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THE AFFINENESS CRITERION FOR QUANTUM HOM-YETTER–DRINFEL'D MODULES

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Abstract. Quantum integrals associated to quantum Hom-Yetter–Drinfel'd modules are defined, and the affineness criterion for quantum Hom-Yetter–Drinfel'd modules is proved in the following form. Let (H,α) be a monoidal Hom-Hopf algebra, (A,β) an (H,α)-Hom-bicomodule algebra and $B=A^{\mathrm{co}\,H}$. Under the assumption that there exists a total quantum integral $\gamma: H\to \mathrm{Hom}(H,A)$ and the canonical map $\beta: A\otimes_B A\to A\otimes H$, $a\otimes_B b\mapsto S^{-1}(b_{[1]})\alpha(b_{[0][-1]})\otimes \beta^{-1}(a)\beta(b_{[0][0]})$, is surjective, we prove that the induction functor $A\otimes_B - : \widetilde{\mathscr{H}}(\mathscr{M}_k)_B\to {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ is an equivalence of categories.

1. Introduction. Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov [19] as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of multiplication is replaced by the Hom-associativity, and similarly for Hom-coassociativity. They also defined the structures of Hom-bialgebras and Hom-Hopf algebras, and described some of their properties in [21] by extending properties of ordinary bialgebras and Hopf algebras. Recently, many properties and structures of Hom-Hopf algebras have been developed: see [3]–[7], [9], [11]–[17], [27]–[31] and references cited therein.

Caenepeel and Goyvaerts [1] studied Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively; these are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. They also introduced the notion of Hom-Hopf modules and proved the fundamental theorem on Hom-Hopf modules, and also presented the Hom-Hopf algebraic structures of the enveloping algebras of monoidal Hom-Lie algebras.

The category of Yetter–Drinfel'd modules is an important category of modules in the theory of Hopf algebras. Under favourable conditions (e.g., if H is a Hopf algebra with a bijective antipode), the category of Yetter–

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Drinfel'd modules is indeed a braided monoidal category by the Drinfel'd double construction. Makhlouf and Panaite [18] defined Yetter–Drinfel'd modules over Hom-bialgebras, and showed that Yetter–Drinfel'd modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang–Baxter equation.

Apart from Makhlouf and Panaite's work, Liu and Shen [16] studied Hom-Yetter–Drinfel'd modules over monoidal Hom-bialgebras, and showed that the category of Hom-Yetter–Drinfel'd modules is a braided monoidal category. Also, Chen and Zhang [7] defined the category of Hom-Yetter–Drinfel'd modules in a slightly different way to [16], and showed that it is a full monoidal subcategory of the left center of the left Hom-module category. Later, You and Wang [31] extended the notion of Hom-Yetter–Drinfel'd modules of generalized Hom-Yetter–Drinfel'd modules.

Total integral is an important notion in representation theories. Chen and Zhang [4] introduced integrals of monoidal Hom-Hopf algebras and investigated the existence and uniqueness of integrals for finite-dimensional monoidal Hom-Hopf algebras. The first named author and Chen [11] introduced the notion of relative Hom-Hopf modules and proved that the forgetful functor F from the category of relative Hom-Hopf modules to the category of right (A,β)-Hom-modules has a right adjoint. In [13], the notion of total integral was introduced for any Hom-comodule algebra (A,β) over a monoidal Hom-Hopf algebra (H,α), which has strong ties both to $\widetilde{\mathscr{H}}(\mathscr{M}_k)^H$ (i.e., the corepresentation of (H,α)) and to the representation of the pair (H,A) (i.e., the category of relative Hom-Hopf modules $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$), and the well-known necessary and sufficient criterion for the existence of a total integral was presented.

Menini and Militaru [22] interpreted the criterion for the existence of a total integral with the help of forgetful functors $F:\mathcal{M}(H)_A^C\to\mathcal{M}_A$ and $G:\mathcal{M}(H)_A^C\to\mathcal{M}^C$. Inspired by their ideas, we introduce the category of left-right quantum Hom-Yetter–Drinfel'd modules ${}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$, which can be viewed as both a category containing the category ${}^H\mathcal{HYD}_H$ of Hom-Yetter–Drinfel'd modules and a quantization of the category of relative Hom-Hopf modules; so it is necessary to study quantum Hom-Yetter–Drinfel'd modules, and in this paper we investigate the criterion for the existence of a total integral of such modules.

The paper is organized as follows. In Section 2, we recall some definitions and properties relating to monoidal Hom-Hopf algebras which are needed later. In Section 3, we introduce the concept of quantum Hom-Yetter–Drinfel'd modules in the sense of [12], which can be interpreted as special Doi Hom-Hopf modules. In Section 4, quantum integrals associated to quantum Hom-Yetter–Drinfel'd modules are defined. Then we prove the affineness criterion for quantum Hom-Yetter–Drinfel'd-modules (Theorem 4.9).

Throughout this paper we freely use the Hopf algebra and coalgebra terminology introduced in [2], [10], [23], [25] and [26].

2. Preliminaries. In this section we recall some basic definitions and results. Throughout, all algebraic systems are supposed to be over a commutative ring k. The reader is referred to [1] and [11] as general references for Hom-structures.

Let $\mathcal{C}$ be a category. Then there is a new category $\mathscr{H}(\mathcal{C})$ as follows: Objects are couples (M,μ) with $M\in\mathcal{C}$ and $\mu\in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $f:(M,\mu)\to(N,\nu)$ is a morphism $f:M\to N$ in $\mathcal{C}$ such that $\nu\circ f=f\circ\mu$.

Let $\mathscr{M}_k$ denote the category of k-modules. Then $\mathscr{H}(\mathscr{M}_k)$ will be called the *Hom-category* associated to $\mathscr{M}_k$. If $(M,\mu)\in\mathscr{M}_k$, then $\mu:M\to M$ is obviously a morphism in $\mathscr{H}(\mathscr{M}_k)$. It is easy to show that $\widetilde{\mathscr{H}}(\mathscr{M}_k)=(\mathscr{H}(\mathscr{M}_k),\otimes,(I,I),\widetilde{a},\widetilde{l},\widetilde{r}))$ is a monoidal category by [1, Proposition 1.1]. The tensor product of (M,μ) and (N,ν) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is given by the formula $(M,\mu)\otimes(N,\nu)=(M\otimes N,\mu\otimes\nu)$, and for $(M,\mu),(N,\nu),(P,\pi)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$, the associativity and unit constraints are given by

$$\widetilde{a}_{M,N,P}((m\otimes n)\otimes p)=\mu(m)\otimes(n\otimes\pi^{-1}(p)),$$
$$\widetilde{l}_M(x\otimes m)=\widetilde{r}_M(m\otimes x)=x\mu(m).$$

An algebra in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ will be called a *monoidal Hom-algebra*.

DEFINITION 2.1. A *monoidal Hom-algebra* is an object $(A,\alpha)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $m_A:A\otimes A\to A$ and an element $1_A\in A$ such that

$$\alpha(ab)=\alpha(a)\alpha(b),\quad \alpha(1_A)=1_A,$$
$$\alpha(a)(bc)=(ab)\alpha(c),\qquad a1_A=1_Aa=\alpha(a),$$

for all $a,b,c\in A$. Here we use the notation $m_A(a\otimes b)=ab$.

DEFINITION 2.2. A *monoidal Hom-coalgebra* is an object $(C,\gamma)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with k-linear maps $\Delta:C\to C\otimes C$, $\Delta(c)=c_{(1)}\otimes c_{(2)}$ (summation implicitly understood) and $\gamma:C\to C$ such that

$$\Delta(\gamma(c))=\gamma(c_{(1)})\otimes\gamma(c_{(2)}),\quad \varepsilon(\gamma(c))=\varepsilon(c),$$

and

$$\gamma^{-1}(c_{(1)})\otimes c_{(2)(1)}\otimes c_{(2)(2)}=c_{(1)(1)}\otimes c_{(1)(2)}\otimes\gamma^{-1}(c_{(2)}),$$
$$\varepsilon(c_{(1)})c_{(2)}=\varepsilon(c_{(2)})c_{(1)}=\gamma^{-1}(c),$$

for all $c\in C$.

DEFINITION 2.3. A *monoidal Hom-bialgebra* $H=(H,\alpha,m,\eta,\Delta,\varepsilon)$ is a bialgebra in the category $\widetilde{\mathscr{H}}(\mathscr{M}_k)$. This means that (H,α,m,η) is a monoidal Hom-algebra, (H, Δ, α) is a monoidal Hom-coalgebra, and Δ and ε are morphisms of monoidal Hom-algebras, that is,

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(1_H) = 1_H.$$

DEFINITION 2.4. A *monoidal Hom-Hopf algebra* is a monoidal Hom-bialgebra (H, α) together with a linear map $S : H \to H$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$S * I = I * S = \eta\varepsilon, \qquad S\alpha = \alpha S.$$

DEFINITION 2.5. Let (A, α) be a monoidal Hom-algebra. A *right* (A, α)*-Hom-module* is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ consisting of a k-module and a linear map $\mu : M \to M$ together with a morphism $\psi : M \otimes A \to M$, $\psi(m \cdot a) = m \cdot a$, in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab), \qquad m \cdot 1_A = \mu(m),$$

for all $a \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f : (M, \mu) \to (N, \nu)$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is said to be *right* A*-linear* if it preserves the A-action, that is, $f(m \cdot a) = f(m) \cdot a$. We denote by $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ the category of right (A, α)-Hom-modules and A-linear morphisms.

DEFINITION 2.6. Let (C, γ) be a monoidal Hom-coalgebra. A *right* (C, γ)*-Hom-comodule* is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$ $(\rho_M(m) = m_{[0]} \otimes m_{[1]})$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}),$$
$$m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m),$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right (C, γ)-Hom-comodule are defined in the obvious way. The category of right (C, γ)-Hom-comodules will be denoted by $\widetilde{\mathscr{H}}(\mathscr{M}_k)^C$.

DEFINITION 2.7. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a *right* (H, α)*-Hom-comodule algebra* if (A, β) is a right (H, α) Hom-comodule with a coaction $\rho^r_A : A \to A \otimes H$, $\rho^r_A(a) = a_{[0]} \otimes a_{[1]}$, such that

$$\rho^r_A(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}, \qquad \rho^r_A(1_A) = 1_A \otimes 1_H,$$

for all $a, b \in A$.

Definition 2.8. Let (H,α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A,β) is called a *left* (H,α)*-Hom-comodule algebra* if (A,β) is a right (H,α) Hom-comodule with a coaction $\rho^l_A : A \to H\otimes A$, $\rho_A(a) = a_{[-1]}\otimes a_{[0]}$, such that

$$\rho^l_A(ab) = a_{[-1]}b_{[-1]}\otimes a_{[0]}b_{[0]}, \quad \rho^l_A(1_A) = 1_H\otimes 1_A,$$

for all $a,b\in A$.

Definition 2.9. Let (H,α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A,β) is called a *bicomodule algebra* if (A,β) is not only a right (H,α)-Hom-comodule algebra with a coaction ρ_A, but also a left (H,α)-Hom-comodule algebra with a coaction ρ^l_A such that

$$\alpha^{-1}(a_{[-1]})\otimes a_{[0][0]}\otimes a_{[0][1]} = a_{[0][-1]}\otimes a_{[0][0]}\otimes \alpha^{-1}(a_{[1]})$$

for all $a\in A$.

3. Quantum Hom-Yetter–Drinfel'd modules

Definition 3.1. Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode, and (A,β) an (H,α)-Hom-bicomodule algebra. A *quantum Hom-Yetter–Drinfel'd module* (M,μ) is a right (A,β)-Hom-module which is also a left (H,α)-Hom-comodule with the coaction structure $\rho_M : M\to H\otimes M$ defined by $\rho_M(m) = m_{[-1]}\otimes m_{[0]}$, and satisfies the following compatibility condition: for all $m\in M$ and $a\in A$,

$$m_{[-1]}a_{[-1]}\otimes m_{[0]}\cdot a_{[0]} = a_{[1]}(\mu^{-1}(m)\cdot a_{[0]})_{[-1]}\otimes \mu((\mu^{-1}(m)\cdot a_{[0]})_{[0]}). \tag{3.1}$$

We denote by ${}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ the category of left-right quantum Hom-Yetter–Drinfel'd modules, morphisms being right (A,β)-linear left (H,α)-colinear maps.

Proposition 3.2. *Let (M,μ) be a right (A,β)-Hom-module and a left (H,α)-Hom-comodule. Then the compatibility relation* (3.1) *is equivalent to*

$$\rho(m\cdot a) = S^{-1}(a_{[1]})(\alpha^{-1}(m_{[-1]})a_{[0][-1]})\otimes m_{[0]}\cdot\beta(a_{[0][0]}) \tag{3.2}$$

for all $a\in A$ and $m\in M$.

Proof. (3.1)⇒(3.2): For $h\in H$ and $m\in M$, we have

$$\begin{aligned}
&S^{-1}(a_{[1]})(\alpha^{-1}(m_{[-1]})a_{[0][-1]})\otimes m_{[0]}\cdot\beta(a_{[0][0]})\\
&\quad= S^{-1}(a_{[1]})(\alpha^{-1}(m_{[-1]})a_{[0][-1]})\otimes \mu(\mu^{-1}(m_{[0]})\cdot a_{[0][0]})\\
&\quad\overset{(3.1)}{=} S^{-1}(a_{[1]})\big(a_{[0][1]}(\mu^{-2}(m)\cdot a_{[0][0]})_{[-1]}\big)\otimes \mu\big(\mu((\mu^{-2}(m)\cdot a_{[0][0]})_{[0]})\big)\\
&\quad= (S^{-1}(a_{[1](2)})a_{[1](1)})(\mu^{-1}(m)\cdot a_{[0]})_{[-1]}\otimes \mu\big(\mu((\mu^{-2}(m)\cdot \beta^{-1}(a_{[0]}))_{[0]})\big)\\
&\quad= (m\cdot a)_{[-1]}\otimes (m\cdot a)_{[0]}.
\end{aligned}$$

(3.2)⇒(3.1): We compute

$$
\begin{aligned}
&a_{[1]}(\mu^{-1}(m)\cdot a_{[0]})_{[-1]}\otimes\mu((\mu^{-1}(m)\cdot a_{[0]})_{[0]})\\
&\overset{(3.2)}{=} a_{[1]}\big(S^{-1}(a_{[1]})(\alpha^{-2}(m_{[-1]})a_{[0][0][-1]})\big)\otimes\mu\big((\mu^{-2}(m_{[0]})\cdot\beta(a_{[0][0][0]}))\big)\\
&= (\alpha^{-1}(a_{[1]})S^{-1}(a_{[1]}))\big(\alpha^{-1}(m_{[-1]})\alpha(a_{[0][0][-1]})\big)\\
&\quad\otimes\mu\big((\mu^{-1}(m_{[0]})\cdot\beta(a_{[0][0][0]}))\big)\\
&= (a_{[1](2)}S^{-1}(a_{[1](1)}))\big(\alpha^{-1}(m_{[-1]})a_{[0][-1]}\big)\otimes\mu\big((\mu^{-1}(m_{[0]})\cdot a_{[0][0]})\big)\\
&= m_{[-1]}a_{[-1]}\otimes m_{[0]}\cdot a_{[0]}.\ \blacksquare
\end{aligned}
$$

EXAMPLE 3.3. (1) Let $A = H$ and $\rho = \rho^l = \Delta$. Then ${}^H\mathscr{HYD}_H$ is the category of Yetter–Drinfel'd modules introduced in [16].

(2) If $\rho_A : A \to A\otimes H$ is the trivial coaction, that is, $\rho_A(a) = \beta^{-1}(a)\otimes 1_H$, then ${}^H\mathscr{HYD}_A = {}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$, the category of relative Hom-Hopf modules introduced in [11].

PROPOSITION 3.4. *Under the hypotheses of Definition* 3.1*:*

(1) *(A, β) is a left $(H \otimes H^{\rm op}, \alpha\otimes\alpha^{\rm op})$-Hom-comodule algebra. The coaction $A \to (H \otimes H^{\rm op}) \otimes A$ is given by*

$$a \mapsto \big(a_{[0][-1]} \otimes S^{-1}(\alpha^{-1}(a_{[1]}))\big) \otimes \beta(a_{[0][0]}).$$

(2) *(H, α) is a right $(H \otimes H^{\rm op}, \alpha \otimes \alpha^{\rm op})$-Hom-module coalgebra. The action of $H^{\rm op} \otimes H$ on H is given by*

$$g \triangleleft (h \otimes k) = \alpha(k)(\alpha^{-1}(g)h).$$

(3) *The category ${}^H\mathscr{HYD}_A$ of left-right quantum Hom-Yetter–Drinfel'd modules is isomorphic to a category of Doi Hom-Hopf modules, namely ${}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H \otimes H^{\rm op})_A$.*

Proof. (1) Let us first prove that (A, β) is a left $(H \otimes H^{\rm op}, \alpha \otimes \alpha^{\rm op})$-Hom-comodule. For all $h \in H$,

$$
\begin{aligned}
(\Delta_{H\otimes H^{\rm op}} \otimes \beta^{-1})\rho_H(h) &= \Delta_{H\otimes H^{\rm op}}(a_{[0][-1]} \otimes S^{-1}(\alpha^{-1}(a_{[1]}))) \otimes a_{[0][0]}\\
&= a_{[0][-1](1)} \otimes S^{-1}(\alpha^{-1}(a_{[1](2)})) \otimes a_{[0][-1](2)} \otimes S^{-1}(\alpha^{-1}(a_{[1](1)})) \otimes a_{[0][0]}\\
&= \alpha^{-1}(a_{[0][-1]}) \otimes S^{-1}(\alpha^{-2}(a_{[1]}))\\
&\quad \otimes \alpha(a_{[0][0][0][-1]}) \otimes S^{-1}(a_{[0][0][1]}) \otimes \beta^2(a_{[0][0][0][0]})\\
&= \alpha^{-1}(a_{[0][-1]}) \otimes S^{-1}(\alpha^{-2}(a_{[1]})) \otimes \rho_A(\beta(a_{[0][0]})) = (\alpha^{-1} \otimes \rho_A)\rho_A(a).
\end{aligned}
$$

Therefore (A, β) is a right $(H \otimes H^{\rm op}, \alpha\otimes\alpha^{\rm op})$-Hom-comodule, and it is easy to check that $\rho_A(ab) = \rho_A(a)\rho_A(b)$ for all $a, b \in A$.

(2) We will prove that (H,α) is a right $(H\otimes H^{\mathrm{op}},\alpha\otimes\alpha^{\mathrm{op}})$-Hom-comodule. For all $h,l,k,m,c\in H$, we have

$$\begin{aligned}
&[c\triangleleft(h\otimes k)]\triangleleft(\alpha(l)\otimes\alpha(m)) = [\alpha(k)(\alpha^{-1}(c)h)]\triangleleft(\alpha(l)\otimes\alpha(m))\\
&\quad=\alpha^2(m)\big[[k(\alpha^{-2}(c)\alpha^{-1}(h))]\alpha(l)\big]=\alpha^2(m)\big[[(\alpha^{-1}(k)\alpha^{-2}(c))h]\alpha(l)\big]\\
&\quad=\alpha^2(m)[(k\alpha^{-1}(c))(hl)]=[c(mk)](\alpha(h)\alpha(l))\\
&\quad=\alpha(c)\triangleleft(hl\otimes mk)=\alpha(c)\triangleleft[(h\otimes k)(l\otimes m)],
\end{aligned}$$

and this implies that (H,α) is a right $(H\otimes H^{\mathrm{op}},\alpha\otimes\alpha^{\mathrm{op}})$-Hom-comodule.

Since (H,α) is an (H,α)-Hom-bimodule algebra, it follows that (H,α) is a left $(H\otimes H^{\mathrm{op}},\alpha\otimes\alpha^{\mathrm{op}})$-Hom-module coalgebra.

(3) Let $(M,\cdot,\mu)$ be a right (A,β)-module and (M,ρ_M,μ) be a right (H,α)-comodule. Then $M\in {}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H\otimes H^{\mathrm{op}})_A$ if and only if

$$\begin{aligned}
\rho_M(m\cdot a)&=m_{[-1]}\triangleleft(a_{[0][-1]}\otimes S^{-1}(\alpha^{-1}(a_{[1]})))\otimes m_{[0]}\cdot\beta(a_{[0][0]})\\
&=S^{-1}(a_{[1]})(\alpha^{-1}(m_{[-1]})a_{[0][-1]})\otimes m_{[0]}\cdot\beta(a_{[0][0]})
\end{aligned}$$

for all $h\in H$ and $m\in M$. This shows that ${}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H\otimes H^{\mathrm{op}})_A$ is isomorphic to ${}^H\mathscr{HYD}_A$. ∎

4. The affineness criterion for quantum Hom-Yetter–Drinfel'd modules. In the section, we first introduce the notion of quantum integrals associated to quantum Hom-Yetter–Drinfel'd modules. Then we show our main result, the affineness criterion for quantum Hom-Yetter–Drinfel'd modules, via the quantum integrals.

Definition 4.1. Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode and (A,β) a (H,α)-Hom-bicomodule algebra. A k-linear map $\gamma: H\to \mathrm{Hom}(H,A)$ (i.e., the set of homomorphisms from (H,α) to (A,β)) satisfying $\gamma(\alpha(g))(\alpha(h))=\beta\circ\gamma(g)(h)$ is called a *quantum integral* if

$$\begin{aligned}
(4.1)\qquad &\alpha(h_{(1)})\otimes\gamma(h_{(2)})(\alpha^{-1}(g))=\\
&S^{-1}([\gamma(h)(\alpha(g_{(1)}))]_{[1]})\big(\alpha^{-1}(g_{(2)})\otimes[\gamma(h)(\alpha(g_{(1)}))]_{[0][-1]}\big)[\gamma(\alpha(h))\alpha^2(g_{(1)})]_{[0][0]}
\end{aligned}$$

for all $g,h\in H$. A quantum integral $\gamma: H\to\mathrm{Hom}(H,A)$ is called *total* if

$$\gamma(h_{(1)})(h_{(2)})=\varepsilon(h)1_A \tag{4.2}$$

for all $h\in H$.

Remark 4.2. Let $\gamma: H\to\mathrm{Hom}(H,A)$ be a quantum integral. Then the map

$$\phi:(H,\alpha)\to(A,\beta),\qquad \phi(h)=\gamma(h)(1_H),$$

satisfies the condition

$$h_{(1)} \otimes \phi(h_{(2)}) = S^{-1}(\alpha^{-1}(\phi(h)_{[1]}))\phi(h)_{[0][-1]} \otimes \beta(\phi(h)_{[0][0]})$$

for all $h \in H$, that is, $\phi : (H, \alpha) \to (A, \beta)$ is left (H, α)-colinear.

It is not hard to check that $H \otimes A$ is an object in ${}^{H}\mathscr{HYD}_{A}$ via the following structures:

$$(h \otimes b)a = S^{-1}(a_{[1]})(h\alpha^{-1}(a_{[0][-1]})) \otimes b\beta(a_{[0][0]}), \tag{4.3}$$

$$\rho_{H\otimes A}(h \otimes a) = \alpha(h_{(1)}) \otimes h_{(2)} \otimes \beta^{-1}(b). \tag{4.4}$$

Proposition 4.3. *Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode and (A, β) an (H, α)-Hom-bicomodule algebra. Assume that there exists a total quantum integral $\gamma : H \to \mathrm{Hom}(H, A)$. Then $\widetilde{\rho} : A \to H \otimes A$, $\widetilde{\rho}(a) = S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]} \otimes \beta(a_{[0][0]})$, splits in ${}^{H}\mathscr{HYD}_{A}$.*

Proof. We consider the map

$$\lambda : H \otimes A \to A,$$
$$\lambda(h \otimes a) = \beta^2(a_{[0][0]})\gamma(\alpha^{-1}(h))\big(S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]}\big),$$

for all $a \in A$ and $h \in H$. It is easy to see that λ is a left (H, α)-colinear retraction of $\widetilde{\rho}$. In particular, $\lambda(1_H \otimes 1_A) = 1_A$ and

$$\begin{aligned} h_{(1)} \otimes \lambda(h_{(2)} \otimes 1_A) \\ = S^{-1}\big(\alpha^{-1}(\lambda(h \otimes a)_{[1]})\big)\lambda(h \otimes a)_{[0][-1]} \otimes \beta(\lambda(h \otimes a)_{[0][0]}). \end{aligned} \tag{4.5}$$

Now define

$$\Lambda : H \otimes A \to A, \quad \Lambda(h \otimes a) = \lambda\big(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]})) \otimes 1_A\big)\beta^2(a_{[0][0]}),$$

for all $h \in H$ and $a \in A$. Then Λ is still a retraction of $\widetilde{\rho}$. In fact,

$$\begin{aligned}
(\Lambda \circ \widetilde{\rho})(a) &= \Lambda(S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]} \otimes \beta(a_{[0][0]})) \\
&= \lambda\big(S^{-2}(\alpha(a_{[0][0][1]}))(S^{-1}(\alpha^{-3}(a_{[1]}))\alpha^{-2}(a_{[0][-1]})S(\alpha(a_{[0][0][0][-1]}))) \otimes 1_A\big)\beta^3(a_{[0][0][0][0]}) \\
&= \lambda\big(S^{-2}(\alpha(a_{[0][0][1]}))(S^{-1}(\alpha^{-2}(a_{[1]}))\{\alpha^{-2}(a_{[0][-1]})S(a_{[0][0][0][-1]})\}) \otimes 1_A\big)\beta^3(a_{[0][0][0][0]}) \\
&= \lambda\big(S^{-2}(\alpha^2(a_{[0][0][0][1]}))(S^{-1}(a_{[0][0][1]})\{\alpha^{-3}(a_{[-1]})S(\alpha^{-2}(a_{[0][-1]}))\}) \otimes 1_A\big)\beta^3(a_{[0][0][0][0]}) \\
&= \lambda\big(S^{-2}(\alpha(a_{[0][0][1]}))(S^{-1}(\alpha^{-1}(a_{[0][1]}))\{\alpha^{-2}(a_{[-1](1)})S(\alpha^{-2}(a_{[-1](2)}))\}) \otimes 1_A\big)\beta^2(a_{[0][0][0]}) \\
&= \lambda(S^{-2}(a_{[0][1]})S^{-1}(\alpha^{-1}(a_{[1]})) \otimes 1_A)\beta(a_{[0][0]}) \\
&= \lambda(S^{-2}(a_{[1](1)})S^{-1}(a_{[1](2)}) \otimes 1_A)a_{[0]} = \lambda(1_H \otimes 1_A)\beta^{-1}(a) = a.
\end{aligned}$$

It remains to show that Λ is a morphism in ${}^{H}\mathscr{HYD}_{A}$. For this purpose, we take $h \in H$ and $a, b \in A$ and calculate

$$\begin{aligned}
\Lambda((h\otimes b)a) &= \Lambda\big(S^{-1}(a_{[1]})(h\alpha^{-1}(a_{[0][-1]}))\otimes b\beta(a_{[0][0]})\big)\\
&= \lambda\big(S^{-2}(b_{[1]}\alpha(a_{[0][0][1]}))(S^{-1}(\alpha^{-2}(a_{[1]}))(\alpha^{-2}(h)\alpha^{-3}(a_{[0][-1]}))\\
&\quad S(b_{[0][1]}\alpha(a_{[0][0][0][1]})))\otimes 1_A\big)\beta^2(b_{[0]})\beta^3(a_{[0][0][0][0]})\\
&= \lambda\big(S^{-2}(\alpha(b_{[1]}))\{S^{-2}(\alpha^{-2}(a_{[0][1]}))(S^{-1}(\alpha^{-3}(a_{[1]})))\}(\alpha^{-1}(h)\alpha^{-1}(a_{[0][-1]}))\\
&\quad S(\alpha(b_{[0][1]})\alpha^2(a_{[0][0][0][1]}))\otimes 1_A\big)\beta^2(b_{[0]})\beta^3(a_{[0][0][0][0]})\\
&= \lambda\big(S^{-2}(b_{[1]})(\alpha^{-2}(h)S(b_{[0][-1]}))\otimes 1_A\big)\beta^2(b_{[0][0]})a \;=\; \Lambda(h\otimes b)a,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\rho}\Lambda(h\otimes a) &= \widetilde{\rho}\big(\lambda(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]}))\otimes 1_A)\beta^2(a_{[0][0]})\big)\\
&= S^{-1}(\alpha(a_{[0][0][1]}))S^{-1}\big(\alpha^{-1}(\lambda(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]}))\otimes 1_A)_{[1]})\big)\\
&\quad \lambda\big(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]}))\otimes 1_A\big)_{[0][-1]}\alpha^2(a_{[0][0][0][-1]})\\
&\quad \otimes\beta\big(\lambda(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]}))\otimes 1_A)_{[0][0]}\big)\beta^3(a_{[0][0][0][0]})\\
&= S^{-1}(\alpha(a_{[0][0][1]}))\alpha^{-1}\big\{S^{-1}\big(\alpha^{-1}(\lambda(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]}))\otimes 1_A)_{[1]})\big)\\
&\quad \lambda(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]}))\otimes 1_A)_{[0][-1]}\big\}\alpha^3(a_{[0][0][0][-1]})\\
&\quad \otimes\beta\big(\lambda(S^{-2}(a_{[1]})(\alpha^{-2}(h)S(a_{[0][-1]}))\otimes 1_A)_{[0][0]}\big)\beta^3(a_{[0][0][0][0]})\\
&\overset{(4.5)}{=} S^{-1}(\alpha(a_{[0][0][1]}))\alpha^{-1}\big(S^{-2}(a_{[1](1)})(\alpha^{-2}(h_{(1)})S(a_{[0][-1](2)}))\big)\alpha^3(a_{[0][0][0][-1]})\\
&\quad \otimes\beta\big(\lambda(S^{-2}(a_{[1](2)})(\alpha^{-2}(h_{(2)})S(a_{[0][-1](1)}))\otimes 1_A)_{[0][0]}\big)\beta^3(a_{[0][0][0][0]})\\
&= \alpha(h_{(1)})\otimes\lambda\big(S^{-2}(\alpha^{-1}(a_{[1]}))(\alpha^{-2}(h_{(2)})S(\alpha^{-1}(a_{[0][-1]})))\otimes 1_A\big)\beta(a_{[0][0]})\\
&= (\mathrm{id}_H\otimes\Lambda)\rho_{H\otimes A}(h\otimes a).
\end{aligned}$$

So Λ is a retraction of $\widetilde{\rho}$ in ${}^{H}\mathscr{H}\mathscr{Y}\mathscr{D}_A$, as required, and this completes the proof. ∎

Define the coinvariants of (A,β) as

$$\begin{aligned}
B = A^{\mathrm{co}\,H} &= \{a\in A \mid \widetilde{\rho}(a) = 1_H\otimes\beta^{-1}(a)\}\\
&= \{a\in A \mid S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]}\otimes\beta(a_{[0][0]}) = 1_H\otimes\beta^{-1}(a)\}.
\end{aligned}$$

Then (B,β) is a Hom-subalgebra of (A,β).

Proposition 4.4. *Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode and (A,β) an (H,α)-Hom-bicomodule algebra. Assume that there exists a total quantum integral $\gamma : H \to \mathrm{Hom}(H,A)$. Then*

(1) *B is a direct summand of A as a left B-submodule;*
(2) *B is a direct summand of A as a right B-submodule.*

Proof. We shall prove that there exists a well-defined left trace given by the formula

$$t^l : (A,\beta) \to (B,\beta),$$
$$t^l(a) = \lambda(1_H \otimes a) = \beta(a_{[0][0]})\gamma(1_H)\big(S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]}\big),$$

for all $a \in A$. By (3.2) we obtain $\widetilde{\rho}(t^l(a)) = \beta^{-1}(t^l(a)) \otimes 1_H$, i.e., $t^l(a) \in B$. Now for any $b \in B$ and $a \in A$, we have

$$\begin{aligned} t^l(ba) &= \beta(b_{[0][0]}a_{[0][0]})\gamma(1_H)\big(S^{-1}(\alpha^{-1}(b_{[1]})\alpha^{-1}(a_{[1]}))b_{[0][-1]}a_{[0][-1]}\big) \\ &= (\beta^{-1}(b)\beta(a_{[0][0]}))\gamma(1_H)\big(S^{-1}(a_{[1]})\alpha(a_{[0][-1]})\big) = bt^l(a). \end{aligned}$$

Hence t^l is a left (B,β)-module map satisfying

$$t^l(1_A) = 1_A\gamma(1_H)(1_H) = 1_A.$$

It follows that t^l is a left (B,β)-module retraction of the inclusion $B \subset A$, as desired.

(2) Similarly, one can prove that the map

$$t^r : (A,\beta) \to (B,\beta),$$
$$t^r(a) = \Lambda(1_H \otimes a) = \gamma\big(S^{-2}(\alpha^{-1}(a_{[1]}))S(a_{[0][-1]})(1_H)\big)\beta(a_{[0][0]}),$$

for all $a \in A$, is a right (B,β)-module retraction of the inclusion $B \subset A$. ∎

Definition 4.5. Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode and (A,β) an (H,α)-Hom-bicomodule algebra. Assume that there exists a total quantum integral $\gamma : H \to \mathrm{Hom}(H,A)$. The map

$$t^l : (A,\beta) \to (B,\beta), \quad t^l(a) = \beta(a_{[0][0]})\gamma(1_H)\big(S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]}\big),\ a \in A,$$

is called a *quantum trace* associated to γ.

Next we will construct functors connecting ${}^H\mathscr{HYD}_A$ and $\widetilde{\mathscr{H}}(\mathscr{M}_k)_B$. First, if $(M,\mu) \in {}^H\mathscr{HYD}_A$, then

$$M^{\mathrm{co}\,H} = \{m \in M \mid \rho_M(m) = 1_H \otimes \mu^{-1}(m)\}$$

is the right (B,β)-module of coinvariants of (M,μ). Furthermore, $M \to M^{\mathrm{co}\,H}$ gives us a covariant functor

$$(-)^{\mathrm{co}\,H} : {}^H\mathscr{HYD}_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_B.$$

Now, for any $(N,\nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_B$, $N \otimes_B A$ is an object in ${}^H\mathscr{HYD}_A$ via the structures

$$(n \otimes_B a)a' = \nu(n) \otimes_B a\beta^{-1}(a'),$$
$$\rho_{N\otimes_B A}(n \otimes_B a) = S^{-1}(\alpha^{-2}(a_{[1]}))\alpha^{-1}(a_{[0][-1]}) \otimes \nu^{-1}(n) \otimes \alpha(a_{[0][0]}),$$

for all $n \in N$ and $a, a' \in A$. In this way, we have constructed a covariant functor (called the *induction functor*)

$$A \otimes_B - : \widetilde{\mathscr{H}}(\mathscr{M}_k)_B \to {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A.$$

PROPOSITION 4.6. *Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode and (A, β) an (H, α)-Hom-bicomodule algebra. Then the induction functor $A \otimes_B - : \widetilde{\mathscr{H}}(\mathscr{M}_k)_B \to {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ is a left adjoint of the coinvariant functor $(-)^{\mathrm{co}\,H} : {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_B$.*

Proof. Similar to [22]. Details are left to the reader. ∎

We have shown that $H \otimes A \in {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ and $(A \otimes H)^{\mathrm{co}\,H} \cong A$ via $a \otimes 1_H \mapsto a$. Then the adjunction map can be viewed as a map in ${}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$:

$$\psi : A \otimes_B A \to A \otimes H, \quad \psi(a \otimes b) = S^{-1}(b_{[1]})\alpha(b_{[0][-1]}) \otimes \beta^{-1}(a)\beta(b_{[0][0]}),$$

for all $a, b \in A$. Here $A \otimes_B A \in {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ with the structures

$$(a \otimes_B b) \cdot a' = \beta^{-1}(a) \otimes_B b\beta^{-1}(a'),$$
$$\rho_{A\otimes_B A}(a \otimes_B b) = S^{-1}(b_{[1]})\alpha(b_{[0][-1]}) \otimes \beta^{-1}(a) \otimes \beta(b_{[0][0]}),$$

for all $a, a', b \in A$.

DEFINITION 4.7. Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode, (A, β) an (H, α)-Hom-bicomodule algebra and $B = A^{\mathrm{co}\,H}$. A/B is called a *quantum Galois extension* if the canonical map

$$\psi : A \otimes_B A \to A \otimes H, \quad \psi(a \otimes b) = S^{-1}(b_{[1]})\alpha(b_{[0][-1]}) \otimes \beta^{-1}(a)\beta(b_{[0][0]}),$$

is bijective.

THEOREM 4.8. *Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode, (A, β) an (H, α)-Hom-bicomodule algebra and $B = A^{\mathrm{co}\,H}$. Assume that there exists a total quantum integral $\gamma : H \to \mathrm{Hom}(H, A)$. Then*

$$\eta_N : N \to (N \otimes_B A)^{\mathrm{co}\,H}, \quad \eta_N(n) = n \otimes_B 1_A,$$

is an isomorphism of right (B, β)-modules for all $(N, \nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_B$.

Proof. Using the left quantum trace $t^l : (A, \beta) \to (B, \beta)$ we construct an inverse of η_N as follows. Define the map

$$\theta_N : (N \otimes_B A)^{\mathrm{co}\,H} \to N, \quad \theta_N(n_i \otimes_B a_i) = \sum n_i t^l(a_i),$$

for any $a_i \otimes_B n_i \in (N \otimes_B A)^{\mathrm{co}\,H}$. Since $t^l(1_A) = 1_A$, we get $\theta_N \circ \eta_N = \mathrm{id}_N$. Let $n_i \otimes_B a_i \in (N \otimes_B A)^{\mathrm{co}\,H}$. Then

$$1_H \otimes \nu^{-1}(n_i) \otimes_B \alpha^{-1}(a_i) = S^{-1}(a_{[1]})\alpha(a_{[0][-1]}) \otimes \nu^{-1}(n_i) \otimes_B \beta(a_{[0][0]}).$$

It follows that

$$\nu^{-1}(n_i)\otimes_B\alpha^{-1}(a_i)\otimes 1_A = \nu^{-1}(n_i)\otimes_B\beta(a_{[0][0]})\otimes\gamma(1_H)\big(S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]}\big).$$

Furthermore, we have

$$\nu^{-1}(n_i)\otimes_B\alpha^{-1}(a_i)1_A = \nu^{-1}(n_i)\otimes_B\beta(a_{[0][0]})\gamma(1_H)\big(S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]}\big).$$

Thus, we get $n_i\otimes_B a_i = n_i\otimes_B t^l(a_i)$ and

$$(\eta_N\circ\theta_N)(n_i\otimes_B a_i)=\sum n_i t^l(a_i)\otimes_B 1_A=\sum n_i\otimes_B t^l(a_i)=n_i\otimes_B a_i.$$

Hence θ_N is the inverse of η_N, as desired. ∎

We now prove the main result of this section, that is, the affineness criterion for quantum Hom-Yetter–Drinfel'd modules.

Theorem 4.9. *Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode, (A,β) an (H,α)-Hom-bicomodule algebra and $B=A^{\operatorname{co}H}$. Assume that*

(1) *there exists a total quantum integral $\gamma: H\to \operatorname{Hom}(H,A)$;*
(2) *the canonical map*

$$\beta: A\otimes_B A\to A\otimes H,\quad a\otimes_B b\mapsto S^{-1}(b_{[1]})\alpha(b_{[0][-1]})\otimes\beta^{-1}(a)\beta(b_{[0][0]}),$$

is surjective.

Then the induction functor $A\otimes_B -: \widetilde{\mathscr{H}}(\mathscr{M}_k)_B\to {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ is an equivalence of categories.

Proof. In Theorem 4.8 we have shown that the adjunction map $\eta_N: N\to(N\otimes_B A)^{\operatorname{co}H}$ is an isomorphism for all $(N,\nu)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)_B$. It remains to prove that the other adjunction map

$$\beta_M: M^{\operatorname{co}H}\otimes_B A\to M,\quad \beta_M(m\otimes_B a)=ma,$$

is also an isomorphism.

Let (V,ω) be a k-module. Then $A\otimes V\in {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ via the structures induced by A, i.e.,

$$(a\otimes v)a' = a\beta^{-1}(a')\otimes\omega(v),$$
$$\rho_{A\otimes V}(a\otimes v)=S^{-1}(a_{[1]})\alpha(a_{[0][-1]})\otimes\beta(a_{[0][0]})\otimes\omega^{-1}(v),$$

for all $a,b\in A$ and $v\in V$. In particular, for $V=A$, $A\otimes A\in {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ via

$$(a\otimes b)\cdot a' = a\beta^{-1}(a')\otimes\beta(b); \tag{4.6}$$

$$\rho_{A\otimes A}(a\otimes b)=S^{-1}(a_{[1]})\alpha(a_{[0][-1]})\otimes\beta(a_{[0][0]})\otimes\beta^{-1}(b), \tag{4.7}$$

for all $a,b,a'\in A$.

Now we prove that the adjunction map $\beta_{A\otimes V}:(A\otimes V)^{\operatorname{co}H}\otimes_B A\to A\otimes V$ is an isomorphism for any k-module V.

First, $V \otimes B$ and $B \otimes V$ are both objects in $\widetilde{\mathscr{H}}(\mathscr{M}_k)_B$ via the usual B-actions:

$$(v \otimes a) \cdot b = \omega(v) \otimes a\beta^{-1}(b), \quad a' \cdot (b' \otimes v') = \beta^{-1}(a')b' \otimes \omega(v'),$$

for all $a, b, a', b' \in B$ and $v, v' \in V$. The flip map $\tau : V \otimes B \to B \otimes V$, $\tau(v \otimes b) = b \otimes v$, is an isomorphism in $\widetilde{\mathscr{H}}(\mathscr{M}_k)_B$. On the other hand, $V \otimes A \in {}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$ via the structures induced by A, i.e.,

$$(v \otimes a) \cdot b = \omega(v) \otimes a\beta^{-1}(b),$$
$$\rho_{V\otimes A}(v \otimes a) = S^{-1}(a_{[1]})\alpha(a_{[0][-1]}) \otimes \omega^{-1}(v) \otimes \beta(a_{[0][0]}).$$

It is easy to see that the flip map $\tau : A \otimes V \to V \otimes A$, $\tau(a \otimes v) = v \otimes a$, is an isomorphism in ${}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$.

Applying Theorem 4.8 for $N = V \otimes B \cong B \otimes V$, we obtain the following isomorphisms in $\mathcal{M}_B$:

$$B \otimes V \cong V \otimes B \cong (V \otimes B \otimes_B A)^{\mathrm{co}\, H} \cong (V \otimes A)^{\mathrm{co}\, H} \cong (A \otimes V)^{\mathrm{co}\, H}.$$

Hence, $(A \otimes V)^{\mathrm{co}\, H} \otimes_B A \cong A \otimes V$.

Define

$$\widetilde{\psi} : A \otimes_B A \to H \otimes A, \quad a \otimes_B b \mapsto S^{-1}(b_{[1]})\alpha(b_{[0][-1]}) \otimes \beta^{-1}(a)\beta(b_{[0][0]}),$$

for all $a, b \in A$. As ψ is surjective, $\widetilde{\psi}$ is surjective since $\widetilde{\psi} = \psi \circ \mathrm{can}$, where $\mathrm{can} : A \otimes A \to A \otimes_B A$ is the canonical surjection.

Define

$$\xi : A \otimes A \to A \otimes H,$$
$$\xi(a \otimes b) = (\widetilde{\psi} \circ \tau)(a \otimes b) = S^{-1}(a_{[1]})\alpha(a_{[0][-1]}) \otimes \beta^{-1}(b)\beta(a_{[0][0]}),$$

for any $a, b \in A$. Then the map ξ is surjective since $\widetilde{\psi}$ and τ are surjective. We will prove that ξ is a morphism in ${}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$, where $A \otimes A$ and $A \otimes H$ are quantum Hom-Yetter–Drinfel'd modules respectively. Indeed, we have

$$\begin{aligned}
&\xi((a \otimes b)c) = \xi(a\beta^{-1}(c) \otimes \beta(b)) \\
&= S^{-1}(a_{[1]}\alpha^{-1}(c_{[1]}))\alpha(a_{[0][-1]})c_{[0][-1]} \otimes b\beta(a_{[0][0]})c_{[0][0]} \\
&= S^{-1}(c_{[1]})\{(S^{-1}(\alpha^{-1}(a_{[1]}))a_{[0][-1]})\alpha^{-1}(c_{[0][-1]})\} \otimes \{\beta^{-1}(b)\beta(a_{[0][0]})\}\beta(c_{[0][0]}) \\
&= \xi(a \otimes b)c
\end{aligned}$$

and

$$\begin{aligned}
\rho_{H\otimes A}\xi(a \otimes b) &= \rho_{H\otimes A}\big(S^{-1}(a_{[1]})\alpha(a_{[0][-1]}) \otimes \beta^{-1}(b)\beta(a_{[0][0]})\big) \\
&= S^{-1}(\alpha(a_{[1](2)}))\alpha^2(a_{[0][-1](1)}) \otimes \{S^{-1}(a_{[1](1)})\alpha(a_{[0][-1](2)}) \otimes \beta^{-2}(b)a_{[0][0]}\} \\
&= (\mathrm{id}_H \otimes \xi)\big(S^{-1}(a_{[1]})\alpha(a_{[0][-1]}) \otimes \beta(a_{[0][0]}) \otimes \beta^{-1}(b)\big) \\
&= (\mathrm{id}_H \otimes \xi)\rho_{A\otimes A}(a \otimes b).
\end{aligned}$$

Hence, ξ is a surjective morphism in ${}^H\mathscr{HYD}_A$. Moreover, $H\otimes A$ is projective as a left (A,β)-module, where $H\otimes A$ is a left A-module given by (3.4). By Proposition 3.1, the map

$$u : H\otimes A \to H\otimes A, \qquad h\otimes a \mapsto S^{-1}(a_{[1]})(h\alpha^{-1}(a_{[0][-1]}))\otimes \beta(a_{[0][0]}),$$

is an isomorphism of right (A,β)-modules. It follows that there exists $\zeta : H\otimes A\to A\otimes A$ such that $\xi\circ\zeta = \mathrm{id}_{H\otimes A}$ since $A\otimes A\to A\otimes H$ is surjective. Hence, ξ splits in the category of right A-modules. In particular, ξ is a k-split epimorphism in ${}^H\mathscr{HYD}_A$.

Let $(M,\mu)\in {}^H\mathscr{HYD}_A$. Then $A\otimes A\otimes M\in {}^H\mathscr{HYD}_A$ via the structures arising from $A\otimes A$:

$$(4.8)\qquad (a\otimes b\otimes m)\cdot c = a\beta^{-2}(c)\otimes\beta(b)\otimes\mu(m),$$

$$(4.9)\qquad \rho_{A\otimes A\otimes M}(a\otimes b\otimes m) = S^{-1}(a_{[1]})(h\alpha^{-1}(a_{[0][-1]}))\otimes\beta(a_{[0][0]})\otimes\beta^{-1}(b)\otimes\mu^{-1}(m),$$

for all $a,b,c\in A$ and $m\in M$. Also, $H\otimes A\otimes M$ is an object in ${}^H\mathscr{HYD}_A$ via the structures arising from $H\otimes A$:

$$(h\otimes a\otimes m)b = S^{-1}(\alpha(b_{[1]}))(hb_{[0][-1]})\otimes a\beta^2(b_{[0][0]})\otimes\mu(m),$$
$$\rho_{H\otimes A\otimes M}(h\otimes a\otimes m) = \alpha(h_{(1)})\otimes h_{(2)}\otimes\beta^{-1}(a)\otimes\mu^{-1}(m),$$

for all $a,b\in A$, $h\in H$ and $m\in M$. Then

$$\xi\otimes\mathrm{id}_M : A\otimes A\otimes M\to H\otimes A\otimes M$$

is a k-split epimorphism in ${}^H\mathscr{HYD}_A$.

Since ${}^H\mathscr{HYD}_A = {}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)(H\otimes H^{\mathrm{op}})_A$, the map

$$f : H\otimes A\otimes M\to M,$$
$$h\otimes a\otimes m \mapsto \mu(m_{[0]})\gamma\big(S^{-2}(\alpha^{-1}(a_{[1]}))(\alpha^{-2}(h)S(a_{[0][-1]}))\big)(\alpha^{-1}(m_{[-1]}))\beta^2(a_{[0][0]}),$$

is a k-split epimorphism in ${}^H\mathscr{HYD}_A$. Thus the composition

$$g = f\circ(\xi\otimes\mathrm{id}_M) : A\otimes A\otimes M\to M,$$
$$a\otimes b\otimes m\mapsto \mu(m_{[0]})\gamma\big(S^{-2}(\alpha^{-1}(b_{[1]}))S(b_{[0][-1]})\big)(\alpha^{-1}(m_{[-1]}))\beta(b_{[0][0]})\beta^{-1}(a),$$

is a k-split epimorphism in ${}^H\mathscr{HYD}_A$. Note that the structure of $A\otimes A\otimes M$ as an object in ${}^H\mathscr{HYD}_A$ is of the form $A\otimes V$ for the k-module $V = A\otimes M$.

To conclude, we have constructed a k-split epimorphism in ${}^H\mathscr{HYD}_A$

$$A\otimes A\otimes M = (M_1,\pi)\xrightarrow{g}(M,\mu)\to 0$$

such that the adjunction map ψ_{M_1} for (M_1,π) is bijective. Since g is k-split and there exists a total quantum integral $\gamma : H \to \mathrm{Hom}(H, A)$, g also splits in ${}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)$. In particular, the sequence

$$(M_1^{\mathrm{co}\,H}, \pi) \xrightarrow{g^{\mathrm{co}\,H}} (M^{\mathrm{co}\,H}, \mu) \to 0$$

is exact. Continuing the resolution with $\mathrm{Ker}(g)$ instead of M, we obtain an exact sequence in ${}^H\mathscr{H}\mathscr{Y}\mathscr{D}_A$

$$(M_2, P) \to (M_1, \pi) \to (M, \mu) \to 0$$

which splits in ${}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)$, and the adjunction maps for (M_1, π) and (M_2, P) are bijective. Using the Five Lemma we conclude that the adjunction map for (M, μ) is bijective. ■

Finally, we consider a special case. Assume that $A = H$. Then (A, β) is a right (H, α)-comodule algebra in a natural way. The coinvariants of (H, α) are

$$\begin{aligned} B = H^{\mathrm{co}\,H} &= \{h \in H \mid \widetilde{\rho}(h) = 1_H \otimes \alpha^{-1}(h)\} \\ &= \{a \in A \mid S^{-1}(\alpha^{-1}(h_{(2)}))h_{(1)(1)} \otimes \alpha(h_{(1)(2)}) = 1_H \otimes \alpha^{-1}(h)\}. \end{aligned}$$

Then (B, α) is a subalgebra of (H, α). Hence we can obtain the following result.

Corollary 4.10. *Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode S and $B = H^{\mathrm{co}\,H}$. Assume that:*

(1) *there exists a total quantum integral $\gamma : H \to \mathrm{Hom}(H, H)$;*
(2) *the canonical map*

$$\begin{aligned} &\psi : H \otimes_B H \to H \otimes H, \\ &h \otimes_B g \mapsto S^{-1}(h_{(2)})\alpha(h_{(1)(1)}) \otimes \alpha^{-1}(h)\beta(g_{(1)(2)}), \end{aligned}$$

is surjective. Then the induction functor $- \otimes_B H : \widetilde{\mathscr{H}}(\mathscr{M}_k)_B \to {}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)_H$ is an equivalence of categories.

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