t)}\,|dt| = \int_\Gamma (S\phi)(t)\overline{\psi(t)h(t)}\,dt \\ &= -\int_\Gamma \phi(t)(S\overline{h\psi})(t)\,dt = -\int_\Gamma \phi(t)\overline{(HSH\psi)(t)}\,|dt| = -(\phi, HSH\psi). \qquad (2.7)\end{aligned}$$

Since $\mathcal{X}(\Gamma)$ is reflexive, $\mathcal{X}(\Gamma)$ and $\mathcal{X}'(\Gamma) = \mathcal{X}^*(\Gamma)$ are separable and by (2.6), $\mathcal{R}(\Gamma)$ is dense in $\mathcal{X}(\Gamma)$ and $\mathcal{X}'(\Gamma)$, from (2.4) and (2.7) we conclude that $S^* = -HSH$. ∎

On the other hand, from [38, Ch. I, Corollary 1.2] we have $(S^2 r)(t) = r(t)$ for all $r \in \mathcal{R}(\Gamma)$. Since $\mathcal{R}(\Gamma)$ is dense in $C^\infty(\Gamma)$ and, by assumption (2.6), $C^\infty(\Gamma)$ is dense in $\mathcal{X}(\Gamma)$, we conclude that $S^2 = I$ in $\mathcal{X}(\Gamma)$. Hence, from (2.4) and (2.6), the operators

$$P_\pm := \tfrac{1}{2}(I \pm S)$$

define bounded complementary projections in the space $\mathcal{X}(\Gamma)$. Thus, we define the subspaces

$$\begin{aligned}&\mathcal{X}_+(\Gamma) := P_+\mathcal{X}(\Gamma), \qquad \mathring{\mathcal{X}}_-(\Gamma) := P_-\mathcal{X}(\Gamma),\\ &\mathcal{X}_-(\Gamma) := \mathring{\mathcal{X}}_-(\Gamma) \dotplus \mathbb{C}.\end{aligned}$$

Set

$$\begin{aligned}L^1_+(\Gamma) :&= \left\{ f \in L^1(\Gamma) : \int_\Gamma f(\tau)\tau^n\,d\tau = 0 \text{ for } n \geq 0 \right\},\\ \mathring{L}^1_-(\Gamma) :&= \left\{ f \in L^1(\Gamma) : \int_\Gamma f(\tau)\tau^{-n}\,d\tau = 0 \text{ for } n \geq 1 \right\},\\ L^1_-(\Gamma) :&= \mathring{L}^1_-(\Gamma) \dotplus \mathbb{C}.\end{aligned}$$

Lemma 2.4. *Let Γ be a Lyapunov curve and let $\mathcal{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.6). *Then:*

(a) $\mathcal{X}_+(\Gamma) = L^1_+(\Gamma) \cap \mathcal{X}(\Gamma)$, $\mathring{\mathcal{X}}_-(\Gamma) = \mathring{L}^1_-(\Gamma) \cap \mathcal{X}(\Gamma)$, $\mathcal{X}_-(\Gamma) = L^1_-(\Gamma) \cap \mathcal{X}(\Gamma)$.

(b) *If $f \in \mathcal{X}_\pm(\Gamma)$, $g \in \mathcal{X}'_\pm(\Gamma)$, then $fg \in L^1_\pm(\Gamma)$. Moreover, if $f \in \mathcal{X}_-(\Gamma)$, $g \in \mathring{\mathcal{X}}'_-\Gamma)$ or $f \in \mathring{\mathcal{X}}_-(\Gamma)$, $g \in \mathcal{X}'_-(\Gamma)$, then $fg \in \mathring{L}^1_-(\Gamma)$.*

Proof. (a) follows from [38, Ch. II, Theorem 1.1] and assumptions (2.2) and (2.4), taking into consideration the decomposition $\mathcal{R}(\Gamma) = \mathcal{R}_+(\Gamma) \dotplus \mathcal{R}_-(\Gamma)$. The proof of (b) is analogous to the proof of [2, Lemma 6.11], from the denseness of $\mathcal{R}(\Gamma)$ in $C(\Gamma)$, assumption (2.6) and the Hölder inequality. ■

Now, we can introduce a factorization of an invertible essentially bounded measurable function a in Γ ($a \in \mathcal{G}(L^\infty(\Gamma))$), where $a \in \mathcal{G}(L^\infty(\Gamma))$ if $\operatorname{ess\,inf}_{t\in\Gamma} |a(t)| > 0$. Let Γ be a Lyapunov curve and $\mathcal{X}(\Gamma)$ be a BFS satisfying (2.2)–(2.6). We say that a function $a \in \mathcal{G}(L^\infty(\Gamma))$ admits a *factorization in* $\mathcal{X}(\Gamma)$ if it can be written in the form

$$a(t) = a_-(t)t^{\aleph}a_+(t), \qquad \text{a.e. on } \Gamma, \tag{2.8}$$

where $\aleph \in \mathbb{Z}$,

(i) $a_- \in \mathcal{X}_-(\Gamma)$, $a_-^{-1} \in \mathcal{X}'_-(\Gamma)$, $a_+ \in \mathcal{X}'_+(\Gamma)$, $a_+^{-1} \in \mathcal{X}_+(\Gamma)$,
(ii) the operator $a_+^{-1}Sa_+I$ is bounded in $\mathcal{X}(\Gamma)$.

The integer $\aleph$ will be called the *index* of the function a and denoted by $\operatorname{ind} a$. We can prove that $\aleph$ is uniquely determined.

3. Binomial boundary value problems on $\mathcal{X}(\Gamma)$

In this chapter we study the Riemann problem

$$\Psi^+(t) = G(t)\Psi^-(t) + g(t), \tag{3.1}$$

or equivalently, the associated problem

$$\Psi^-(t) + G(t)\Psi^+(t) = g(t) \tag{3.2}$$

for analytic functions represented by integrals of Cauchy type with density on the space $\mathcal{X}(\Gamma)$, assuming the function G is continuous, piecewise continuous or $G \in L^\infty(\Gamma)$ admitting a factorization (2.8).

The problem is stated in the following way: find functions $\Psi^+(t)$ and $\Psi^-(t)$ analytic in D^+ and D^- respectively, vanishing at infinity, satisfying condition (3.1) or (3.2) on their boundary values on the contour Γ.

The pair $\{\Psi^-, \Psi^+\}$ is referred to as a solution of problem (3.1) (or (3.2)). The space of all solutions of the homogeneous problem is called its *kernel*, and the space of functions g for which the inhomogeneous problem is solvable is said to be its *image*. The dimension α of the first of them and the codimension (in $\mathcal{X}(\Gamma)$) β of the closure of the second are called the *defect numbers* of the boundary value problem. If at least one of the numbers α and β is finite, the difference $\alpha - \beta$ is referred to as the *index* of the problem. Problem (3.1), or (3.2), is said to be *normally solvable* if its image is closed; it is called *Fredholm* if it is normally solvable and has finite index.

Let Ψ be an analytic function of Cauchy integral type with nontangential limit $\varphi \in \mathcal{X}(\Gamma)$,

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - z}\, d\tau, \qquad z \notin \Gamma,$$

with boundary values $\Psi^+(t)$ (resp. $\Psi^-(t)$) as $z \to t$, $t \in \Gamma$, $z \in D^+$ (resp. $t \in \Gamma$, $z \in D^-$). According to the Sokhotski–Plemelj formulas, these boundary values are expressed by

$$\Psi^+(t) = \tfrac{1}{2}[(I\varphi)(t) + (S\varphi)(t)], \qquad \Psi^-(t) = \tfrac{1}{2}[(-I\varphi)(t) + (S\varphi)(t)].$$

Thus,

$$\Psi^+(t) - \Psi^-(t) = \varphi(t), \qquad \Psi^+(t) + \Psi^-(t) = (S\varphi)(t).$$

For a simply connected domain D, bounded by a rectifiable curve Γ, we denote by $E^\delta(D)$, $\delta > 0$, the *Smirnov class* of functions $\Psi(z)$ in D for which

$$\sup_r \int_{\Gamma_r} |\Psi(z)|^\delta\, |dz| < \infty,$$

where Γ_r is the image of $\gamma_r = \{z : |z| = r\}$ under the conformal mapping of $U = \{z : |z| < 1\}$ onto D. When D is an infinite domain, the conformal mapping means the one which transforms 0 into infinity. A function $\Psi \in E^\delta(D)$ has angular boundary values almost everywhere on Γ and the boundary function belongs to $L^\delta(\Gamma)$.

On the other hand, let us introduce the notation

$$\mathcal{E}(D) = \left\{\Psi(z) : \Psi(z) = (K\varphi)(z) = \frac{1}{2\pi}\int_\Gamma \frac{\varphi(\tau)}{\tau - z}\, d\tau,\ z \notin \Gamma, \text{ with } \varphi \in \mathcal{X}(\Gamma)\right\}.$$

As is known, $E^1(D)$ coincides with the class of analytic functions represented by Cauchy integrals. So, for the function $\Psi(z)$ which is analytic on the plane cut along the closed curve Γ and belongs to $E^1(D^\pm)$, we have

$$\Psi(z) = K(\Psi^+ - \Psi^-)(z).$$

This, together with the inclusion $\mathcal{X}(\Gamma) \subset L^1(\Gamma)$, given by assumption (2.2), allows us to define the following subsets of $\mathcal{E}(D)$:

$$\mathcal{E}^1(D^\pm) = \{\varphi \in E^1(D^\pm) : \varphi \text{ has definite limiting values on } \mathcal{X}(\Gamma)\}.$$

Denote by $\mathcal{E}^1_+(D^+)$ the set of *analytic functions on* D^+ *with definite limiting on* $\mathcal{X}_+(\Gamma)$. $\mathring{\mathcal{E}}^1_-(D^-)$ is referred to as the set of *analytic functions on* D^- *vanishing at infinity with boundary value on* $\mathring{\mathcal{X}}_-(\Gamma)$. Finally, we set $\mathcal{L}^1(\Gamma) := L^1_+(\Gamma) \dotplus \mathring{L}^1_-(\Gamma)$.

Since the boundary value problem is posed in $\mathcal{X}(\Gamma)$, we are looking for solutions $\{\varphi^+, \varphi^-\}$ represented by integrals of Cauchy type, i.e.,

$$\varphi^\pm(z) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi^\pm(\zeta)}{\zeta - z}\, d\zeta \quad \text{in } D^\pm,$$

with nontangential limits a.e. on $\mathcal{X}(\Gamma)$. Then, the solutions are such that $\varphi^+ \in \mathcal{E}^1(D^+)$ and $\varphi^- \in \mathcal{E}^1(D^-)$ for the boundary value problem with continuous coefficient G. In the case of G essentially bounded and admitting a factorization (2.8), the solutions $\varphi^\pm$ represented by integrals of Cauchy type have nontangential limits a.e. on $\mathcal{X}_+(\Gamma)$ and $\mathring{\mathcal{X}}_-(\Gamma)$ respectively; in this case the solutions are such that $\varphi^+ \in \mathcal{E}^1_+(D^+)$ and $\varphi^- \in \mathring{\mathcal{E}}^1_-(D^-)$.

3.1. The two-term boundary value problem with continuous coefficients. In this section we study the solvability of problem (3.1) for G a nonvanishing continuous function on Γ, with index $\aleph = \frac{1}{2\pi}[\arg G(t)]_\Gamma$ and with $g \in \mathcal{X}(\Gamma)$.

We first establish the following auxiliary result:

PROPOSITION 3.1.1. *Let* Γ *be a closed curve and* $\mathcal{X}(\Gamma)$ *a BFS satisfying* (2.2)–(2.4) *and* (2.6). *Assume that* $z_0 \in D^+$. *Then there exists an integer* $k \geq 0$ *such that*

$$\exp\{(K\varphi)(z)\} =: X(z) \in \mathcal{E}^1(D^+) \quad \textit{and} \quad \frac{X(z) - 1}{(z - z_0)^k} \in \mathcal{E}^1(D^-).$$

Proof. Let $\delta > 0$ and Γ_r be the image of $\gamma_r = \{z : |z| = r\}$, $r < 1$, under the conformal mapping of $U = \{z : |z| < 1\}$ onto D^+. We have

$$\int_{\Gamma_r} |X(z)|^\delta\, |dz| \leq \int_{\Gamma_r} \sum_{n=0}^\infty \frac{1}{n!}|\delta\Psi(z)|^n\, |dz|, \quad \text{where} \quad \Psi(z) = \frac{1}{2\pi}\int_\Gamma \frac{\varphi(\tau)}{\tau - z}\, d\tau. \tag{3.3}$$

Since $\Psi(z) \in E^1(D^+)$, it is known that

$$\int_{\Gamma_r} |\Psi(z)|^n \, |dz| \le \int_\Gamma |\Psi^+(t)|^n \, |dt|.$$

From (3.3) we obtain

$$\begin{aligned}\int_{\Gamma_r} |X(z)|^\delta \, |dz| &\le \sum_{n=0}^\infty \int_\Gamma |\delta\Psi^+(t)|^n \, |dt| \le \sum_{n=0}^\infty \frac{1}{n!} \int_\Gamma \left| \frac{\delta\varphi(t)}{2} + \frac{\delta}{2}(S\varphi)(t) \right|^n |dt| \\ &\le \sum_{n=0}^\infty \frac{1}{n!} \int_\Gamma |\delta\varphi(t)|^n \, |dt| + \sum_{n=0}^\infty \frac{1}{n!} \int_\Gamma |\delta(S\varphi)(t)|^n \, |dt| \\ &\le \ell e^{\delta M} + \sum_{n=0}^\infty \frac{1}{n!} \int_\Gamma |\delta(S\varphi)(t)|^n \, |dt|,\end{aligned}$$

where $M = \sup_{t\in\Gamma} |\varphi(t)|$. It remains to show that the series $\sum_{n=0}^\infty \frac{1}{n!} \int_\Gamma |\delta(S\varphi)(t)|^n \, |dt|$ converges. Since S is an involution, we have

$$(S\varphi)^n(t) = \begin{cases} (S\varphi)(t), & n \text{ odd}, \\ \varphi(t), & n \text{ even}. \end{cases}$$

For n odd, from the Hölder inequality, we have

$$\sum_{\substack{n=1 \\ n\,\text{odd}}}^\infty \frac{1}{n!}\delta^n \int_\Gamma |(S\varphi)(t)| \, |dt| \le \sum_{n=0}^\infty \frac{1}{n!}\delta^n \|S\|_{\mathcal{X}(\Gamma)} \|1\|_{\mathcal{X}'(\Gamma)} = \ell \|S\|_{\mathcal{X}(\Gamma)} \sum_{n=0}^\infty \frac{\delta^n}{n!}.$$

The last series converges if $\delta \le 1$. The case of n even was shown above. This proves that $X(z) \in E^\delta(D^+)$ when $\delta \le 1$.

In the case of D^- it is necessary to consider two cases: $0 < r < r_0$ and $r_0 < r < 1$ for some fixed r_0. The needed inequalities are obtained by choosing $k > [1/\delta]$ and proceeding as before.

Now, notice that

$$\int_\Gamma |\Psi^\pm(t)| \, |dt| = \int_\Gamma |e^{\pm\delta\varphi(t)/2}| \, |e^{\delta/2}(S\varphi)(t)| \, |dt| \le e^{\delta M/2} \sum_{n=0}^\infty \frac{1}{n!} \int_\Gamma \left| \frac{\delta}{2}(S\varphi)(t) \right|^n |dt|.$$

As before, we can prove that this series converges if $\delta \le 2$.

Finally, we apply the following Smirnov Theorem: *If $\Psi \in E^{\gamma_1}(D)$ and $\Psi^+ \in L^{\gamma_2}(\Gamma)$ where $\gamma_2 > \gamma_1$, then $\Psi \in E^{\gamma_2}(D)$.* In our case, $X(z) \in E^\delta(D^+)$, $0 < \delta < 1$, and assumption (2.2) gives $\Psi^+ \in L^1(\Gamma)$, so $X(z) \in E^1(D^+)$ and $\frac{X(z)-1}{(z-z_0)^k} \in E^1(D^-)$. Even more, we are considering analytic functions with nontangential limits in $\mathcal{X}(\Gamma)$, more precisely, $\Psi^+ \in \mathcal{X}(\Gamma)$; then we conclude that $X(z) \in \mathcal{E}^1(D^+)$ and $\frac{X(z)-1}{(z-z_0)^k} \in \mathcal{E}^1(D^-)$. ∎

THEOREM 3.1.2. *Let Γ be a Lyapunov curve and $\mathcal{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.4) *and* (2.6). *Assume $G \in C(\Gamma)$ and $G(t) \ne 0$ for $t \in \Gamma$. Then:*

(a) *For $\aleph \ge 0$, problem* (3.1) *is unconditionally solvable in the class $\mathcal{E}(D)$, and all its solutions are given by*

$$\Psi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z} + X(z)\rho(z) \tag{3.4}$$

with

$$X(z) = \begin{cases} \exp h(z), & z \in D^+, \\ (z-z_0)^{\aleph} \exp h(z), & z \in D^-,\ z_0 \in D^+, \end{cases}$$

where

$$h(z) = K\big(\ln G(t)(t-z_0)^{\aleph}\big)(z)$$

and ρ is an arbitrary polynomial of degree $\aleph - 1$.

(b) *For $\aleph < 0$, problem* (3.1) *is solvable in this class if and only if*

$$\int_\Gamma \frac{g(\tau)\tau^\kappa}{X^+(\tau)}\, d\tau = 0, \qquad \kappa = 0, \dots, |\aleph| - 1, \tag{3.5}$$

and under these conditions the unique solution is given by (3.4) *with $\rho \equiv 0$.*

Proof. Consider first the case $\aleph = 0$. We choose a rational function $\widetilde{G}(t)$ such that

$$\sup_{t \in \Gamma} \left| \frac{G(t)}{\widetilde{G}(t)} - 1 \right| < \tfrac{1}{2}(1 + \|S\|_{\mathcal{X}(\Gamma)})^{-1}.$$

Clearly, $\operatorname{ind} \widetilde{G} = 0$ and therefore $\widetilde{X}(z) = \exp(K(\ln \widetilde{G}))(z)$ is continuous in $D^\pm$. Since $\Psi(z) = (K(\Psi^+ - \Psi^-))(z)$, we have

$$\left(\frac{\Psi}{\widetilde{X}}\right)^+ + \frac{G}{\widetilde{G}}\left(\frac{\Psi}{\widetilde{X}}\right)^- = \frac{g}{\widetilde{X}^+}. \tag{3.6}$$

Notice that $\Psi/\widetilde{X} \in \mathcal{E}(D)$. In fact, because $\Psi \in E^1(D^\pm)$ and $1/X$ is bounded, we have $\Psi/\widetilde{X} \in E^1(D^\pm)$ and therefore

$$\Psi/\widetilde{X} = K(\Psi^+/\widetilde{X}^+ - \Psi^-/\widetilde{X}^-).$$

From the Sokhotski–Plemelj formulas it follows that $\Psi^+ \in \mathcal{X}(\Gamma)$, and hence $\Psi/\widetilde{X} \in \mathcal{E}(D)$. Let

$$\Psi(z)/\widetilde{X}(z) = (K\psi)(z), \qquad \psi \in \mathcal{X}(\Gamma);$$

then equality (3.6) yields

$$\psi(t) = \left(\frac{G(t)}{\widetilde{G}(t)} - 1\right)\left(-\frac{1}{2}\psi(t) + \frac{1}{2}(S\psi)(t)\right) + \frac{g(t)}{\widetilde{X}(t)}. \tag{3.7}$$

That means the function ψ is a solution of the equation $\psi = K\psi$ in the space $\mathcal{X}(\Gamma)$, where K is a contractive operator. Therefore, equation (3.7) and consequently problem (3.1) has a unique solution in $\mathcal{E}(D)$. Now, we are going to construct that solution.

Let

$$X(z) = \exp(K(\ln G)(z)).$$

Since $\aleph = 0$,

$$\ln G(t) = \ln |G(t)| + i \arg G(t)$$

is a continuous function, and from Proposition 3.1.1,

$$1/X(z) - 1 \in \mathcal{E}^1(D^\pm).$$

If Ψ is a solution of problem (3.1), then $\Psi \in \mathcal{E}(D)$ and therefore $\Psi \in E^1(D^\pm)$, even more $\Psi/X \in \mathcal{E}(D)$. Also,

$$(\Psi/X)^+ - (\Psi/X)^- = g/X^+.$$

Since this problem has a unique solution in $\mathcal{E}(D)$, the function

$$\Psi(z) = X(z)K(g/X^+)(z)$$

is the solution of (3.1) in the class $\mathcal{E}(D)$.

Let now $\aleph > 0$. We choose $z_0 \in D^+$ and rewrite problem (3.1) as

$$\Psi^+(t) = G_1(t)(t-z_0)^{\aleph}\Psi^-(t) + g(t)$$

where $G_1(t) = (t-z_0)^{-\aleph}G(t)$ is a continuous function with index zero. Let

$$F(z) = \begin{cases} \Psi(z), & z \in D^+, \\ (z-z_0)^{\aleph}\Psi(z), & z \in D^-. \end{cases} \tag{3.8}$$

There exists a polynomial $\rho(z)$ of degree $\aleph - 1$ such that

$$\Xi(z) = F(z) - \rho(z) \in E^1(D^-); \tag{3.9}$$

then $\Xi(z) = K(\Xi^+ - \Xi^-)(z)$. But

$$\Xi^+(t) - \Xi^-(t) = F^+(t) - F^-(t) = \Psi(t) - (t-z_0)^{\aleph}\Psi^-(t) \in \mathcal{X}(\Gamma),$$

thus $\Xi \in \mathcal{E}(D)$ and moreover

$$\Xi^+(t) = G_1(t)\Xi^-(t) + g_1(t)$$

where $g_1(t) = g(t) - \rho(t) + G_1(t)\rho(t)$. Since $\operatorname{ind} G_1 = 0$, from the previous part

$$\Xi(z) = X_1(z)K(g_1/X_1^+)(z), \qquad X_1(z) = \exp\bigl(K(\ln G_1)(z)\bigr).$$

On the other hand,

$$K(g_1/X_1^+)(z) = K(g/X_1^+)(z) - \frac{1}{2\pi i}\int_\Gamma \frac{\rho(t)}{X_1^+(t)}\,\frac{dt}{t-z} + \frac{1}{2\pi i}\int_\Gamma \frac{\rho(t)}{X_1^-(t)}\,\frac{dt}{t-z}.$$

But

$$\frac{1}{2\pi i}\int_\Gamma \frac{\rho(t)}{X_1^+(t)}\,\frac{dt}{t-z} = \begin{cases} \rho(z)/X_1(z), & z \in D^+, \\ 0, & z \in D^-, \end{cases}$$

and

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma \frac{\rho(t)}{X_1^-(t)}\,\frac{dt}{t-z} &= \frac{1}{2\pi i}\int_\Gamma \left[\frac{\rho(t)}{X_1^-(t)} - \rho(t)\right]\frac{dt}{t-z} + \frac{1}{2\pi i}\int_\Gamma \frac{\rho(t)}{t-z}\,dt \\ &= \begin{cases} \rho(z), & z \in D^+, \\ -\rho(z)/X_1(z) + \rho(z), & z \in D^-. \end{cases} \end{aligned}$$

Thus,

$$\Xi(z) = X_1(z)K(g_1/X_1^+)(z) = X_1(z)K(g/X_1^+)(z) + X_1(z)\rho(z) - \rho(z).$$

From (3.8) and (3.9) we arrive at (3.4).

It can verified that in the solution provided for problem (3.1) the arbitrary polynomial ρ does not depend on the choice of the point z_0. Finally, for $\aleph < 0$, the function given by (3.8) belongs to $\mathcal{E}(D)$. Moreover, $F^+ = G_1F^- + g$. Hence,

$$F(z) = X_1(z)K(g/X_1^+)(z),$$

and since the Cauchy formula [17, Ch. X, §4, Th. 1] states that for any analytic function ϕ with nontangential limit a.e. on Γ to be representable by an integral of Cauchy type in D it is necessary and sufficient that

$$\int_\Gamma \phi(\zeta)\zeta^n \, d\zeta = 0, \qquad n = 0, 1, \dots,$$

the condition $\Psi(z) = (z - z_0)^{-\aleph} F(z) \in E^1(D^-)$ is fulfilled if and only if conditions (3.5) are satisfied. ∎

3.2. The boundary value problem (3.1) **for piecewise continuous coefficients.** In this section we will study the solvability of the two-term boundary value problem (3.1) with piecewise continuous functions as coefficients. To do so, we first introduce two necessary axioms on $\mathcal{X}(\Gamma)$ as well as some auxiliary results proved in [32].

Axiom 1. *For the space $\mathcal{X}(\Gamma)$ there exist two functions α and β with $0 < \alpha(t), \beta(t) < 1$ such that*

$$|t - t_0|^{\gamma(t_0)} S |t - t_0|^{-\gamma(t_0)} I, \qquad t_0 \in \Gamma,$$

is bounded in the space $\mathcal{X}(\Gamma)$ for all $\gamma(t_0)$ such that

$$-\alpha(t_0) < \gamma(t_0) < 1 - \beta(t_0),$$

and is unbounded in $\mathcal{X}(\Gamma)$ if $\gamma(t_0) \notin (-\alpha(t_0), 1 - \beta(t_0))$.

The functions α and β are called the *index functions* of the space $\mathcal{X}(\Gamma)$.

Axiom 2. *For any $\gamma < 1 - \beta(t_0)$ the embedding $\mathcal{X}(\Gamma, |t - t_0|^\gamma) \subset L^1(\Gamma)$ is valid and $C^\infty(\Gamma)$ is dense in $\mathcal{X}(\Gamma, |t - t_0|^\gamma)$, for any $t_0 \in \Gamma$.*

From Axiom 1 the following result holds.

Lemma 3.2.1. *Suppose $\mathcal{X}(\Gamma)$ satisfies conditions* (2.2)–(2.3), *and $t_1, \dots, t_n \in \Gamma$. Then*

$$\prod_{k=1}^n |t - t_k|^{\gamma_k} \in \mathcal{X}(\Gamma) \qquad \text{for all } \gamma_k > -\alpha_k,\ k = 1, \dots, n.$$

Lemma 3.2.2. *Let $\mathcal{X}(\Gamma)$ be a Banach function space satisfying conditions* (2.2)–(2.3) *and Axioms* 1–2. *Then the space $\mathcal{X}(\Gamma, \varrho)$ for $\varrho(t) = \prod_{k=1}^n |t - t_k|^{\gamma_k}$, $t_1, \dots, t_k \in \Gamma$, satisfies conditions* (2.2)–(2.3) *as well if*

$$-\alpha(t_k) < \gamma_k < 1 - \beta(t_k), \qquad k = 1, \dots, n. \tag{3.10}$$

Let G be a piecewise continuous function on Γ (written $G \in \mathrm{PC}(\Gamma)$) with $\inf_{t\in\Gamma} |G(t)| > 0$, and let $t_1, \dots, t_n$ be the points of discontinuity of G. As usual, we set

$$\gamma(t) = \frac{1}{2\pi i} \ln\left(\frac{G(t-0)}{G(t+0)}\right), \tag{3.11}$$

$$\omega(t) = \prod_{k=1}^n (t - z_0)^{\gamma(t_k)},$$

where $z_0 \in D^+$, the t_k are the discontinuity points of G and the functions $\omega_k(z) = (z - z_0)^{\gamma(t_k)}$ are univalent analytic functions in the complex plane with cut from z_0 to infinity through $t_k \in \Gamma$. The function

$$G_1(t) = G(t)/\omega(t) \tag{3.12}$$

is continuous on Γ independently of the choice of

$$\alpha_k := \Re\gamma(t_k) = \frac{1}{2\pi i} \arg\left(\frac{G(t-0)}{G(t+0)}\right).$$

Consider now the function

$$\varpi(z) = \prod_{k=1}^{m} \varpi_k(z) \quad \text{with} \quad \varpi_k(z) = \begin{cases} (z-t_k)^{\gamma(t_k)}, & z \in D^+, \\ \dfrac{(z-t_k)^{\gamma(t_k)}}{(z-z_0)^{\gamma(t_k)}}, & z \in D^-, \end{cases}$$

where the branch of the function $\left(\frac{z-t_k}{z-z_0}\right)^{\gamma(t_k)}$ is chosen so that it tends to 1 as $z \to \infty$, $z \in D^-$ so that ϖ_k is analytic in $D^\pm$.

Assuming that the curve Γ has at least one-sided tangents at all t_k. For $\beta_k := \Im\gamma(t_k)$, with γ_k chosen as in (3.10), and taking into account Lemmas 3.2.1 and 3.2.2, from the equality

$$\varpi_k(z) = e^{\alpha_k \ln|z-t_k| - \beta_k \arg(z-t_k)} e^{i(\beta_k \ln|z-t_k| + \alpha_k \arg(z-t_k))}$$

we conclude that

$$\varpi(z), 1/\varpi(z) \in \mathcal{E}^1(D^\pm). \tag{3.13}$$

Let $X(z) = \varpi(z)X_1(z)$, where

$$X_1(z) = \exp(K(\ln(G_1(t)))),$$

and introduce a new function

$$\Psi_1(z) = \Psi(z)/\varpi(z). \tag{3.14}$$

If $\Psi \in \mathcal{E}^1(D^\pm)$ and (3.13) holds, according to Smirnov's Theorem we have $\Psi_1 \in \mathcal{E}^1(D^\pm)$. Since $\Psi_1 = K(\Psi_1^+ - \Psi_1^-)$, we get $\Psi_1 \in \mathcal{E}(D)$.

Now, for the function G_1 given in (3.12) and Ψ_1 in (3.14), consider the problem

$$\Psi_1^+(t) = G_1(t)\Psi_1^-(t) + \varpi(t)g(t). \tag{3.15}$$

If we resolve (3.15), we find that all solutions of problem (3.1) in the case $\aleph = \operatorname{ind} G_1(t) \geq 0$ are given by

$$\Psi(z) = \varpi(z)X_1(z)K\left(\frac{g}{\varpi^+ X_1^+}\right)(z) + \varpi(z)X(z)\rho(z) \tag{3.16}$$

where ρ is an arbitrary polynomial of degree $\aleph$.

Notice that from Lemma 3.2.2, the function Ψ in (3.16) is such that $\Psi^\pm \in \mathcal{X}(\Gamma)$. Therefore, (3.16) with an arbitrary polynomial ρ provides solutions of (3.1) in $\mathcal{E}(\Gamma)$. The case of negative index is considered in the standard way.

Thus, we arrive at the following result.

THEOREM 3.2.3. *Let Γ be a Lyapunov curve and $\mathcal{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.4), (2.6) *and Axioms* 1–2. *Let $G \in \mathrm{PC}(\Gamma)$ be such that $\inf_{t\in\Gamma} |G(t)| > 0$ with points of discontinuity $t_1, \dots, t_n$, and suppose that the curve Γ has at least one-sided tangents at the points t_k. Let $\aleph = \operatorname{ind} G_1$, where G_1 is given by* (3.12). *For $\gamma(t_k)$ given in* (3.11) *satisfying* (3.10), *the statement of Theorem* 3.1.2 *holds for problem* (3.1) *if* $X(z)$ *is replaced by* $X_1(z)$ *and formula* (3.4) *is replaced by formula* (3.16).

3.3. The Riemann problem (3.2) **with factorizable essentially measurable coefficients.** Now, we are going to consider the boundary value problem (3.2) with essentially measurable coefficients admitting a factorization in the space $\mathfrak{X}(\Gamma)$.

THEOREM 3.3.1. *Let* Γ *be a Lyapunov curve and let* $\mathfrak{X}(\Gamma)$ *be a BFS satisfying* (2.2)–(2.6). *Suppose that the function* G *admits a factorization* $G(t) = G_-(t)t^{\aleph}G_+(t)$ *in* $\mathfrak{X}(\Gamma)$. *Then*

(a) *Problem* (3.2) *is solvable if and only if*

$$G_-^{-1}g \in \mathcal{L}^1(\Gamma), \qquad \varphi_0^- = G_-P_-G_-^{-1}g \in \mathring{\mathfrak{X}}_-^1(\Gamma), \qquad \varphi_0^+ = G_+^{-1}t^{-\aleph}P_+G_-^{-1}g \in \mathfrak{X}_+(\Gamma). \tag{3.17}$$

(b) *If conditions* (3.17) *are fulfilled, then the general solution of problem* (3.2) *is*

$$\varphi^+ = \varphi_0^+ + G_+^{-1}t^{-\aleph}\rho, \qquad \varphi^- = \varphi_0^- - G_-\rho, \tag{3.18}$$

where ρ *is a polynomial of degree* $\le -\aleph - 1$ *if* $\aleph < 0$, *and is equal to zero if* $\aleph \ge 0$.

Proof. Let $\{\varphi^+, \varphi^-\}$ be a solution of problem (3.2). Substituting the representation (2.8) of G into the boundary condition (3.2) we obtain

$$\varphi^-(t) + G_-(t)t^{\aleph}G_+(t)\varphi^+(t) = g(t),$$
$$f^-(t) + t^{\aleph}f^+(t) = G_-^{-1}(t)g(t), \tag{3.19}$$

where $f^- = G_-^{-1}\varphi^-$ and $f^+ = G_+\varphi^+$. Since $G_-^{-1} \in \mathfrak{X}'_-(\Gamma)$ and $\varphi^- \in \mathring{\mathfrak{X}}_-(\Gamma)$, from the Hölder inequality (2.1) we see that $f^- \in \mathring{L}_-^1(\Gamma)$. Analogously, $f^+ \in L_+^1(\Gamma)$. Thus, equation (3.3) means that $G_-^{-1}g \in \mathcal{L}^1(\Gamma)$, thus $P_+G_-^{-1}g \in L_+^1(\Gamma)$ and $P_-G_-^{-1}g \in \mathring{L}_-^1(\Gamma)$ are well-defined.

Rewriting equation (3.3), we get

$$f^-(t) - P_-G_-^{-1}(t)g(t) = -t^{\aleph}f^+(t) + P_+G_-^{-1}(t)g(t).$$

Since f^- and $P_-G_-^{-1}g$ vanish at infinity, and we have $L_+^1(\Gamma) \cap L_-^1(\Gamma) = \text{Const.}$ and $L_+^1(\Gamma) \cap \mathring{L}_-^1(\Gamma) = \{0\}$, it follows that $\rho(t) = t^{\aleph}f^+(t) - P_+G_-^{-1}(t)g(t)$ is identically zero for $\aleph \ge 0$ and a polynomial of order $\le -\aleph - 1$ for $\aleph < 0$.

Set

$$f^-(t) = P_-G_-^{-1}(t)g(t) - \rho(t), \qquad f^+(t) = t^{-\aleph}P_+G_-^{-1}(t)g(t) + t^{-\aleph}\rho(t);$$

returning to the functions $\varphi^{\pm}$ we have formulas (3.18). By the assumptions, $\{\varphi^+, \varphi^-\}$ is a solution of problem (3.2), so $\varphi^+ \in \mathfrak{X}_+(\Gamma)$ and $\varphi^- \in \mathring{\mathfrak{X}}_-(\Gamma)$. Since $\rho \in \mathring{L}_-^{\infty}(\Gamma)$ and $t^{-\aleph}\rho \in L_+^{\infty}(\Gamma)$, the conditions $G_-\rho \in \mathring{\mathfrak{X}}_-(\Gamma)$ and $G_+^{-1}t^{-\aleph}\rho \in \mathfrak{X}_+(\Gamma)$ are satisfied, and therefore

$$\varphi_0^+ \in \mathfrak{X}_+(\Gamma), \qquad \varphi_0^- \in \mathring{\mathfrak{X}}_-(\Gamma),$$

which proves the necessity of (3.17) for the solvability of problem (3.2) as well as the fact that every solution is of the form (3.18).

To prove that conditions (3.17) imply that every pair $\{\varphi^+, \varphi^-\}$ defined by (3.18) is a solution of problem (3.2) is direct by inserting the functions (3.18) into (3.2) taking into consideration the boundedness of the functions on the corresponding spaces, which is given by (3.17). ■

From the Cauchy formula [17, Ch. X, §4, Th. 1] for any analytic function representable by an integral of Cauchy type on a domain D, we immediately obtain:

Corollary 3.3.2. *For the solvabiliy of problem* (3.2) *it is necessary that, for* $\aleph \geq 0$,

$$\int_\Gamma G_-^{-1}(\tau) g(\tau) \tau^{-\kappa}\, d\tau = 0, \qquad \kappa = 1, \dots, \aleph.$$

Remark 3.3.3. (1) Notice that to establish Theorem 3.3.1, only condition (i) in the definition of factorization of an essentially bounded function in $\mathcal{X}(\Gamma)$ was necessary. Thus, in this sense, the factorization used in Theorem 3.3.1 is weaker than that defined on page 10.

(2) Similar conclusions to those in Theorem 3.3.1 can be given for problem (3.1) by considering the associate space $\mathcal{X}'(\Gamma)$ of $\mathcal{X}(\Gamma)$. In this case we define the factorization in $\mathcal{X}(\Gamma)$ for $a \in \mathcal{G}(L^\infty(\Gamma))$ as

$$a(t) = a_+(t) t^{\aleph} a_-(t)$$

with $\aleph \in \mathbb{N}$, $a_+ \in \mathcal{X}_+(\Gamma)$, $a_+^{-1} \in \mathcal{X}'_+(\Gamma)$, $a_- \in \mathcal{X}'_-(\Gamma)$, $a_-^{-1} \in \mathcal{X}_-(\Gamma)$. The proof is similar to the proof of Theorem 3.3.1 with obvious changes.

4. Solvability of singular integral equations with factorizable coefficients

Now, we will give conditions guaranteeing the existence of solutions for a class of singular integral equations with essentially bounded functions as coefficients, admitting a factorization in $\mathcal{X}(\Gamma)$ as in (2.8). To do so, the Fredholmness of the associated singular integral operator is studied.

4.1. Simonenko's criterion for the Fredholm property for SIO's with essentially bounded coefficients. We are going to establish a Fredholm criterion for the operator $\mathcal{A} = aP_+ + bP_-$ on $\mathcal{X}(\Gamma)$ with $a, b \in \mathcal{G}(L^\infty(\Gamma))$, by adapting the classical Simonenko scheme for singular integral operators with generalized factorizable functions on $L^p(\Gamma)$.

First, recall that for a bounded linear operator $\mathbb{A} \in \mathcal{B}(X, Y)$, the set $\ker \mathbb{A}$ of all solutions of the homogeneous equation

$$\mathbb{A}x = 0 \tag{4.1}$$

is the *kernel* of $\mathbb{A}$. Its dimension is called the *nullity* of $\mathbb{A}$ and denoted by $\alpha(\mathbb{A})$. A bounded operator $\mathbb{A}$ is called *normally solvable* (in the sense of Hausdorff) if the equation

$$\mathbb{A}x = y$$

is soluble only for those elements y which are orthogonal to the solution space of the equation $\mathbb{A}^*u = 0$, where $\mathbb{A}^*$ is the conjugate operator $\mathbb{A}^* : Y^* \to X^*$ defined by

$$(\mathbb{A}^*u)x = u(\mathbb{A}x).$$

That is, $\mathbb{A}^*u = 0$ if and only if $u(y) = 0$ for all $y \in \operatorname{Im} \mathbb{A}$, where $\operatorname{Im} \mathbb{A} = \{\mathbb{A}x : x \in X\}$. This is equivalent to saying that $\operatorname{Im} \mathbb{A}$ is a closed set.

For a normally solvable operator $\mathbb{A}$ the *cokernel* of $\mathbb{A}$, $\operatorname{Coker} \mathbb{A}$, is defined as

$$\operatorname{Coker} \mathbb{A} = Y/\operatorname{Im} \mathbb{A}.$$

Its dimension (called the *deficiency of* $\mathbb{A}$) is denoted by

$$\beta(\mathbb{A}) := \dim \operatorname{Coker} \mathbb{A}.$$

$\alpha(\mathbb{A})$ and $\beta(\mathbb{A})$ are frequently called the *deficiency numbers* of $\mathbb{A}$.

An operator $\mathbb{A}$ is called a *Fredholm operator*, or a Φ-operator, if $\alpha(\mathbb{A})$ and $\beta(\mathbb{A})$ are finite. In this case, the *Fredholm index* is

$$\operatorname{Ind} \mathbb{A} := \alpha(\mathbb{A}) - \beta(\mathbb{A}).$$

The operator $\mathbb{A}$ is called *semi-Fredholm* if at least one of $\alpha(\mathbb{A})$ or $\beta(\mathbb{A})$ is finite.

Let us first consider the singular integral operator $\mathcal{A}_a := aP_+ + P_-$ on $\mathcal{X}(\Gamma)$ with $a \in L^\infty(\Gamma)$.

Proposition 4.1.1. *If $a \in L^\infty(\Gamma)$ and $\mathcal{A}_a$ is semi-Fredholm, then $a \in \mathcal{G}(L^\infty(\Gamma))$.*

Proposition 4.1.2. *If $a \in \mathcal{G}(L^\infty(\Gamma))$, then $\min(\alpha(\mathcal{A}_a), \beta(\mathcal{A}_a)) = 0$.*

Propositions 4.1.1 and 4.1.2 can be proved as the classical case of $L^p(\Gamma)$ with minor modifications, by using the well-known Lusin–Privalov Theorem which can be applied in this framework in view of assumption (2.2) and Lemma 2.3.

The following is a Fredholmness criterion for the operator $\mathcal{A}_a$ on the space $\mathcal{X}(\Gamma)$. This result was established on the spaces $L^p(\Gamma)$ by I. B. Simonenko [45, 46] (see also e.g. [38]) and by A. Yu. Karlovich in the case of reflexive rearrangement-invariant spaces [20]. The proof that we are going to give is analogous to those cases.

Theorem 4.1.3. *Let Γ be a Lyapunov curve. A function $a \in L^\infty(\Gamma)$ admits a factorization in $\mathcal{X}(\Gamma)$ if and only if $\mathcal{A}_a = aP_+ + P_-$ is a Φ-operator on $\mathcal{X}(\Gamma)$. In that case* $\operatorname{Ind} \mathcal{A}_a = -\operatorname{ind} a$.

Proof. Necessity. Let $0 \in D^+$. First we assume that a admits a factorization $a = a_- a_+$ in $\mathcal{X}(\Gamma)$. Let $r \in \mathcal{R}(\Gamma)$. From Lemma 2.4 and the definition of factorization we know that $a_+^{-1} P_+ a_-^{-1} r \in \mathcal{X}_+(\Gamma)$ and $a_- P_- a_-^{-1} r \in \mathring{\mathcal{X}}_-(\Gamma)$. Consider the bounded linear operator

$$\mathcal{B} := (a_+^{-1} P_+ + a_- P_-) a_-^{-1} I = I + (1-a) a_+^{-1} P_+ a_+ a^{-1} I.$$

Then

$$\mathcal{A}_a \mathcal{B} r = (aP_+ + P_-)(a_+^{-1} P_+ + a_- P_-) a_-^{-1} I r = a a_+^{-1} P_+ a_-^{-1} r + a_- P_- a_-^{-1} r = r.$$

Analogously, $\mathcal{B}\mathcal{A}_a r = r$. Since $\mathcal{B}$ is bounded in $\mathcal{X}(\Gamma)$ due to assumptions (2.3) and (2.4) and because $\mathcal{R}(\Gamma)$ is dense in $\mathcal{X}(\Gamma)$, we conclude that $\mathcal{A}_a$ is invertible with inverse $\mathcal{A}_a^{-1} = \mathcal{B}$. Hence, $\operatorname{Ind} \mathcal{A}_a = 0$.

Now, let $a(t) = a_-(t) t^{\aleph} a_+(t)$ be a factorization in $\mathcal{X}(\Gamma)$. Then the function $at^{-\aleph}$ admits the factorization $at^{-\aleph} = a_- a_+$ in $\mathcal{X}(\Gamma)$. Thus, the operator $at^{-\aleph} P_+ + P_-$ is invertible in $\mathcal{X}(\Gamma)$. If $\aleph > 0$ then $\mathcal{A}_a$ can be represented in the form

$$\mathcal{A}_a = (at^{-\aleph} P_+ + P_-)(t^{\aleph} P_+ + P_-).$$

From the 3rd step in the proof of Theorem B in [32] (see Theorem 5.1.1) we see that $t^{\aleph} P_+ + P_- = t^{\aleph}(P_+ + t^{-\aleph} P_-)$ is a Φ-operator with index $-\aleph$; then from the Atkinson Theorem for Fredholm operators we conclude that $\mathcal{A}_a$ is a Φ-operator with index $-\aleph$.

In the case $\aleph < 0$ we have

$$at^{-\aleph} P_+ + P_- = \mathcal{A}_a (t^{-\aleph} P_- + P_-)$$

with $t^{-\aleph} P_+ + P_- = t^{-\aleph}(P_+ + t^{\aleph} P_-)$ a Φ-operator with index $\aleph$, thus $\mathcal{A}_a$ is a Φ-operator with index $-\aleph$.

Sufficiency. Suppose that $\mathcal{A}_a$ is a Fredholm operator with index $-\aleph$. By Proposition 4.1.1, $a \in \mathcal{G}(L^\infty(\Gamma))$. Consider the operator $\mathcal{A}_q$, where $q(t) = a(t) t^{-\aleph}$. By the compactness of the commutator $aS + SaI$ for $a \in C(\Gamma)$ given by the 1st step of the proof of Theorem B in [32], $\mathcal{A}_q = \mathcal{A}_a \mathcal{A}_{t^{-\aleph}} + \mathcal{K}$ where $\mathcal{K}$ is a compact operator. Hence $\mathcal{A}_q$ is Fredholm, and $\operatorname{Ind} \mathcal{A}_q = 0$. From the Atkinson Theorem and the fact that the Fredholm index is invariant under compact perturbations, by Proposition 4.1.2 it follows that $\mathcal{A}_q$ is invertible in $\mathfrak{X}(\Gamma)$. Taking into account that the operator S^* in the dual space $\mathfrak{X}^*(\Gamma)$ is given by $S^* = -HSH$, applying Lemma 2.3 we can show that in this case the operator $\mathcal{A}_{q^{-1}}$ is invertible in the associate space $\mathfrak{X}'(\Gamma)$.

Let $\varphi_0 \in \mathfrak{X}(\Gamma)$ and $\psi_0 \in \mathfrak{X}'(\Gamma)$ be the solutions of the equations $\mathcal{A}_q\varphi = 1$ and $\mathcal{A}_{q^{-1}}\psi = 1$ respectively. Applying Lemma 2.4 one can show that $a_+ := P_+\psi_0$ and $a_- := 1 - P_-\psi_0$ are the factors of the factorization $a(t) = a_-(t)t^{\aleph}a_+(t)$, that is, $a_- \in \mathfrak{X}_-(\Gamma)$, $a_+ \in \mathfrak{X}'_+(\Gamma)$, $a_-^{-1} \in \mathfrak{X}'_-(\Gamma)$, $a_+^{-1} \in \mathfrak{X}_+(\Gamma)$. To prove the boundedness of $a_+^{-1}Sa_+I$ in $\mathfrak{X}(\Gamma)$, assume without loss of generality that $\|q\|_\infty < 1$ and consider the operator

$$\mathcal{B} := (a_+^{-1}P_+ + a_-P_-)a_-^{-1}I = I + (1-q)a_+P_+a_-^{-1}I.$$

As above, $\mathcal{B} = \mathcal{A}_q^{-1}$ is bounded and therefore $a_+^{-1}P_+a_-^{-1}I$ is bounded too, which is equivalent to $a_+^{-1}Sa_-^{-1}I$ being bounded in $\mathfrak{X}(\Gamma)$. ■

4.2. Effective solution of singular integral equations with essentially bounded factorizable coefficients. The solvability theory in the space $\mathfrak{X}(\Gamma)$ of the equation

$$\mathcal{A}\varphi(t) := u(t)\varphi(t) + \frac{v(t)}{\pi i}\,\text{p.v.}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau = f(t), \qquad t \in \Gamma,\ u, v \in L^\infty(\Gamma), \tag{4.2}$$

or alternatively

$$\mathcal{A} = aP_+ + bP_-, \qquad a := u + v,\ b := u - v,$$

is given in the following results.

THEOREM 4.2.1. *Let Γ be a Lyapunov curve, $a, b \in L^\infty(\Gamma)$ and let $\mathfrak{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.6)*. Then, for the operator $\mathcal{A} = aP_+ + bP_-$ to be a Φ_+- or Φ_--operator on $\mathfrak{X}(\Gamma)$ it is necessary that $a, b \in \mathcal{G}(L^\infty(\Gamma))$. Let a, b invertible functions on $L^\infty(\Gamma)$. Then $\mathcal{A}$ is a Φ-operator if and only if the function ab^{-1} admits a factorization* (2.8)*. Let $\mathcal{A}$ be a Φ-operator and $\aleph = \operatorname{ind}(a/b)$. Then $\operatorname{Ind}\mathcal{A} = -\aleph$ and the operator $\mathcal{A}$ is left-invertible, right-invertible, or two-sided invertible if $\aleph > 0$, $\aleph < 0$ or $\aleph = 0$ respectively. The corresponding* (*one- or two-sided*) *inverse is of the form*

$$\mathcal{A}^{-1} = (t^{-\aleph}P_+ + P_-)(c_+^{-1}P_+ + c_-P_-)c_-^{-1}b^{-1}I$$

where $ab^{-1} = c_-t^{\aleph}c_+$ is the factorization in $\mathfrak{X}(\Gamma)$ of the function ab^{-1}.

Proof. Let $0 \in D^+$ and assume $a, b \in \mathcal{G}(L^\infty(\Gamma))$. Then $\mathcal{A}$ can be written as

$$\mathcal{A} = b(ab^{-1}P_+ + P_-),$$

where the multiplicator operator bI is bounded in $\mathfrak{X}(\Gamma)$ by assumption (2.3), and invertible with inverse $b^{-1}I$. Thus it is a Φ-operator.

Since $ab^{-1} \in \mathcal{G}(L^\infty(\Gamma))$, from Proposition 4.1.1 the operator $ab^{-1}P_+ + P_-$ is semi-Fredholm. Therefore, by the Atkinson Theorem, $\mathcal{A}$ is a semi-Fredholm operator.

On the other hand, Theorem 4.1.3 guarantees that $ab^{-1}P_+ + P_-$ is a Φ-operator if and only if ab^{-1} admits a factorization. Then reasoning as before we conclude that $\mathcal{A} = b(ab^{-1}P_+ + P_-)$ is a Φ-operator iff ab^{-1} admits a factorization in $\mathfrak{X}(\Gamma)$.

Now, suppose $\mathcal{A}$ is a Φ-operator and $\aleph = \operatorname{ind} ab^{-1}$. Since bI is invertible, we have $\operatorname{Ind} bI = 0$, and from the Atkinson Theorem,

$$\operatorname{Ind} \mathcal{A} = \operatorname{Ind} bI + \operatorname{Ind}(ab^{-1}P_+ + P_-) = \operatorname{Ind}(ab^{-1}P_+ + P_-).$$

Then Theorem 4.1.3 asserts that $\operatorname{Ind}(ab^{-1}P_+ + P_-) = -\operatorname{ind} ab^{-1} = -\aleph$.

To prove the (one- or two-sided) invertibility of $\mathcal{A}$ notice that for $ab^{-1} = c_- t^{\aleph} c_+$,

$$\mathcal{A} = bc_-(t^{\aleph}c_+P_+ + c_-^{-1}P_-) = bc_-(t^{\aleph}P_+ + P_-)(c_+P_+ + c_-^{-1}P_-) \tag{4.3}$$

with bc_-I and $c_+P_+ + c_-^{-1}P_-$ invertible operators with inverses

$$(bc_-I)^{-1} = b^{-1}c_-^{-1}I \quad \text{and} \quad (c_+P_+ + c_-^{-1}P_-)^{-1} = c_+^{-1}P_+ + c_-P_-.$$

From the 3rd step of the proof of Theorem B in [32], the operator $t^{\aleph}P_+ + P_-$ is left-, right- or two-sided invertible (so, by (4.3), so is $\mathcal{A}$) if $\aleph > 0$, $\aleph < 0$ or $\aleph = 0$ respectively. Direct computations show that the inverses of $\mathcal{A}$ are in fact $(t^{-\aleph}P_+ + P_-)(c_+^{-1}P_+ + c_-P_-) \times c_-^{-1}b^{-1}I$. ■

The following result gives the dimension of $\ker \mathcal{A}$ and $\operatorname{Coker} \mathcal{A}$, as well as the solvability conditions for (4.2).

THEOREM 4.2.2. *Let Γ be a Lyapunov curve, let $\mathcal{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.6) *and let $a, b \in \mathcal{G}(L^\infty(\Gamma))$. Moreover, assume that the function ab^{-1} admits a factorization $ab^{-1} =: c = c_- t^{\aleph} c_+$ in the space $\mathcal{X}(\Gamma)$. Then if $\aleph = \operatorname{ind} c < 0$,*

$$\ker(aP_+ + bP_-) = \operatorname{span}\{g, gt, \dots, gt^{|\aleph|-1}\} \tag{4.4}$$

where $g = c_+^{-1} - c_- t^{\aleph}$. In the case $\aleph > 0$,

$$\operatorname{Coker}(aP_+ + bP_-) = \operatorname{span}\{bc_-, bc_-t, \dots, bc_-t^{\aleph-1}\} \tag{4.5}$$

and the equation $aP_+\varphi + bP_-\varphi = f$ has a solution if and only if

$$\int_\Gamma f(t)b^{-1}(t)c_-^{-1}(t)t^{-j}\,dt = 0, \qquad j = 1, \dots, \aleph. \tag{4.6}$$

Proof. Let $\mathcal{A} = aP_+ + bP_-$ and assume $0 \in D^+$. From (4.3) we see that $\mathcal{A}$ and $t^{\aleph}P_+ + P_-$ are equivalent operators, therefore

$$\dim\ker \mathcal{A} = \dim\ker(t^{\aleph}P_+ + P_-) = \dim\ker(P_+ + t^{-\aleph}P_-).$$

The 3rd step of the proof of Theorem B in [32] shows that if $\aleph \le 0$ then $\dim\ker \mathcal{A} = |\aleph|$, and if $\aleph \ge 0$ then $\dim \operatorname{Coker} \mathcal{A} = \aleph$.

Now, suppose $\aleph < 0$. We are going to find the set $\ker(P_+ + t^{-\aleph}P_-)$, that is, $\{\varphi \in \mathcal{X}(\Gamma) : P_+\varphi + t^{-\aleph}P_-\varphi = 0\}$. Since $\dim\ker(P_+ + t^{-\aleph}P_-) = |\aleph|$, there is a polynomial $p_{|\aleph|-1}(t) = a_{|\aleph|-1}t^{|\aleph|-1} + \dots + a_1t + a_0$ of degree at most $|\aleph| - 1$ such that $t^{|\aleph|}P_-\varphi + p_{|\aleph|-1} \in P_-(\mathcal{X}(\Gamma))$. From the above we have

$$P_+\varphi = p_{|\aleph|-1}, \qquad P_-\varphi = -p_{|\aleph|-1}/t^{|\aleph|}.$$

Thus,

$$P_+\varphi(t) + P_-\varphi(t) = \varphi(t) = p_{|\aleph|-1}(t)[1 - 1/t^{|\aleph|}],$$

and therefore

$$\ker(P_+ + t^{-\aleph}P_-) = \operatorname{span}\{t^{|\aleph|-1} - 1/t, t^{|\aleph|-2} - 1/t^2, \dots, 1 - 1/t\}.$$

On the other hand, by (4.3) we obtain

$$\ker \mathcal{A} = (c_+^{-1} P_+ + c_- P_-) \ker(t^{\aleph} P_+ + P_-) = \operatorname{span}\{g_1, \ldots, g_{|\aleph|}\}$$

with $g_j = (c_+^{-1} P_+ + c_- P_-)(t^{|\aleph|-j} - t^{-j})$, $j = 1, \ldots, |\aleph|$. Here

$$\begin{aligned} g_j &= (c_+^{-1} P_+ + c_- P_-)(t^{|\aleph|-j} - t^{-j}) = (c_+^{-1} P_+ + c_- P_-) t^{|\aleph|-j} - (c_+^{-1} P_+ + c_- P_-) t^{-j} \\ &= c_+^{-1} P_+ t^{|\aleph|-j} - c_- P_- t^{-j} = c_+^{-1} t^{|\aleph|-j} - c_- t^{-j}, \end{aligned}$$

proving (4.4).

Now, assume $\aleph > 0$. From (4.3) we have $\operatorname{Im} \mathcal{A} = bc_- \operatorname{Im}(T^{\aleph} P_+ + P_-)$, which gives (4.5) because $\operatorname{Im}(T^{\aleph} P_+ + P_-)$ consists of all functions $\varphi \in \mathfrak{X}(\Gamma)$ such that $P_+\varphi$ has a zero of order at most $\aleph$ at $t = 0$. On the other hand, from (4.3) and the 3rd step of Theorem B in [32], $\mathcal{A}$ is a Φ-operator, left-invertible and therefore normally solvable, so the equation $\mathcal{A}\varphi = f$ has a solution if and only if

$$\int_{\Gamma} f(t) \overline{y_j(t)}\, |dt| = 0, \qquad j = 1, \ldots, m, \tag{4.7}$$

where $y_1, \ldots, y_m$ is a basis of solutions of the adjoint homogeneous equation $\mathcal{A}^* y = 0$ in $\mathfrak{X}(\Gamma)$. From Lemma 2.3 the adjoint operator of $\mathcal{A}$ is defined by $\mathcal{A}^* = H(P_+ b + P_- a)H$, because $P_+^* = HP_- H$ and $P_-^* = HP_+ H$. Therefore

$$H\mathcal{A}^* z_j = (P_+ b + P_- a) c_-^{-1} b^{-1} t^{-j} = P_+ c_-^{-1} t^{-j} + P_- c_+ t^{\aleph - j} = 0,$$

so $z_j \in \ker \mathcal{A}^*$, $j = 1, \ldots, \aleph$. But the functions z_j are linearly independent, and since

$$\dim \ker \mathcal{A}^* = \dim \operatorname{Coker} \mathcal{A} = \aleph,$$

we conclude that $\aleph = m$ and $z_j = y_j$ for all j.

Moreover, $\overline{y_j(t)} |dt| = h(t) c_-^{-1}(t) b^{-1}(t) t^{-j} |dt| = c_-^{-1}(t) b^{-1}(t) t^{-j} dt$, and so (4.6) and (4.7) coincide, which completes the proof. ∎

5. Fredholmness of boundary value problems and explicit representation of solutions of (4.2)

Let Ψ be an analytic function of Cauchy integral type with nontangential limit $\varphi \in \mathfrak{X}(\Gamma)$. According to the Sokhotski–Plemelj formulas, the boundary values $\Psi^+(t)$ (resp. $\Psi^-(t)$) with $z \to t$, $t \in \Gamma$, $z \in D^+$ (resp. $t \in \Gamma$, $z \in D^-$) are expressed by

$$\Psi^+(t) = \tfrac{1}{2}[(I\varphi)(t) + (S\varphi)(t)], \qquad \Psi^-(t) = \tfrac{1}{2}[(-I\varphi)(t) + (S\varphi)(t)].$$

Therefore, $\Psi^+(t) - \Psi^-(t) = \varphi(t)$ and $\Psi^+(t) + \Psi^-(t) = (S\varphi)(t)$, which allows us to reduce the equation

$$(a(t)P_+ + b(t)P_-)\varphi(t) = g(t), \qquad b \in \mathcal{G}(L^\infty(\Gamma)),$$

or equivalently

$$(a(t)b^{-1}(t)P_+ + P_-)\varphi(t) = b^{-1}(t)g(t), \tag{5.1}$$

to the Riemann boundary value problem

$$\Psi^-(t) + (-a(t)b^{-1}(t))\Psi^+(t) = -b^{-1}(t)g(t). \tag{5.2}$$

Analogously, the equation

$$(a(t)P_+ + b(t)P_-)\varphi(t) = g(t), \qquad a \in \mathcal{G}(L^\infty(\Gamma)),$$
$$(P_+ + b(t)a^{-1}(t)P_-)\varphi(t) = a^{-1}(t)g(t) \tag{5.3}$$

reduces to the Riemann boundary value problem

$$\Psi^+(t) = (b(t)a^{-1}(t))\Psi^-(t) + a^{-1}(t)g(t), \tag{5.4}$$

for a piecewise analytic function $\{\Psi^+(z), \Psi^-(z)\}$ vanishing at $z = \infty$. That is, equation (5.1) and problem (5.2), as well as equation (5.3) and problem (5.4) with the additional condition $\Psi^-(\infty) = 0$, are equivalent. This means that there exists a one-to-one correspondence between the solutions of problem (5.2) (resp. (5.4)) and the solutions of equation (5.1) (resp. (5.3)).

5.1. The case of continuous coefficients. We can characterize the Fredholmness of the boundary value problem (5.4) with an invertible continuous coefficient ba^{-1} through the Fredholmness of the operator $\mathcal{A} = aP_+ + bP_-$ given in the following criterion [32, Theorem B]:

THEOREM 5.1.1. *Let $\mathfrak{X}(\Gamma)$ be any BFS satisfying assumptions* (2.2)–(2.4) *and* (2.6)*. The operator $\mathcal{A} = aP_+ + bP_-$ with $a, b \in C(\Gamma)$ is Fredholm in the space $\mathfrak{X}(\Gamma)$ if and only if $a(t) \neq 0$ and $b(t) \neq 0$ for all $t \in \Gamma$. In this case,* $\operatorname{Ind} \mathcal{A} = \operatorname{ind}(b/a) = \aleph$.

Thus, we have the following result:

PROPOSITION 5.1.2. *Let* Γ *be a Lyapunov curve and let* $\mathcal{X}(\Gamma)$ *be any BFS satisfying assumptions* (2.2)–(2.4) *and* (2.6). *The boundary value problem* (5.4) *with continuous coefficient* ba^{-1} *is Fredholm with index* $\aleph$ *if and only if* $ba^{-1} \in \mathcal{G}(C(\Gamma))$ *with* $\operatorname{ind} ba^{-1} = \aleph$.

On the other hand, using the equivalence mentioned above, we can give an explicit representation of solutions of (4.2) with continuous coefficients a and b, satisfying the assumptions of Theorem 5.1.1, through the two-term boundary value problem (5.4).

THEOREM 5.1.3. *Let* Γ *be a Lyapunov curve and let* $\mathcal{X}(\Gamma)$ *be any BFS satisfying assumptions* (2.2)–(2.4) *and* (2.6). *Equation* (4.2) *with continuous coefficients* a *and* b *has solutions if and only if* $a(t) \neq 0$ *and* $b(t) \neq 0$ *for all* $t \in \Gamma$. *The solutions are described, according to the case, by:*

($\aleph \geq 0$)

$$\varphi(t) = \frac{(1-(t-z_0)^{\aleph})e^{h(t)}}{2\pi i} \int_\Gamma \frac{f(\tau)}{e^{h(\tau)}} \frac{d\tau}{\tau - z} + (1-(t-z_0)^{\aleph})e^{h(t)}\rho(t) \tag{5.5}$$

where $h(t) = K\big(\ln \frac{b(\tau)}{a(\tau)}(\tau - z_0)^{\aleph}\big)(t)$ *and* ρ *is an arbitrary polynomial of degree* $\aleph - 1$.

($\aleph < 0$) *The unique solution in this case is as in* (5.5) *with* $\rho(t) \equiv 0$. *In addition, it is necessary that*

$$\int_\Gamma \frac{f(\tau)\tau^{\kappa}}{e^{h(\tau)}}\, d\tau = 0, \qquad \kappa = 0, \dots, |\aleph| - 1. \tag{5.6}$$

Here, $\aleph := \operatorname{ind} ba^{-1}$.

Proof. From Theorem 5.1.1, the conditions $a(t) \neq 0$ and $b(t) \neq 0$ for all $t \in \Gamma$ are equivalent to the Fredholmness of the associated operator $\mathcal{A} = aP_+ + bP_-$. Therefore equation (4.2) is solvable. From Theorem 3.1.2, the solutions of the boundary problem (5.4) are given by (3.4). The solvability conditions (5.6) are necessary for the solvability of (5.4) as stated in Theorem 3.1.2. Finally, from the Sokhotski–Plemelj formulas we have (5.5). ∎

5.2. The case of piecewise continuous coefficients. The Fredholmness of problem (5.4) with piecewise continuous coefficients is characterized by using the Fredholmness of the operator $\mathcal{A} = aP_+ + bP_-$ with coefficients in the same class. To establish this we will use the so-called Khvedelidze–Gohberg–Krupnik investigation scheme, so first we are going to recall the reformulations of the notions of p-no singularity and p-index in the framework of Banach function spaces introduced in [32].

For a BFS $\mathcal{X}(\Gamma)$ satisfying Axiom 1, a function $G \in \mathrm{PC}(\Gamma)$ is called $\mathcal{X}(\Gamma)$-*nonsingular* if $\inf_{t\in\Gamma} |G(t)| > 0$ and

$$\frac{1}{2\pi} \arg \frac{G(t_k - 0)}{G(t_k + 0)} \notin [\alpha(t_k), \beta(t_k)] + \mathbb{Z}$$

where $[\cdots] + \mathbb{Z}$ stands for the set $\bigcup_{\xi\in[\cdots]}\{\xi, \xi \pm 1, \xi \pm 2, \dots\}$ and α and β are the index functions of the space $\mathcal{X}(\Gamma)$. For an $\mathcal{X}(\Gamma)$-nonsingular function, the integer

$$\operatorname{ind} a = \sum_{k=1}^{n} [\theta(t_k) - \Re\gamma(t_k)],$$

where $\theta(t_k)$ are the increments

$$\theta(t_k) = \frac{1}{2\pi} \int_{t_k+0}^{t_{k+1}-0} d \arg G(t),$$

is referred to as the $\mathcal{X}(\Gamma)$-*index* of the function G.

The Fredholmness criterion for the operator $\mathcal{A} = aP_+ + bP_-$ reads as follows [32, Theorem C]:

Theorem 5.2.1. *Let* $\mathcal{X}(\Gamma)$ *be any BFS satisfying* (2.2)–(2.4), (2.6) *and Axioms* 1–2. *The operator* $\mathcal{A} = aP_+ + bP_-$ *with* $a, b \in \mathrm{PC}(\Gamma)$ *is Fredholm in the space* $\mathcal{X}(\Gamma)$ *if*

$$\inf_{t\in\Gamma} |a(t)| \neq 0, \qquad \inf_{t\in\Gamma} |b(t)| \neq 0 \tag{5.7}$$

and the function

$$a/b \textit{ is } \mathcal{X}(\Gamma)\textit{-nonsingular.} \tag{5.8}$$

In this case,

$$\operatorname{Ind} \mathcal{A} = -\operatorname{ind}(a/b).$$

Condition (5.7) *is also necessary for the operator* $\mathcal{A}$ *to be Fredholm in* $\mathcal{X}(\Gamma)$. *If the index functions* α *and* β *of the space* $\mathcal{X}(\Gamma)$ *coincide at the points* t_k *of discontinuity of the coefficients* a, b*:*

$$\alpha(t_k) = \beta(t_k), \qquad k = 1, \dots, n,$$

then condition (5.8) *is necessary as well.*

The Fredholm property of problem (5.4) with a piecewise continuous coefficient is established in the following result.

Proposition 5.2.2. *Let* Γ *be a Lyapunov curve and let* $\mathcal{X}(\Gamma)$ *be any BFS satisfying* (2.2)–(2.4), (2.6) *and Axioms* 1–2. *The boundary value problem* (5.4) *with piecewise continuous coefficient* ab^{-1} *is Fredholm with index* $\aleph$ *if*

$$\inf_{t\in\Gamma} |a(t)| \neq 0, \qquad \inf_{t\in\Gamma} |b(t)| \neq 0 \tag{5.9}$$

and the function

$$a/b \textit{ is } \mathcal{X}(\Gamma)\textit{-nonsingular} \tag{5.10}$$

with index $\aleph = -\operatorname{ind}(a/b)$. *Condition* (5.9) *is also necessary for the Fredholmness of problem* (5.4). *If at the points* t_k *of discontinuity of the coefficients* a, b,

$$\alpha(t_k) = \beta(t_k), \qquad k = 1, \dots, n,$$

then condition (5.10) *is necessary as well.*

The representation of the solutions of (4.2) with piecewise continuous coefficients a and b satisfying the assumptions of Theorem 5.2.1 is given in the following result.

Theorem 5.2.3. *Let* Γ *be a Lyapunov curve and let* $\mathcal{X}(\Gamma)$ *be any BFS satisfying* (2.2)–(2.4), (2.6) *and Axioms* 1–2. *Equation* (4.2) *with piecewise continuous coefficients* a *and* b *has solutions if* a *and* b *satisfy* (5.7), *and* ab^{-1} *is* $\mathcal{X}(\Gamma)$*-nonsingular with discontinuity points* $t_1, \dots, t_n$, *at which the curve* Γ *has at least one-sided tangents. The solutions are described, according to the case, by:*

($\aleph \geq 0$)

$$\varphi(t) = \big(\omega^+(t)X_1^+(t) - \omega^-(t)X_1^-(t)\big)K\bigg(\frac{f}{\omega^+ X_1^+}\bigg)(t) + \big(\omega^+(t)X_1^+(t) - \omega^-(t)X_1^-(t)\big)\rho(t) \tag{5.11}$$

where

$$\omega^+(t) = \prod_{k=1}^{n}(t-t_k)^{\gamma(t_k)}, \qquad \omega^-(t) = \prod_{k=1}^{n}\bigg(\frac{t-t_k}{t-z_0}\bigg)^{\gamma(t_k)},$$

$$X_1(t) = e^{K(\ln G_1(t))}, \qquad G_1(t) = \frac{a(t)b^{-1}(t)}{\prod_{k=1}^{n}(t-t_k)^{\gamma(t_k)}}$$

with $z_0 \in D^+$, *and* ρ *is an arbitrary polynomial of degree* $\aleph - 1$.

($\aleph < 0$) *The unique solution in this case is as in* (5.11) *with* $\rho(t) \equiv 0$. *In addition, it is necessary that*

$$\int_\Gamma \frac{f(\tau)\tau^\kappa}{X_1^+(\tau)}\,d\tau = 0, \qquad \kappa = 0, \dots, |\aleph| - 1. \tag{5.12}$$

Here, $\aleph := \operatorname{ind} G_1$.

Proof. From Theorem 5.2.1, the assumptions of the theorem are sufficient for the Fredholmness of the associated operator $\mathcal{A}$, and thus equation (4.2) is solvable. From Theorem 3.2.3, the solutions of the boundary problem (5.4) are given by (3.16). The solvability conditions (5.12) are necessary for the solvability of problem (5.4) as is stated in Theorem 3.2.3. Finally, from the Sokhotski–Plemelj formulas we have (5.11). ∎

5.3. The case of essentially bounded factorizable coefficients. The Fredholmness of problem (5.2), for an essentially measurable function admitting a factorization in $\mathcal{X}(\Gamma)$ as in (2.8), can be characterized through Theorem 4.1.3:

PROPOSITION 5.3.1. *Let* Γ *be a Lyapunov curve and let* $\mathcal{X}(\Gamma)$ *be any BFS satisfying* (2.2)–(2.6). *Then problem* (5.2) *is Fredholm iff its coefficient function* ab^{-1} *admits a factorization in* $\mathcal{X}(\Gamma)$ *given in* (2.8).

Notice that from Theorem 4.2.1, the invertibility of the coefficient in problem (3.2) is necessary for its normal solvability.

THEOREM 5.3.2. *Let* Γ *be a Lyapunov curve and let* $\mathcal{X}(\Gamma)$ *be any BFS satisfying assumptions* (2.2)–(2.6). *Equation* (4.2) *with essentially bounded coefficients* a *and* b *has solutions if and only if* $a, b \in \mathcal{G}(L^\infty(\Gamma))$ *and* ab^{-1} *admits a factorization* (2.8). *The solutions are described, according to the case, by:*

($\aleph < 0$)

$$\varphi(t) = \big(c_+^{-1}(t)t^{-\aleph}P_+ + c_-(t)P_-\big)c_-^{-1}(t)b^{-1}(t)g(t) + \big(c_+^{-1}(t)t^{-\aleph} + c_-(t)\big)\rho(t) \tag{5.13}$$

where ρ *is an arbitrary polynomial of degree* $|\aleph| - 1$.

($\aleph \geq 0$) *The unique solution in this case is as in* (5.13) *with* $\rho(t) \equiv 0$. *In addition, it is necessary that*

$$\int_\Gamma c_-^{-1}(\tau)b(\tau)f(\tau)\tau^{-\kappa}\,d\tau = 0, \qquad \kappa = 1, \dots, \aleph. \tag{5.14}$$

Here, $\aleph := \operatorname{ind} ab^{-1}$.

Proof. From Theorem 4.2.2, equation (4.2) is solvable. Notice that condition (3.17) holds for the factorizable function $ab^{-1} \in \mathcal{G}(L^\infty(\Gamma))$. Therefore, from Theorem 3.3.1, the solutions of problem (5.2) have the form (3.18). In this last case, from Corollary 3.3.2, for the solvability of the boundary value problem (5.2) it is necessary that (5.14) holds, therefore, from the Sokhotski–Plemelj formulas, the general solutions of equation (4.2) have the representation (5.13) with an arbitrary polynomial function ρ if $\aleph < 0$, and $\rho(t) \equiv 0$ if $\aleph \geq 0$, in which case condition (5.14) should be satisfied. See Theorem 4.2.2, equality (4.6). ∎

6. Singular integral equations with Carleman shift

Let $\alpha(t)$ be a homeomorphism of Γ onto itself which may preserve or change the orientation of Γ, and suppose that at every point t the derivative $\alpha'(t)$ exists and satisfies $\alpha'(t) \neq 0$ and the Hölder condition. In addition, we will assume that $\alpha(t)$ satisfies the so-called *Carleman condition*:

$$\alpha^2(t) = (\alpha \circ \alpha)(t) = t. \tag{6.1}$$

Moreover, we will assume that

$$\alpha(t) \text{ induces a bounded shift operator } (W\varphi)(t) = \varphi(\alpha(t)) \text{ on } \mathcal{X}(\Gamma). \tag{6.2}$$

Notice that (6.1) implies that W satisfy the Carleman condition $W^2 = I$ (see e.g. [19, 35]). Among all kinds of Carleman shift operators, here we are going to consider those satisfying $WS = \gamma SW$, where $\gamma = \pm 1$. When $\gamma = 1$ (α preserves the orientation of Γ), W is called a *commutative Carleman shift operator*, and for $\gamma = -1$ (α reverses the orientation of Γ) an *anti-commutative Carleman shift operator*.

In the present chapter, the solvability of the following class of integral equations will be studied in the space $\mathcal{X}(\Gamma)$ over a Lyapunov curve Γ satisfying (2.2)–(2.4), (2.6) and (6.2):

$$f(t)\varphi(t) + g(t)\frac{1}{\pi i}\,\text{p.v.}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau + g(t)\frac{1}{\pi i}\,\text{p.v.}\int_\Gamma \frac{\varphi(\tau)}{\tau - \alpha(t)}\,d\tau = h(t), \tag{6.3}$$

where $f(t)$ and $g(t)$ are essentially bounded functions with $f(t) \neq 0$ on Γ, and $\alpha(t)$ is a Carleman shift function.

Consider the following complementary projection operators on $\mathcal{X}(\Gamma)$:

$$P_1 := \tfrac{1}{2}(I - W) \quad \text{and} \quad P_2 := \tfrac{1}{2}(I + W).$$

Note that $W^k = \sum_{j=1}^{2}(-1)^{kj}P_j$, $k = 1, 2$, and

$$P_k = \frac{1}{2}\sum_{j=1}^{2}(-1)^{k(1-j)}W^{j+1}, \qquad k = 1, 2. \tag{6.4}$$

The following lemma will be useful.

Lemma 6.0.1. *Let Γ be a Lyapunov curve and let $\mathcal{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.4), (2.6) *and* (6.2). *Let $\psi \in \mathcal{X}(\Gamma)$. Then, for $z \in \mathbb{C}$,*

$$(P_k S\psi)(z) = \begin{cases} (SP_k\psi)(z) & \text{if } W \text{ is a commutative shift operator,} \\ (SP_{3-k}\psi)(z) & \text{if } W \text{ is an anti-commutative shift operator.} \end{cases} \tag{6.5}$$

Proof. We have directly

$$(P_k S\psi)(z) = \tfrac{1}{2}\{(S\psi)(z) + (-1)^k W(S\psi)(z)\} = \tfrac{1}{2}\{(S\psi)(z) + (-1)^k \gamma(WS\psi)(z)\}$$

where $\gamma = \pm 1$ depending on whether W is a commutative or anti-commutative Carleman shift operator. From this, (6.5) follows. ■

6.1. An auxiliary system of equations and solvability of (6.3)**.** We are going to discuss the existence and uniqueness of solutions of (6.3). Moreover, we will provide explicit representations of such solutions. To this end, in particular, we will use projection methods as in [3, 4, 5, 8, 7, 48] so that we will be able to transform the initial equation into a system of equations which can be solved by means of a Riemann boundary value problem technique.

Let us introduce the following functions: for $k = 1, 2$,

$$f_\alpha(t) := f(t)f(\alpha(t)), \tag{6.6}$$

$$[fg]_k(t) := f(\alpha(t))g(t) + (-1)^k f(t)g(\alpha(t)), \tag{6.7}$$

$$[fh]_k(t) := \tfrac{1}{2}\big(f(\alpha(t))h(t) + (-1)^k f(t)h(\alpha(t))\big). \tag{6.8}$$

Notice that with the complementary projections P_k $(k = 1, 2)$, given in (6.4), the functions in (6.7) and (6.8) can be rewritten as $[fg]_k(t) = 2P_k[f(\alpha(t))g(t)]$ and $[fh]_k(t) = P_k[f(\alpha(t))h(t)]$.

Now, we will replace (6.3) with a simpler and equivalent system of equations. First of all, notice that using P_k $(k = 1, 2)$, we can rewrite (6.3) as

$$f(t)\varphi(t) + 2g(t)(P_2S\varphi)(t) = h(t). \tag{6.9}$$

Proposition 6.1.1. *Let $\varphi \in \mathcal{X}(\Gamma)$. Then φ is a solution of* (6.9) *if and only if $\{\varphi_k := P_k\varphi$, $k = 1, 2\}$ is a solution of*

$$\begin{cases} f_\alpha(t)\varphi_k(t) + [fg]_k(t)[(S\varphi_2)(t)] = [fh]_k(t) & \textit{if } \alpha \textit{ preserves orientation, or} \\ f_\alpha(t)\varphi_k(t) + [fg]_k(t)(S\varphi_1)(t)] = [fh]_k(t) & \textit{otherwise.} \end{cases} \tag{6.10}$$

Here $f_\alpha(t)$, $[fg]_k(t)$ and $[fh]_k(t)$ $(k = 1, 2)$ are defined in (6.6)–(6.8) *respectively.*

Proof. Suppose that $\varphi \in \mathcal{X}(\Gamma)$ is a solution of (6.9). Multiplying by $f(\alpha(t))$ we have

$$f(\alpha(t))f(t)\varphi(t) + 2f(\alpha(t))g(t)(P_2S\varphi)(t) = f(\alpha(t))h(t).$$

Applying the projections P_k $(k = 1, 2)$, we get

$$P_k[f(\alpha(t))f(t)\varphi](t) + 2P_k[f(\alpha(t))g(t)(P_2S\varphi)](t) = P_k[f(\alpha(t))h(t)]. \tag{6.11}$$

By using (6.4) and the fact that $WP_2 = P_2$, we can verify that

$$P_k[f(\alpha(t))f(t)\varphi](t) = f(\alpha(t))f(t)(P_k\varphi)(t),$$

$$P_k[f(\alpha(t))g(t)(P_2S\varphi)](t) = P_k[f(\alpha(t))g(t)](P_2S\varphi)(t).$$

Therefore, we can rewrite (6.11) as

$$f(\alpha(t))f(t)(P_k\varphi)(t) + 2P_k\big(f(\alpha(t))g(t)\big)(P_2S\varphi)(t) = P_k[f(\alpha(t))h(t)]. \tag{6.12}$$

Now, by Lemma 6.0.1 we see that $P_2 S = SP_2$ for W a commutative Carleman shift operator, and $P_2 S = SP_1$ for W an anti-commutative shift. In this way, we conclude that $(P_1\varphi, P_2\varphi)$ is a solution of (6.10).

Conversely, suppose that there exists φ such that $(P_1\varphi, P_2\varphi)$ is a solution of (6.10). Then Lemma 6.0.1 guarantees that system (6.10) is equivalent to (6.12), thus summing k from 1 to 2 we directly see using (6.12) that

$$\sum_{k=1}^{2}\big[f_\alpha(t)(\varphi_k)(t) + 2P_k[f(\alpha(t))g(t)](P_2 S\varphi)(t)\big] = \sum_{k=1}^{2} P_k[f(\alpha(t))h(t)]$$

is equivalent to

$$f(\alpha(t))f(t)\varphi(t) + 2f(\alpha(t))g(t)(P_2 S\varphi)(t) = f(\alpha(t))h(t)$$

due to the fact that $f(t) \neq 0$ for $t \in \Gamma$. Then

$$f(t)\varphi(t) + 2g(t)(P_2 S_\Gamma \varphi)(t) = h(t),$$

which completes the proof. ∎

Proposition 6.1.2. *If* (ϕ_1, ϕ_2) *is a solution of system* (6.10), *then so is* $(P_1\phi_1, P_2\phi_2)$.

Proof. Let (ϕ_1, ϕ_2) be a solution of (6.10). Applying the projections P_k to (6.10) we have

$$P_k\big(f_\alpha(t)\phi_k(t) + [fg]_k(t)(S\phi_i)(t)\big) = P_k[fh]_k(t), \qquad k, i = 1, 2.$$

Notice that $P_k[f_\alpha(t)\phi_k](t) = f_\alpha(t)P_k\phi_k(t)$ and

$$\begin{aligned} P_k\big([(fg)]_k(t)(S\phi_i)\big)(t) &= \tfrac{1}{2}\{[fg]_k(t)(S\phi_i)(t) + (-1)^k[fg]_k(\alpha(t))W(S_\Gamma\phi_i)(t)\} \\ &= [fg]_k(t)\tfrac{1}{2}\{(S\phi_i)(t) + W(S\phi_i)(t)\}, \end{aligned} \tag{6.13}$$

because $[fg]_k(t) = (-1)^k[fg]_k(\alpha(t))$.

Since $P_k[fg]_k = [fg]_k$, the right-hand side of (6.13) can be rewritten as $P_k([fg]_k(t)) \times P_2(S\phi_i)(t)$. From (6.10), the value of i depends on whether W is commutative or anti-commutative, therefore

$$P_k\big[[fg]_k(t)(S\phi_i)\big](t) = P_k([fg]_k(t))(SP_i\phi_i)(t).$$

Finally, note that $P_k([fh]_k)(t) = [fh]_k(t)$. Thus, $(P_1\phi_1, P_2\phi_2)$ is a solution of (6.10). ∎

Theorem 6.1.3. *Equation* (6.9) *has solutions in* $\mathfrak{X}(\Gamma)$ *if and only if the equation*

$$\begin{aligned} &f_\alpha(t)\varphi_2(t) + [fg]_2(t)(S\varphi_2)(t) = [fh]_2(t) \quad \textit{if } W \textit{ is commutative, or} \\ &f_\alpha(t)\varphi_1(t) + [fg]_1(t)(S\varphi_1)(t) = [fh]_1(t) \quad \textit{if } W \textit{ is anti-commutative,} \end{aligned} \tag{6.14}$$

has solutions. Moreover, if $\varphi_k(t)$ $(k = 1, 2)$ *is a solution of* (6.14), *then equation* (6.9) *has a solution given by*

$$\varphi(t) = \begin{cases} \dfrac{h(t) - 2g(t)[(S\varphi_2)(t)]}{f(t)} & \textit{if } WS = SW, \\ \dfrac{h(t) - 2g(t)[(S\varphi_1)(t)]}{f(t)} & \textit{if } WS = -SW. \end{cases} \tag{6.15}$$

Proof. Suppose that $\varphi \in \mathfrak{X}(\Gamma)$ is a solution of (6.9). By Proposition 6.1.1 we know that $(P_1\varphi, P_2\varphi)$ is a solution of (6.10). Hence, for W preserving the orientation of Γ, $P_2\varphi$ is a solution of (6.14), and $P_1\varphi$ is the corresponding solution for W reversing the orientation of Γ.

Conversely, suppose that φ_2 is a solution of (6.14). Without loss of generality, we assume that we are in the orientation preserving case (since the other case is dealt with similarly). In this case, (6.10) has a solution (φ_1, φ_2) determined by

$$\varphi_1(t) = \frac{[fh]_1(t) - [fg]_1(t)[(S\varphi_2)(t)]}{f_\alpha(t)}. \tag{6.16}$$

From Proposition 6.1.2, $P_i\varphi_i$ is also a solution of (6.14), so $(P_1\varphi_1, P_2\varphi_2)$ is also a solution of (6.10). Set

$$\varphi = \sum_{k=1}^{2} P_k\varphi_k. \tag{6.17}$$

It is clear that $P_k\varphi = P_k\varphi_k$. This means that $(P_1\varphi, P_2\varphi)$ is a solution of (6.11). From Proposition 6.1.1, φ is a solution of (6.10). Moreover, from (6.16) and (6.17), we obtain

$$\varphi(t) = \sum_{k=1}^{2} P_k\left[\frac{[fh]_k(t) - [fg]_k[(S\varphi_2)(t)]}{f_\alpha(t)}\right]. \tag{6.18}$$

As before, we can see that

$$\sum_{k=1}^{2} P_k[fh]_k(t) = f(\alpha(t))h(t),$$
$$\sum_{k=1}^{2} P_k([fg]_k(t)[(S\varphi_2)(t)]) = 2f(\alpha(t))g(t)[(S\varphi_2)(t)].$$

Substituting these in (6.18), we have

$$\varphi(t) = \frac{h(t) - 2g(t)[(S\varphi_2)(t)]}{f(t)}. \blacksquare$$

6.2. Closed form of solutions. At this point we know that formula (6.15) gives a representation for solutions of (6.3). In order to obtain a closed form of solutions, we must compute $(S\varphi_k)(t)$, $k = 1, 2$, from formula (6.15). We can use the Riemann problems associated to (6.14) to describe the form of that solution in the cases where the coefficients of the equation are continuous, piecewise continuous and essentially bounded factorizable functions.

For continuous coefficients, assume that $f_\alpha \pm [fg]_k \in \mathcal{G}(C(\Gamma))$ $(k = 1, 2)$ and set

$$G(t) = \begin{cases} \dfrac{f_\alpha(t) - [fg]_2(t)}{f_\alpha(t) + [fg]_2(t)} & \text{if } W \text{ commutes,} \\ \dfrac{f_\alpha(t) - [fg]_1(t)}{f_\alpha(t) + [fg]_1(t)} & \text{if } W \text{ anti-commutes,} \end{cases} \tag{6.19}$$

$$H(t) = \begin{cases} \dfrac{[fh]_2(t)}{f_\alpha(t) + [fg]_2(t)} & \text{if } W \text{ commutes,} \\ \dfrac{[fh]_1(t)}{f_\alpha(t) + [fg]_1(t)} & \text{if } W \text{ anti-commutes.} \end{cases} \tag{6.20}$$

Equation (6.14) can now be rewritten as

$$P_+\varphi_k(t) + G(t)P_-\varphi_k(t) = H(t),$$

which is then reduced to

$$\Psi_k^+(t) = G(t)\Psi_k^-(t) + H(t), \qquad G \in \mathcal{G}(C(\Gamma)).$$

Moreover, from the Sokhotski–Plemelj formulas, we have $(S\varphi_k)(t) = \Psi_k^+(t) + \Psi_k^-(t)$, thus the representations of the solutions of equations (6.14) for commuting or anti-commuting shift with continuous coefficients are given in the following result.

THEOREM 6.2.1. *Let Γ be a Lyapunov curve and let $\mathfrak{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.4), (2.6) *and* (6.2). *Let $G(t)$ and $H(t)$ be as in* (6.19) *and* (6.20) *respectively. Then equation* (6.3) *has solutions in $\mathfrak{X}(\Gamma)$ and they are given by*

$$\varphi(t) = \frac{h(t) - 2g(t)(S\varphi_k)(t)}{f(t)}, \qquad k = 1, 2,$$

where $k = 1$ if W is a commutative Carleman shift operator, and $k = 2$ if W is anti-commutative. In addition, for $(S\varphi_k)(t) = \Psi_k^+(t) + \Psi_k^-(t)$, we have the following different situations:

($\aleph \geq 0$) *In this case*

$$\Psi_k^+(t) = \frac{e^{h(t)}}{2\pi i}\int_\Gamma \frac{H(\tau)}{e^{h(\tau)}}\,\frac{d\tau}{\tau - t} + e^{h(t)}\rho(t), \tag{6.21}$$

$$\Psi_k^-(t) = \frac{(t-z_0)^{\aleph}e^{h(t)}}{2\pi i}\int_\Gamma \frac{H(\tau)}{e^{h(\tau)}}\,\frac{d\tau}{\tau - t} + (t-z_0)^{\aleph}e^{h(t)}\rho(t), \tag{6.22}$$

where

$$h(t) = \frac{1}{2\pi}\int_\Gamma \frac{\ln G(\tau)(\tau - z_0)^{\aleph}}{\tau - t}\,d\tau, \qquad z_0 \in D^+,$$

and $\rho(t) = a_{\aleph-1}t^{\aleph-1} + a_{\aleph-2}t^{\aleph-2} + \cdots + a_0$.

($\aleph < 0$) *For this case the solution is unique and $\Psi_k^\pm$ are as in* (6.21) *and* (6.22) *with $\rho(t) \equiv 0$, and in addition it is necessary that*

$$\int_\Gamma \frac{H(\tau)\tau^{\kappa}}{e^{h(\tau)}} = 0, \qquad \kappa = 0, \ldots, |\aleph| - 1.$$

Proof. From Theorem 6.1.3 we know that (6.3) has solutions if and only if (6.14) does. Furthermore, the solutions of (6.3) are given by (6.15). Thus, we will compute the solutions of (6.14). We will use the associated Riemann boundary value problem. Namely, by the Sokhotski–Plemelj formulas, (6.14) reduces to the following boundary problem: Find a sectionally analytic function $\Psi_k(z)$ ($\Psi_k(z) = \Psi_k^+(z)$ for $z \in D^+$ and $\Psi_k(z) = \Psi_k^-(z)$ for $z \in D^-$) vanishing at infinity and satisfying

$$\Psi_k^+(t) = G(t)\Psi_k^-(t) + H(t) \tag{6.23}$$

on Γ, where $G(t)$ and $H(t)$ are defined in (6.19) and (6.20) respectively.

From Theorem 3.1.2, the solutions of problem (6.23) read as follows:

(1) *Case* $\aleph \geq 0$. In this case the solutions are given by (cf. (3.4))

$$\Psi_k^{\pm}(t) = \frac{X^{\pm}(t)}{2\pi i} \int_{\Gamma} \frac{H(\tau)}{X^{+}(\tau)} \frac{d\tau}{\tau - z} + X^{\pm}(t)\rho(t) \tag{6.24}$$

where $X^{+}(t) = \exp h(t)$, $X^{-}(t) = (t - z_0)^{\aleph} \exp h(t)$ $(z_0 \in D^{+})$, and

$$h(t) = K\big(\ln G(\tau)(\tau - z_0)^{\aleph}\big)(t)$$

and ρ is an arbitrary polynomial of degree $\aleph - 1$. The second term on the right-hand side of (6.24) is the general solution of the homogeneous ($H(t) \equiv 0$) Riemann problem (6.23), and the first term is a particular solution of the corresponding inhomogeneous problem (6.23).

(2) *Case* $\aleph < 0$. For this case, $\Psi_k^{\pm}$ are as in (6.24) and $\rho(z) \equiv 0$. In addition, it is necessary that

$$\int_{\Gamma} \frac{H(\tau)\tau^{\kappa}}{e^{h(\tau)}}\, d\tau = 0, \qquad \kappa = 0, \dots, |\aleph| - 1.$$

This completes the proof. ■

When (6.3) has essentially bounded coefficients, assume $f_\alpha \pm [fg]_k \in \mathcal{G}(L^\infty(\Gamma))$ $(k = 1, 2)$ and define

$$\widetilde{G}(t) = 1/G(t) \tag{6.25}$$

for the function G given in (6.19), and

$$\widetilde{H}(t) = \begin{cases} \dfrac{[fh]_2(t)}{f_\alpha(t) - [fg]_2(t)} & \text{if } W \text{ commutes,} \\[2ex] \dfrac{[fh]_1(t)}{f_\alpha(t) - [fg]_1(t)} & \text{if } W \text{ anti-commutes.} \end{cases} \tag{6.26}$$

These functions allow us to rewrite (6.14) as

$$\widetilde{G}(t) P_{+}\varphi_k(t) + P_{-}\varphi_k(t) = \widetilde{H}(t). \tag{6.27}$$

Theorem 6.2.2. *Let Γ be a Lyapunov curve and let $\mathfrak{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.6) *and* (6.2). *Let $\widetilde{G}(t)$ and $\widetilde{H}(t)$ be as in* (6.25) *and* (6.26) *respectively; moreover, suppose that $\widetilde{G}$ admits a factorization $G_{-}(t)t^{\aleph}G_{+}(t)$ in $\mathfrak{X}(\Gamma)$. Then equation* (6.3) *has solutions in $\mathfrak{X}(\Gamma)$ and they are given by*

$$\varphi(t) = \frac{h(t) - 2g(t)(S\varphi_k)(t)}{f(t)}, \qquad k = 1, 2,$$

where $k = 1$ if W is a commutative Carleman shift operator, and $k = 2$ if W is anti-commutative. In addition, for $(S\varphi_k)(t) = \Psi_k^{+}(t) + \Psi_k^{-}(t)$, we have:

($\aleph < 0$) *In this case*

$$\Psi_k^{+}(t) = G_{+}^{-1}(t)t^{-\aleph}P_{+}G_{-}^{-1}(t)H(t) + G_{+}^{-1}(t)t^{-\aleph}\rho(t), \tag{6.28}$$

$$\Psi_k^{-}(t) = G_{-}(t)P_{-}G_{-}^{-1}(t)H(t) + G_{-}(t)\rho(t), \tag{6.29}$$

where $\rho(t) = a_{-\aleph-1}t^{-\aleph-1} + a_{-\aleph-2}t^{-\aleph-2} + \cdots + a_0$.

($\aleph \geq 0$) *For this case,* $\Psi_k^{\pm}$ *are as in* (6.28) *and* (6.29) *with* $\rho(t) \equiv 0$, *and in addition it is necessary that*

$$\int_\Gamma G_-^{-1}(\tau)\big(f_\alpha(\tau) - [fg]_k(\tau)\big)[fh]_k(\tau)\tau^{-\kappa} = 0, \qquad \kappa = 1, \dots, \aleph,\ k = 1, 2.$$

Proof. From Theorem 6.1.3 we know that (6.3) has solutions if and only if (6.14) does. Furthermore, the solutions of (6.3) are given by (6.15). Thus, we will compute the solutions of (6.14). We will use the Riemann boundary value problem associated to (6.27): Find a sectionally analytic function vanishing at infinity and satisfying

$$\Psi_k^-(t) - G(t)\Psi_k^+(t) = -H(t) \tag{6.30}$$

on Γ, where $\widetilde{G}(t)$ and $\widetilde{H}(t)$ are defined in (6.25) and (6.26) respectively. Since $\widetilde{G}(t)$ admits a factorization in $\mathfrak{X}(\Gamma)$, we are able, as in Chapter 5, to use Theorem 3.3.1. Thus, the solutions of problem (6.30) read as follows:

(1) *Case* $\aleph < 0$. In this case the solutions are given by

$$\Psi_k^+(t) = G_+^{-1}t^{-\aleph}P_+G_-^{-1}(t)H(t) + G_+^{-1}(t)t^{-\aleph}\rho(t), \tag{6.31}$$

$$\Psi_k^-(t) = G_-(t)P_-G_-^{-1}(t)H(t) + G_-(t)\rho(t), \tag{6.32}$$

where $\rho(t) = a_{-\aleph-1}t^{-\aleph-1} + a_{-\aleph-2}t^{-\aleph-2} + \dots + a_0$. The second term on the right-hand side of (6.31) and (6.32) is the general solution of the homogeneous ($H(t) \equiv 0$) Riemann problem (6.30), and the first term is a particular solution of the corresponding inhomogeneous problem (6.30).

(2) *Case* $\aleph \geq 0$. For this case, $\Psi_k^{\pm}$ are as in (6.31) and (6.32), and $\rho(z) \equiv 0$. In addition, it is necessary that

$$\int_\Gamma G_-^{-1}(\tau)\big(f_\alpha(\tau) - [fg]_k(\tau)\big)[fh]_k(\tau)\tau^{-\kappa} = 0, \qquad \kappa = 1, \dots, \aleph,\ k = 1, 2.$$

If $\aleph = 0$, then problem (6.30) has a unique solution. This completes the proof. ∎

6.3. The Fredholmness of the singular integral operator with shift associated to (6.3)**.** Notice that in the operator theory approach, to equation (6.3) is associated the singular integral operator

$$\mathcal{S} := fI + gS + gWS : \mathfrak{X}(\Gamma) \to \mathfrak{X}(\Gamma).$$

The projection method used before will allow us to establish a Fredholmness criterion for the operator $\mathcal{S}$ on $\mathfrak{X}(\Gamma)$ by means of a nonexplicit equivalence operator relation.

THEOREM 6.3.1. *Let* Γ *be a Lyapunov curve and let* $\mathfrak{X}(\Gamma)$ *be a BFS satisfying* (2.2)–(2.4), (2.6) *and* (6.2). *Then the operator* $\mathcal{S} := fI + gS + gWS$ *is a* Φ*-operator on* $\mathfrak{X}(\Gamma)$ *if and only if* $(f_\alpha - [fg]_k)(f_\alpha + [fg]_k)^{-1} \in \mathcal{G}(C(\Gamma))$. *The functions* f_α *and* $[fg]_k$ *are given in* (6.6) *and* (6.7) *respectively, with* $k = 1$ *if* W *anti-commutes and* $k = 2$ *if* W *commutes. Moreover, under the presence of the Fredholm property,* $\operatorname{Ind}\mathcal{S} = \operatorname{ind}(f_\alpha - [fg]_k)(f_\alpha + [fg]_k)^{-1} =: \aleph$.

Proof. From Theorem 6.1.3, we know that equation (6.3) is solvable if and only if the equation

$$f_\alpha(t)\varphi_k(t) + [fg]_k(t)(S\varphi_k)(t) = [fh]_k(t) \qquad (k = 1, 2) \tag{6.33}$$

is solvable, where f_α, $[fg]_k(t)$ and $[fh]_k(t)$ are given in (6.6)–(6.8) respectively, and $k = 1$ or $k = 2$ depending on the commutative nature of the shift operator W. Even more, from (6.15), the dimensions of the sets of solutions of (6.3) and of (6.33) coincide.

On the other hand, by using the functions G and H defined in (6.25) and (6.26), equation (6.33) can be rewritten as

$$P_+\varphi_k(t) + G(t)P_-\varphi_k(t) = H(t).$$

Therefore, the regularity properties of the operators $P_+ + GP_-$ and $\mathcal{S}$ coincide.

Finally, from Theorem 5.1.1, the operator $P_+ + GP_-$ is a Φ-operator with Fredholm index $\aleph = \operatorname{ind} G$ if and only if $G \in \mathcal{G}(C(\Gamma))$, and so this is transferred to the operator $\mathcal{S}$; i.e., we conclude that $f_\alpha \pm [fg]_k \in \mathcal{G}(C(\Gamma))$ (with $\operatorname{ind} G = \aleph$) if and only if $\mathcal{S}$ is a Φ-operator with $\operatorname{Ind} \mathcal{S} = \aleph$. ∎

In a similar way, the Fredholmness of the operator $\mathcal{S}$ with piecewise continuous and factorizable essentially bounded functions as coefficients can be proved by using Theorems 5.2.1 and 4.1.3 respectively.

Theorem 6.3.2. *Let Γ be a Lyapunov curve and let $\mathcal{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.4), (2.6), (6.2) *and Axioms* 1–2. *Then $\mathcal{S} := fI + gS + gWS$ is a Φ-operator on $\mathcal{X}(\Gamma)$ if $f_\alpha \pm [fg]_k \in \mathcal{G}(\mathrm{PC}(\Gamma))$ and the function $(f_\alpha + [fg]_k)(f_\alpha - [fg]_k)^{-1}$ is $\mathcal{X}(\Gamma)$-nonsingular with discontinuity points $t_1, \ldots, t_n$, at which the curve Γ has at least one-sided tangents. The functions f_α and $[fg]_k$ are given in* (6.6) *and* (6.7) *respectively, and $k = 1$ if W anti-commutes and $k = 2$ if W commutes. In this case,*

$$\operatorname{Ind} \mathcal{S} = \operatorname{ind}(f_\alpha + [fg]_k)(f_\alpha - [fg]_k)^{-1}.$$

The condition $f_\alpha \pm [fg]_k \in \mathcal{G}(\mathrm{PC}(\Gamma))$ is also necessary for the Fredholmness of $\mathcal{S}$. On the other hand, if the index functions α and β of the space $\mathcal{X}(\Gamma)$ coincide at the points t_k of discontinuity of the coefficients $(f_\alpha + [fg]_k)(f_\alpha - [fg]_k)^{-1}$, then the $\mathcal{X}(\Gamma)$-nonsingularity of $(f_\alpha + [fg]_k)(f_\alpha - [fg]_k)^{-1}$ is necessary as well.

Theorem 6.3.3. *Let Γ be a Lyapunov curve and let $\mathcal{X}(\Gamma)$ be a BFS satisfying* (2.2)–(2.6) *and* (6.2). *Then $\mathcal{S} := fI + gS + gWS$ is a Φ-operator on $\mathcal{X}(\Gamma)$ if and only if $f_\alpha \pm [fg]_k \in \mathcal{G}(L^\infty(\Gamma))$ and $\widetilde{G} = (f_\alpha + [fg]_k)(f_\alpha - [fg]_k)^{-1}$ admits a factorization* (2.8) *in $\mathcal{X}(\Gamma)$ with $\operatorname{ind} \widetilde{G} = \aleph$. The functions f_α and $[fg]_k$ are given in* (6.6) *and* (6.7) *respectively, and $k = 1$ if W anti-commutes and $k = 2$ if W commutes. Moreover, under the presence of the Fredholm property,* $\operatorname{Ind} \mathcal{S} = -\aleph$.

7. The variable exponent Lebesgue spaces case

The variable exponent Lebesgue spaces are one of the most well-known Banach function spaces. Their fundamental study have been growing rapidly during the last two decades, apart from mathematical interest, due to possible applications to image restoration and to models with the so-called nonstandard local growth in fluid mechanics and elasticity theory; see for instance [6, 12] and the references therein.

In this chapter we are going to show that all the results given in the previous chapters are valid in variable exponent Lebesgue spaces. To do so we will show that conditions (2.2)–(2.6), (6.2) and Axioms 1–2 imposed on $\mathcal{X}(\Gamma)$ are, in fact, well-known results on variable exponent Lebesgue spaces.

The space $L^{p(\cdot)}(\Gamma)$ over a Jordan curve Γ of finite length ℓ is defined as the set of all measurable complex-valued functions f on Γ such that $I_p(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$I_p(f) = \int_\Gamma |f(t)|^{p(t)}\,|dt| = \int_0^\ell |f(t(s))|^{p(t(s))}\,ds.$$

This set becomes a Banach space with respect to the (Luxemburg) norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : I_p(f/\lambda) \le 1\}.$$

For the fundamental properties of these spaces we refer to [10, 11].

Assume $p : \Gamma \to [1, \infty)$ is a measurable function with

$$1 < p_- = \operatorname{ess\,inf} p(t) \le p(t) \le p_+ = \operatorname{ess\,sup} p(t) < \infty, \qquad t \in \Gamma. \tag{7.1}$$

We will need the following condition on $p(t)$:

$$|p(t_1) - p(t_2)| \le \frac{A}{-\ln|t_1 - t_2|}, \qquad |t_1 - t_2| \le 1/2,\ t_1, t_2 \in \Gamma, \tag{7.2}$$

where $A > 0$ does not depend on t_1 and t_2, or on the function $p_*(s) = p(t(s))$:

$$|p_*(s_1) - p_*(s_2)| \le \frac{A}{-\ln|s_1 - s_2|}, \qquad |s_1 - s_2| \le 1/2,\ s_1, s_2 \in [0, \ell]. \tag{7.3}$$

Since $|t(s_1) - t(s_2)| \le |s_1 - s_2|$, condition (7.2) always implies (7.3). Conversely, (7.3) implies (7.2) if there exists $\lambda > 0$ such that $|s_1 - s_2| \le c|t(s_1) - t(s_2)|^\lambda$ with some $c > 0$. Therefore, conditions (7.2) and (7.3) are equivalent on Jordan curves. Moreover, this is valid on general curves satisfying the so-called *chord condition*.

On the suitability of $L^{p(\cdot)}(\Gamma)$ for our results. In order to establish the validity of assumptions (2.2)–(2.6) and (6.2), as well as of Axioms 1 and 2 on the spaces $L^{p(\cdot)}(\Gamma)$, we will assume that properties (7.1)–(7.3) of the exponent $p(t)$ hold.

(2.2) $C(\Gamma) \subset L^{p(\cdot)}(\Gamma) \subset L^1(\Gamma)$. This follows from (7.1).
(2.3) $\|af\|_{L^{p(\cdot)}(\Gamma)} \leq \sup_{t\in\Gamma} |a(t)| \cdot \|f\|_{L^{p(\cdot)}(\Gamma)}$ for $a \in L^\infty(\Gamma)$. Evident.
(2.4) The operator S is bounded in $L^{p(\cdot)}(\Gamma)$. This is proved in [33, Theorem 2].
(2.5) $L^{p(\cdot)}(\Gamma)$ is reflexive. Proved in [34, Corollary 2.7].
(2.6) $C^\infty(\Gamma)$ is dense in $L^{p(\cdot)}(\Gamma)$. Given in [33, Theorem 4.1].
(6.2) $\alpha(t)$ induces a bounded shift operator $(W\varphi)(t) = \varphi(\alpha(t))$ on $L^{p(\cdot)}(\Gamma)$. In fact, for a shift function α as on p. 30 and an exponent function p satisfying (7.1)–(7.3), in [41, Lemma 2] it was proved that the functions $p_\alpha(t) := p(\alpha(t))$ and $\overline{p}_\alpha(t) := \max(p(t), p_\alpha(t))$ satisfy (7.1)–(7.3) as well. Also, in [41] it is shown that $L^{p(\cdot)}(\Gamma) \cap L^{p_\alpha(\cdot)}(\Gamma) = L^{\overline{p}_\alpha(\cdot)}(\Gamma)$, therefore the operators W and S are bounded on $L^{\overline{p}_\alpha(\cdot)}(\Gamma)$.

Axiom 1 in $L^{p(\cdot)}(\Gamma)$ is proved in [33, Theorem 2]. For Axiom 2, the embedding $L^{p(\cdot)}(\Gamma, |t-t_0|^\gamma) \subset L^1(\Gamma)$ if $\gamma < 1/q(t_0)$ follows from the Hölder inequality on $L^{p(\cdot)}(\Gamma)$, and the denseness of $C^\infty(\Gamma)$ in $L^{p(\cdot)}(\Gamma, |t-t_0|^\gamma)$ for $t_0 \in \Gamma$ is a particular case from [33, Theorem 4.1].

On the other hand, the dual space of $L^{p(\cdot)}(\Gamma)$ is $L^{p'(\cdot)}(\Gamma)$ [34, Corollary 2.7], where $p'(t) = \frac{p(t)}{p(t)-1}$. [34, Corollary 2.12] asserts that $L^{p(\cdot)}(\Gamma)$ is separable, so the adjoint operator of S is well-defined in $L^{p'(\cdot)}(\Gamma)$. The denseness of the rational functions in variable exponent Lebesgue spaces is given in [33, Theorem 4.1], so the complementary projections $P_\pm$ are well-defined; hence so are the subspaces

$$L_+^{p(\cdot)}(\Gamma) := P_+ L^{p(\cdot)}(\Gamma), \qquad \mathring{L}_-^{p(\cdot)}(\Gamma) := P_- L^{p(\cdot)}(\Gamma), \qquad L_-^{p(\cdot)}(\Gamma) := \mathring{L}_-^{p(\cdot)}(\Gamma) \dotplus \mathbb{C}.$$

Now, we can introduce a factorization for an invertible function $a \in L^\infty(\Gamma)$ in $L^{p(\cdot)}(\Gamma)$. A function $a \in \mathcal{G}(L^\infty(\Gamma))$ admits a *factorization in* $L^{p(\cdot)}(\Gamma)$ if it can be written in the form

$$a(t) = a_-(t) t^{\aleph} a_+(t), \qquad \text{a.e. on } \Gamma,$$

where $\aleph \in \mathbb{Z}$ and

(i) $a_- \in L_-^{p(\cdot)}(\Gamma)$, $a_-^{-1} \in L_-^{p'(\cdot)}(\Gamma)$, $a_+ \in L_+^{p'(\cdot)}(\Gamma)$, $a_+^{-1} \in L_+^{p(\cdot)}(\Gamma)$,
(ii) the operator $a_+^{-1} S a_+ I$ is bounded in $L^{p(\cdot)}(\Gamma)$.

The integer $\aleph$ is referred to as the *index* of the function a and is denoted by $\operatorname{ind} a$. We can prove that the number $\aleph$ is uniquely determined.

We point out that for this case we can use the Smirnov class $E^p(D)$ instead of $E^1(D)$, because [30, Theorem 3.3] ensures that if S is bounded from $L^{p(\cdot)}(\Gamma)$ to $L^p(\Gamma)$ $(1 < p < \infty)$, then for every $\varphi \in L^{p(\cdot)}(\Gamma)$, the corresponding analytic function of Cauchy integral type whose nontangential limit is φ belongs to $E^p(D)$.

Using this factorization and due to the fact that $L^{p(\cdot)}(\Gamma)$ satisfies all the assumptions imposed on $\mathfrak{X}(\Gamma)$, all the results given in the previous chapters are valid, with obvious modifications, in this case.

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston, MA, 1988.

[2] A. Böttcher and Y. I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Progr. Math. 154, Birkhäuser, Basel, 1997.

[3] L. P. Castro and E. M. Rojas, *Explicit solutions of Cauchy singular integral equations with weighted Carleman shift*, J. Math. Anal. Appl. 371 (2010), 128–133.

[4] L. P. Castro and E. M. Rojas, *On the solvability of singular integral equations with reflection on the unit circle*, Integral Equations Operator Theory 70 (2011), 63–99.

[5] L. P. Castro, E. M. Rojas, S. Saitoh, N. M. Tuan and P. D. Tuan, *Solvability of singular integral equations with rotations and degenerate kernels in the vanishing coefficient case*, Anal. Appl. (Singapore) 13 (2015), 1–21.

[6] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66 (2006), 1383–1406.

[7] L. H. Chuan, N. V. Mau and N. M. Tuan, *On a class of singular integral equations with the linear fractional Carleman shift and the degenerate kernel*, Complex Var. Elliptic Equations 53 (2008), 117–137.

[8] L. H. Chuan and N. M. Tuan, *On singular integral equations with Carleman shifts in the case of the vanishing coefficient*, Acta Math. Vietnam. 28 (2003), 319–333.

[9] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*, Oper. Theory Adv. Appl. 3, Birkhäuser, 1981.

[10] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2013, 312 pp.

[11] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math. 2017, Springer, 2011.

[12] L. Diening and M. Růžička, *Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. 563 (2003), 197–220.

[13] F. D. Gakhov, *Riemann boundary value problem*, Mat. Sb. 2 (1937), 673–683.

[14] F. D. Gakhov, *Boundary Value Problems*, Dover Publ., 1990.

[15] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations: Introduction*, Oper. Theory Adv. Appl. 53, Birkhäuser, 1992.

[16] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations: General Theory and Applications*, Oper. Theory Adv. Appl. 54, Birkhäuser, 1992.

[17] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Transl. Math. Monogr. 26, Amer. Math. Soc., 1969.

[18] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Teubner, Leipzig, 1912.

[19] N. Karapetiants and S. Samko, *Equations with Involutive Operators*, Birkhäuser Boston, Boston, MA, 2001.

[20] A. Yu. Karlovich, *Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces*, Integral Equations Operator Theory 32 (1998), 436–481.

[21] A. Yu. Karlovich, *Remark on the boundedness of the Cauchy singular integral operator on variable Lebesgue spaces with radial oscillating weights*, J. Function Spaces Appl. 7 (2009), 301–311.

[22] A. Yu. Karlovich and A. K. Lerner, *Commutators of singular integrals on generalized L^p spaces with variable exponent*, Publ. Mat. 49 (2005), 111–125.

[23] A. Yu. Karlovich and I. M. Spitkovsky, *The Cauchy singular integral operator on weighted variable Lebesgue spaces*, in: Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation, Oper. Theory Adv. Appl. 236, Birkhäuser, 2014, 275–291.

[24] G. Khuskivadze, V. Kokilashvili and V. Paatashvili, *Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings*, Mem. Differential Equations Math. Phys. 14 (1998), 195 pp.

[25] V. Kokilashvili and V. Paatashvili, *The Dirichlet problem for harmonic functions in the Smirnov class with variable exponent*, Georgian Math. J. 14 (2007), 289–299.

[26] V. Kokilashvili and V. Paatashvili, *The Riemann–Hilbert problem in weighted classes of Cauchy type integrals with density from $L^{p(\cdot)}(\Gamma)$*, Complex Anal. Oper. Theory 2 (2008), 569–591.

[27] V. Kokilashvili and V. Paatashvili, *The Riemann–Hilbert problem in a domain with piecewise-smooth boundaries in weighted classes of Cauchy type integrals with density from variable Lebesgue spaces*, Georgian Math. J. 16 (2009), 289–299.

[28] V. Kokilashvili and V. Paatashvili, *The Dirichlet problem for harmonic functions from variable exponent Smirnov classes in domains with piecewise smooth boundary*, J. Math. Sci. 172 (2011), 401–421.

[29] V. Kokilashvili and V. Paatashvili, *Boundary Value Problems for Analytic and Harmonic Functions on Nonstandard Banach Function Spaces*, Nova Sci. Publ., 2012.

[30] V. Kokilashvili, V. Paatashvili and S. Samko, *Boundary value problems for analytic functions in the class of Cauchy-type integrals with density in $L^{p(\cdot)}(\Gamma)$*, Bound. Value Problems 2005, no. 1, 43–71.

[31] V. Kokilashvili, N. Samko and S. Samko, *Singular operators in variable spaces $L^{p(\cdot)}(\Omega, \rho)$ with oscillating weights*, Math. Nachr. 280 (2007), 1145–1156.

[32] V. Kokilashvili and S. Samko, *Singular integral equations in the Lebesgue spaces with variable exponent*, Proc. A. Razmadze Math. Inst. 131 (2003), 61–78.

[33] V. Kokilashvili and S. Samko, *Singular integral in weighted Lebesgue spaces with variable exponent*, Georgian Math. J. 10 (2003), 145–156.

[34] O. Kovačik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. 41 (1991), 592–618.

[35] V. G. Kravchenko and G. S. Litvinchuk, *Introduction to the Theory of Singular Integral Operators with Shift*, Kluwer, Amsterdam, 1994.

[36] G. S. Litvinchuk, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift*, Math. Appl. 523, Kluwer, Dordrecht, 2000.

[37] G. S. Litvinchuk and I. M. Spitkovsky, *Factorization of Measurable Matrix Functions*, Oper. Theory Adv. Appl. 25, Birkhäuser, 1987.

[38] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators*, Springer, Berlin, 1986.

[39] N. I. Muskhelishvili, *Applications of integrals of Cauchy type to a class of singular integral equations*, Trav. Inst. Math. Tbilissi 10 (1941), 1–43 (in Russian).

[40] N. I. Muskhelishvili, *Singular Integral Equations*, Nauka, Moscow, 1968 (in Russian).

[41] V. Paatashvili, *Haseman's problems in classes of functions representable by the Cauchy type integral with density from* $L^{p(\cdot)}(\Gamma;\rho)$, Mem. Differential Equations Math. Phys. 50 (2010), 139–162.

[42] J. Plemelj, *Ein Ergänzungssatz zur Cauchyschen Integraldarstellung analytischer Funktionen, Randwerte betreffend*, Monatsh. Math. Phys. 19 (1908), 205–210.

[43] H. Poincaré, *Leçons de mécanique céleste professées à la Sorbonne. Tome III: Théorie des marées*, rédigée par R. Fichot, Gauthier-Villars, Paris, 1910.

[44] B. Riemann, *Grundlagen für eine Allgemeine Theorie der Funktionen Einer Veränderlichen Komplexen Grösse*, Werke, Leipzig, 1867.

[45] I. B. Simonenko, *The Riemann boundary value problem for n pairs of functions with measurable coefficients and its application to the investigation of singular integral operators in the spaces* L^p *with weight*, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 277–306 (in Russian).

[46] I. B. Simonenko, *Some general questions in the theory of the Riemann boundary value problem*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 1138–1146 (in Russian); English transl.: Math. USSR Izv. 2 (1968), 1091–1099.

[47] F. G. Tricomi, *Integral Equations*, Dover Publ., 1985.

[48] N. M. Tuan, *On a class of singular integral equations with rotation*, Acta Math. Vietnam. 21 (1996), 201–211.

[49] N. P. Vekua, A. G. Gibbs and J. H. Ferziger, *Systems of Singular Integral Equations*, Noordhoff, Groningen, 1967.