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Local cohomological properties of homogeneous ANR compacta

by

V. Valov (North Bay, ON)

Abstract. In accordance with the Bing–Borsuk conjecture, we show that if X is an n-dimensional homogeneous metric ANR continuum and $x \in X$, then there is a local basis at x consisting of connected open sets U such that the cohomological properties of $\overline{U}$ and $\operatorname{bd} U$ are similar to the properties of the closed ball $\mathbb{B}^n \subset \mathbb{R}^n$ and its boundary $\mathbb{S}^{n-1}$. We also prove that a metric ANR compactum X of dimension n is dimensionally full-valued if and only if the group $H_n(X, X \setminus x; \mathbb{Z})$ is not trivial for some $x \in X$. This implies that every 3-dimensional homogeneous metric ANR compactum is dimensionally full-valued.

1. Introduction. The Bing–Borsuk conjecture [2] asserts that a homogeneous Euclidean neighborhood retract is a topological manifold. In accordance with that conjecture, we show that the local cohomological structure of any n-dimensional homogeneous metric ANR continuum is similar to the local structure of $\mathbb{R}^n$ (see Theorem 1.1 below). We also establish conditions for a metric ANR compactum X to satisfy the equality $\dim(X \times Y) = \dim X + \dim Y$ for all compact metric spaces Y (any such X is said to be *dimensionally full-valued*). It follows from these conditions that every 3-dimensional homogeneous ANR compactum is dimensionally full-valued (Corollary 1.5), thus providing a partial answer to one of the problems accompanying the Bing–Borsuk conjecture (whether homogeneous metric ANRs are dimensionally full-valued).

Everywhere in this paper by a *space* we mean a homogeneous metric ANR continuum X with $\dim_G X = n$, where $n \geq 2$ and G is a fixed countable abelian group or a principal ideal domain (PID) with unity. Reduced Čech

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homology groups $H_n(X;G)$ and cohomology groups $H^n(X;G)$ with coefficient from G are considered everywhere below. Let us recall that for any abelian group G the cohomology groups $H^n(X;G)$, $n \geq 2$, are isomorphic to the groups $[X, K(G,n)]$ of pointed homotopy classes of maps from X to $K(G,n)$, where $K(G,n)$ is the Eilenberg–MacLane space of type (G,n) (see [22]). The cohomological dimension $\dim_G X$ is the largest integer m such that there exists a closed set $A \subset X$ with $H^m(X,A;G) \neq 0$. Equivalently, $\dim_G X \leq n$ iff every map $f\colon A \to K(G,n)$ can be extended to a map $\tilde{f}\colon X \to K(G,n)$.

Suppose (K,A) is a pair of closed subsets of a space X with $A \subset K$. Following [2], we say that K is an *n-homology membrane spanned on A for an element* $\gamma \in H_n(A;G)$ provided γ is homologous to zero in K, but not homologous to zero in any proper closed subset of K containing A. Similarly, K is said to be an *n-cohomology membrane spanned on A for an element* $\gamma \in H^n(A;G)$ if γ is not extendable over K, but it is extendable over every proper closed subset of K containing A. Here, $\gamma \in H^n(A;G)$ is not extendable over K means that γ is not contained in the image $j^n_{K,A}(H^n(K;G))$, where $j^n_{K,A} : H^n(K;G) \to H^n(A;G)$ is the homomorphism induced by the inclusion $A \hookrightarrow K$.

We note the following simple fact, which will be used in this paper and follows from Zorn's lemma and the continuity of Čech cohomology [22]: *If A is a closed subset of a compact space X and γ is an element of $H^n(A;G)$ not extendable over X, then there exists an n-cohomology membrane for γ spanned on A.*

We also say that a closed set $A \subset X$ is *a cohomological carrier* of a non-zero element $\alpha \in H^n(A;G)$ if $j^n_{A,B}(\alpha) = 0$ for every proper closed subset $B \subset A$. If $H^n(A;G) \neq 0$, but $H^n(B;G) = 0$ for every closed proper subset $B \subset A$, then A is called an *(n,G)-bubble.*

Theorem 1.1. *Let X be a homogeneous metric ANR continuum with $\dim_G X = n$, where G is a countable PID with unity and $n \geq 2$. Then every point x of X has a basis $\mathcal{B}_x$ of open sets $U \subset X$ satisfying the following conditions:*

1. $\operatorname{int}\overline{U} = U$ *and the complement of* $\operatorname{bd} U$ *has two components, one of which is U;*
2. $H^{n-1}(\overline{U};G) = 0$ *and $\overline{U}$ is an $(n-1)$-cohomology membrane spanned on* $\operatorname{bd} U$ *for any non-zero $\gamma \in H^{n-1}(\operatorname{bd} U;G)$;*
3. $\operatorname{bd} U$ *is an $(n-1,G)$-bubble and $H^{n-1}(\operatorname{bd} U;G)$ is a finitely generated G-module.*

The restriction $n \geq 2$ in Theorem 1.1 is needed because of Lemma 2.7, which is used in the proof.

Remark. Condition (1) from Theorem 1.1 implies that $\dim_G \operatorname{bd} U = n-1$ (see [13]).

Theorem 1.2. *Let* X *be as in Theorem* 1.1 *and* G *be a countable group. If a closed subset* $K \subset X$ *is an* $(n-1)$*-cohomology membrane spanned on* A *for some closed set* $A \subset K$ *and some* $\gamma \in H^{n-1}(A;G)$*, then* $(K\setminus A)\cap\overline{X \setminus K} = \emptyset$.

Corollary 1.3. *In the setting of Theorem* 1.2*, if* $U \subset X$ *is open and* $f : U \to X$ *is an injective map, then* $f(U)$ *is open in* X.

We already mentioned that a compactum X is dimensionally full-valued if $\dim(X \times Y) = \dim X + \dim Y$ for any compact metric space Y, or equivalently, $\dim_G X = \dim_{\mathbb{Z}} X$ for any abelian group G. Recent work of Bryant [5] was believed to provide a positive answer to the question whether any homogeneous metric ANR is dimensionally full-valued, but Bryant discovered a gap in the proof of one of the theorems from [5]. The question whether $\dim(X \times Y) = \dim X + \dim Y$ if both X and Y are homogeneous compact ANRs was raised in [6] and [10]. Theorem 1.4 below provides some necessary and sufficient conditions for ANR spaces to be dimensionally full-valued.

Theorem 1.4. *The following conditions are equivalent for any metric ANR compactum* X *of dimension* $\dim X = n$*:*

(1) *X is dimensionally full-valued.*
(2) *There is a point* $x \in X$ *with* $H_n(X, X \setminus x; \mathbb{Z}) \neq 0$.
(3) $\dim_{\mathbb{S}^1} X = n$.

Corollary 1.5. *Every homogeneous metric ANR compactum* X *with* $\dim X = 3$ *is dimensionally full-valued.*

2. Some preliminary results. In this section, if not stated otherwise, G is a countable abelian group and X denotes a homogeneous metric ANR continuum with $\dim_G X = n$, $n \geq 2$. If $H^n(X;G) \neq 0$, then $H^n(B;G) = 0$ for all proper closed subsets B of X (see [23]). Obviously, this is true when $H^n(X;G) = 0$. Therefore, all proper closed subsets of X have trivial n-cohomology groups.

We begin with the following analogue of Theorem 8.1 from [2] (it is here that the countability of G is used).

Proposition 2.1. *Theorem* 1.2 *holds under the additional assumption that* K *is contractible in a proper subset of* X.

Proof. According to the duality between homology and cohomology for countable groups [12, viii 4G)], for any compact metric space Y the groups $H_{n-1}(Y, G^*)$ and $H^{n-1}(Y;G)^*$ are isomorphic, where G^* and $H^{n-1}(Y;G)^*$ denote the character groups of G and $H^{n-1}(Y;G)$, respectively. Here both $H^{n-1}(Y;G)$ and G are considered as discrete groups. Using this duality, we can show that K is an $(n-1)$-homology membrane for some $\beta \in H_{n-1}(A, G^*)$ spanned on A.

Indeed, consider the homomorphism $j^{n-1}_{K,A} : H^{n-1}(K; G) \to H^{n-1}(A; G)$. Since γ is not extendable over K, we have $\gamma \notin G_A = j^{n-1}_{K,A}(H^{n-1}(K; G))$. Considering $H^{n-1}(A; G)$ as a discrete group, we can find a character $\beta : H^{n-1}(A; G) \to \mathbb{S}^1$ such that $\beta(\gamma) \neq e$ and $\beta(G_A) = e$, where e is the unit of $\mathbb{S}^1$. On the other hand, γ is extendable over every proper closed subset B of K which contains A. Therefore, γ is contained in the image of $j^{n-1}_{B,A} : H^{n-1}(B; G) \to H^{n-1}(A; G)$ for any such B. Then $j^{n-1}_{K,A} \circ \beta$ is the trivial character of $H^{n-1}(K; G)$, while $j^{n-1}_{B,A} \circ \beta$ is non-trivial for any proper closed subset B of K containing A. So, β is homologous to zero in K, but not homologous to zero in any proper closed subset of K containing A. Hence, K is an $(n-1)$-homology membrane for β spanned on A.

Now, assume that $(K \setminus A) \cap \overline{X \setminus K} \neq \emptyset$. Then following [3, proof of Theorem 16.1] (see also [2, Theorem 8.1]), we can find a proper closed subset Γ of X and a non-zero element $\alpha \in H_n(\Gamma, G^*)$. This means that $H^n(\Gamma; G) \neq 0$, a contradiction. ∎

Since the Bing–Borsuk result used in the proof of Proposition 2.1 was established for locally homogeneous spaces, Proposition 2.1 remains valid for locally homogeneous spaces X such that $H^n(A; G)$ is trivial for any proper closed subset $A \subset X$.

Corollary 2.2. *Let $A \subset X$ be a closed subset and K an $(n-1)$-cohomology membrane for some $\gamma \in H^{n-1}(A; G)$ spanned on A. Then $K \setminus A$ is connected. If, in addition, K is contractible in a proper subset of X, then $K \setminus A$ is an open subset of X.*

Proof. Suppose $K \setminus A$ is the union of two non-empty, disjoint open sets U and V. Then $K \setminus U$ and $K \setminus V$ are closed proper subsets of K such that $(K \setminus U) \cap (K \setminus V) \subset A$. Hence, γ is extendable over each of these sets and, because A contains their common part, γ is extendable over K. The last conclusion contradicts the fact that K is an $(n-1)$-cohomology membrane for γ.

If K is contractible in a proper subset of X, then $(K \setminus A) \cap \overline{X \setminus K} = \emptyset$ (see Proposition 2.1). Hence, $K \setminus A$ is open in X. ∎

Corollary 2.3. *For any closed set $Z \subset X$ one has $\dim_G Z = n$ if and only if Z has a non-empty interior in X.*

Proof. This was established by Seidel [19] for the covering dimension. His arguments can be modified for $\dim_G$. If $\dim_G Z = n$, we may assume that Z is contractible in a proper subset of X (this can be done because X is locally contractible and $\dim_G$ satisfies the countable sum theorem). Since

$\dim_G Z = n$, there exists a closed set $A \subset Z$ such that $H^n(Z, A; G) \neq 0$. On the other hand, $H^n(Z; G) = 0$ (as a proper closed subset of X). So, according to the exact sequence

$$H^{n-1}(Z;G) \xrightarrow{j^{n-1}_{Z,A}} H^{n-1}(A;G) \xrightarrow{\delta} H^n(Z,A;G) \to 0$$

there exists $\gamma \in H^{n-1}(A; G)$ not extendable over Z. Hence, as noted above, we can find a closed subset K of Z such that K is an $(n-1)$-cohomological membrane for γ spanned on A. So, $K \setminus A$ is open in X (by Corollary 2.2) and $K \setminus A \subset Z$.

If Z has a non-empty interior, then it contains an open set U in X with $\dim_G U = n$. So, $\dim_G Z = n$. ∎

LEMMA 2.4. *Let a closed set $F \subset X$ with $H^{n-1}(F; G) \neq 0$ be contractible in an open set $U \subset X$. If $\overline{U}$ is contractible in a proper subset of X, then F separates $\overline{W}$ for any open set $W \subset X$ containing U.*

Proof. Indeed, there is a closed set P in X such that $P \subset U$ and F is contractible in P. Then any non-zero element $\gamma \in H^{n-1}(F; G)$ is not extendable over P (otherwise γ, considered as a map from F to $K(G, n-1)$, would be homotopic to a constant because F is contractible in P). This yields the existence of an $(n-1)$-cohomology membrane $K_\gamma \subset P$ for γ spanned on F. Because $\overline{U}$ is contractible in a proper subset of X, so is K_γ. Hence, by Proposition 2.1, $(K_\gamma \setminus F) \cap \overline{X \setminus K_\gamma} = \emptyset$. The last equality implies that F separates any $\overline{W}$ such that $W \subset X$ is open and contains U. ∎

LEMMA 2.5. *Suppose $U \subset X$ is open and $P \subsetneq X$ is closed such that $\overline{U} \subsetneq P$ and $H^{n-1}(\mathrm{bd}\, U; G)$ contains elements not extendable over $\overline{U}$. Then there exists $\gamma \in H^{n-1}(\mathrm{bd}\, U; G) \setminus L$ extendable over $P \setminus V$, where $V = \mathrm{int}\, \overline{U}$ and $L = j^{n-1}_{\overline{U},\mathrm{bd}\, U}(H^{n-1}(\overline{U}; G))$. Moreover, if $L = 0$, then every $\gamma \in H^{n-1}(\mathrm{bd}\, U; G)$ is extendable over $P \setminus V$.*

Proof. Since $H^{n-1}(\mathrm{bd}\, U; G)$ contains elements not extendable over $\overline{U}$, L is a proper subgroup of $H^{n-1}(\mathrm{bd}\, U; G)$. Consider the homomorphism $j^{n-1}_{P\setminus V,\mathrm{bd}\, U} : H^{n-1}(P \setminus V; G) \to H^{n-1}(\mathrm{bd}\, U; G)$. It suffices to show that the image of $H^{n-1}(P \setminus V; G)$ under $j^{n-1}_{P\setminus V,\mathrm{bd}\, U}$ is not contained in L.

Indeed, suppose otherwise. Consider the Mayer–Vietoris exact sequence, where $A = P \setminus V$ and $\varphi(\gamma_1, \gamma_2) = j^{n-1}_{A,\mathrm{bd}\, U}(\gamma_2) - j^{n-1}_{\overline{U},\mathrm{bd}\, U}(\gamma_1)$ for $\gamma_1 \in H^{n-1}(\overline{U}; G)$ and $\gamma_2 \in H^{n-1}(A; G)$:

$$H^{n-1}(\overline{U};G) \oplus H^{n-1}(A;G) \xrightarrow{\varphi} H^{n-1}(\mathrm{bd}\, U;G) \xrightarrow{\triangle} H^n(P;G) \to \cdots .$$

Obviously, $L_U = \varphi(H^{n-1}(\overline{U}; G) \oplus H^{n-1}(A; G)) \subset L$. Consequently, any $\gamma \in H^{n-1}(\mathrm{bd}\, U; G) \setminus L$ is not contained in L_U. Hence, $\triangle(\gamma) \neq 0$ for all

$\gamma \in H^{n-1}(\operatorname{bd} U; G) \setminus L$. So, $H^n(P; G) \neq 0$, a contradiction (recall that the nth cohomology groups of all proper closed sets in X are trivial).

If $L = 0$, then $j^{n-1}_{\overline{U},\operatorname{bd} U}(\gamma_1) = 0$ for all $\gamma_1 \in H^{n-1}(\overline{U}; G)$, so $\varphi(\gamma_1, \gamma_2) = j^{n-1}_{A,\operatorname{bd} U}(\gamma_2)$. Since $\triangle(H^{n-1}(\operatorname{bd} U; G)) = 0$, we find that for any element γ in $H^{n-1}(\operatorname{bd} U; G)$ there exist $\gamma_1 \in H^{n-1}(\overline{U}; G)$ and $\gamma_2 \in H^{n-1}(A; G)$ such that $\varphi(\gamma_1, \gamma_2) = \gamma$. Hence, $\gamma = j^{n-1}_{A,\operatorname{bd} U}(\gamma_2)$, which means that γ is extendable over A. This completes the proof. ∎

Lemma 2.6. *If $U \subset X$ is a connected open set and $\overline{U}$ is contractible in a proper subset of X, then $\overline{U}$ is an $(n-1)$-cohomology membrane spanned on* $\operatorname{bd} U$ *for every $\gamma \in H^{n-1}(\operatorname{bd} U; G)$ not extendable over $\overline{U}$.*

Proof. Observe first that U is dense in $V = \operatorname{int}(\overline{U})$, so V is also connected. Let γ be an element of $H^{n-1}(\operatorname{bd} U; G)$ not extendable over $\overline{U}$. Then there exists a closed subset $K \subset \overline{U}$ such that K is an $(n-1)$-cohomology membrane for γ spanned on $\operatorname{bd} U$. Since K is contractible in a proper subset of X (as a subset of $\overline{U}$), by Proposition 2.1, $(K \setminus \operatorname{bd} U) \cap \overline{X \setminus K} = \emptyset$. Hence, $K \setminus \operatorname{bd} U$ is open in X. This implies that $K = \overline{U}$, otherwise V would be the union of the non-empty disjoint open sets $V \setminus K$ and $(K \setminus \operatorname{bd} U) \cap V$. Therefore, $\overline{U}$ is an $(n-1)$-cohomology membrane spanned on $\operatorname{bd} U$ for γ. ∎

The last two statements of this section (Lemmas 2.7 and 2.8) hold for an arbitrary compactum X.

Lemma 2.7. *Let X be an arbitrary compactum and $A \subset X$ be a carrier for a non-zero element $\gamma \in H^{n-1}(A; G)$ with* $\dim_G A \leq n-1$, $n \geq 2$. *Then A is connected.*

Proof. Suppose A is not connected, so A is the union of two closed disjoint non-empty sets A_1 and A_2. Then $H^{n-1}(A; G)$ is isomorphic to the direct sum $H^{n-1}(A_1; G) \oplus H^{n-1}(A_2; G)$ and γ is identified with the pair (γ_1, γ_2), where $\gamma_i = j^{n-1}_{A,A_i}(\gamma)$, $i = 1, 2$. Because A is a carrier of γ and A_i are proper closed non-empty subsets of A, $\gamma_1 = \gamma_2 = 0$. So, $\gamma = 0$, a contradiction. ∎

Since $\dim_G A = 0$ is equivalent to $\dim A = 0$, Lemma 2.7 is not valid for $n = 1$. For example, if A consists of two different points, then there exists a non-trivial element of $\gamma \in H^0(A; \mathbb{Z})$ such that A is a carrier of γ.

Suppose G is a group (resp., a ring). Let $F \subset Z \subset X$ be compact sets. We say that F is an $(n-1, G)$-*bubble with respect to a subgroup* (resp., *a submodule*) $L \subset H^{n-1}(Z; G)$ if the group (resp., the submodule) $j^{n-1}_{Z,F}(L) \subset H^{n-1}(F; G)$ is non-trivial, but $j^{n-1}_{Z,B}(L) \subset H^{n-1}(B; G)$ is trivial for any closed proper subset $B \subset F$.

Lemma 2.8. *Let G be a group* (*resp., a ring*). *If Z is a closed subset of an arbitrary compactum X and $L \subset H^{n-1}(Z; G)$ is a non-trivial and finitely* *generated subgroup (resp., a submodule), then Z contains a non-empty closed subset F such that F is an $(n-1,G)$-bubble with respect to L.*

Proof. If L has one generator γ, we just take a closed set $F \subset Z$ which is a carrier for γ. Then $\beta = j^{n-1}_{Z,F}(\gamma)$ and $\beta_B = j^{n-1}_{Z,B}(\gamma)$ are generators, respectively, of $j^{n-1}_{Z,F}(L) \subset H^{n-1}(F;G)$ and $j^{n-1}_{Z,B}(L) \subset H^{n-1}(B;G)$ for any closed set $B \subset Z$. So, $j^{n-1}_{Z,B}(L) = 0$ for every proper closed subset B of F because $j^{n-1}_{Z,B}(\gamma) = j^{n-1}_{F,B}(\beta) = 0$. Hence, F is an $(n-1,G)$-bubble with respect to L.

Suppose our lemma is true for any such set Z and a subgroup (resp., a submodule) $L \subset H^{n-1}(Z;G)$ with $\le k$ generators. In case L has $k+1$ generators $\gamma_1,\dots,\gamma_{k+1}$, we first take a closed non-empty set $F_1 \subset Z$ which is a carrier for γ_1. So, $j^{n-1}_{Z,B}(\gamma_1) = 0$ for any proper closed subset B of F_1. If $H^{n-1}(B;G) = 0$ for all closed $B \subsetneq F_1$, then F_1 is as required. If $j^{n-1}_{Z,B^*}(L) \neq 0$ for some closed proper set $B^* \subset F_1$, then $j^{n-1}_{Z,B^*}(L)$ is generated by the set $\{j^{n-1}_{Z,B^*}(\gamma_i) : i = 2,\dots,k+1\}$. According to our inductive assumption, there exists a closed non-empty set $F \subset B^*$ which is an $(n-1,G)$-bubble in B^* with respect to $j^{n-1}_{Z,B^*}(L)$. Then F is an $(n-1,G)$-bubble in Z with respect to L. ∎

3. Proof of Theorems 1.1, 1.2 and Corollary 1.3. In this section, X continues to be as in Section 2, but G is assumed to be a countable PID (the last condition is used in the proof of Claim 1).

Proof of Theorem 1.1. As in the proof of Proposition 2.1, we may suppose that X is connected and $H^n(C;G) = 0$ for any closed proper subset C of X. Moreover, we equip X with a convex metric d generating its topology (such a metric exists, see [1]). According to [16, Theorem 2], there exists a closed subset $Y \subset X$ with $\dim_G Y = n$ and a dense open subset D of Y satisfying the following property: any $y \in D$ has sufficiently small neighborhoods U_y in Y such that the homomorphism $j^{n-1}_{\overline{U}_y,\mathrm{bd}_Y \overline{U}_y}$ is not surjective (here $\mathrm{bd}_Y \overline{U}_y$ denotes the boundary of $\overline{U}_y$ in Y). Because Y has a non-empty interior in X (by Corollary 2.3), there exists a point $x \in \mathrm{int}(Y) \cap D$, a connected open neighborhood W_x of x in X, and an element $\alpha_x \in H^{n-1}(\mathrm{bd}\,\overline{W}_x;G)$ such that α_x is not extendable over $\overline{W}_x$. We can suppose that $\overline{W}_x$ is contractible in a proper subset of X. So, by Lemma 2.6, $\overline{W}_x$ is an $(n-1)$-cohomology membrane for α_x spanned on $\mathrm{bd}\,\overline{W}_x$. Because X is homogeneous, it suffices to construct the required base $\mathcal{B}_x$ at that particular point x. We define $\mathcal{B}'_x$ to be the family of all open connected subsets $U \subset X$ containing x such that $U = \mathrm{int}(\overline{U})$ and $\overline{U}$ is contractible in W_x. Then $\mathcal{B}'_x$ is a local base at x and $\mathrm{bd}\,U = \mathrm{bd}\,\overline{U}$ for all $U \in \mathcal{B}'_x$.

Claim 1. *Every $U \in \mathcal{B}'_x$ has the following properties:*

(i) *$\overline{U}$ is an $(n-1)$-cohomology membrane for some element of the group $H^{n-1}(\operatorname{bd} U; G)$;*
(ii) *the module $L_U = j^{n-1}_{\overline{W}_x \setminus U, \operatorname{bd} U}(H^{n-1}(\overline{W}_x \setminus U; G)) \subset H^{n-1}(\operatorname{bd} U; G)$ is non-trivial and finitely generated;*
(iii) *the module $H^{n-1}(\operatorname{bd} U; G)$ is finitely generated provided the homomorphism $j^{n-1}_{\overline{U}, \operatorname{bd} U}$ is trivial.*

We fix $U \in \mathcal{B}'_x$ and a non-zero element $\alpha_x \in H^{n-1}(\operatorname{bd} \overline{W}_x; G)$ such that $\overline{W}_x$ is an $(n-1)$-cohomology membrane for α_x spanned on $\operatorname{bd} \overline{W}_x$. Then α_x is not extendable over $\overline{W}_x$ but it is extendable over every closed proper subset of $\overline{W}_x$. Next, extend α_x to an element $\widetilde{\alpha}_x \in H^{n-1}(\overline{W}_x \setminus U; G)$. Obviously, $\operatorname{bd} U \subset \overline{W}_x \setminus U$. Hence, the element $\gamma_U = j^{n-1}_{\overline{W}_x \setminus U, \operatorname{bd} U}(\widetilde{\alpha}_x) \in H^{n-1}(\operatorname{bd} U; G)$ is not extendable over $\overline{U}$ (otherwise α_x would be extendable over $\overline{W}_x$), in particular $\gamma_U \neq 0$. Since U is connected, by Lemma 2.6, $\overline{U}$ is an $(n-1)$-cohomology membrane for γ_U spanned on $\operatorname{bd} U$.

To prove (ii), let U_0 be an open subset of X with $\overline{U}_0 \subset U$. Since $\gamma_U \in L_U$ and $\gamma_U \neq 0$, we have $L_U \neq 0$. For any $\gamma \in L_U$ there are two possibilities: either γ is extendable over $\overline{U}$ or it is not extendable over $\overline{U}$. In both cases γ is extendable over the set $\overline{U} \setminus U_0$. Indeed, this is clear if γ is extendable on $\overline{U}$. If γ is not extendable over $\overline{U}$, then $\overline{U}$ is an $(n-1)$-cohomology membrane for γ spanned on $\operatorname{bd} U$ (Lemma 2.6). Consequently, γ is extendable over $\overline{U} \setminus U_0$ because $\overline{U} \setminus U_0$ is a proper subset of $\overline{U}$ containing $\operatorname{bd} U$. Hence, every $\gamma \in L_U$ is extendable over the set $\overline{W}_x \setminus U_0$, which is closed in X and contains $\operatorname{bd} U$ in its interior. Therefore, by [4, Theorem 17.4 and Corollary 17.5, p. 127], L_U is finitely generated. If $j^{n-1}_{\overline{U}, \operatorname{bd} U}(H^{n-1}(\overline{U}; G)) = 0$, then every $\gamma \in H^{n-1}(\operatorname{bd} U; G)$ is extendable over $\overline{W}_x \setminus U$ (see Lemma 2.5). Hence, $H^{n-1}(\operatorname{bd} U; G) \subset L_U$, and item (ii) yields (iii).

Let $\mathcal{B}''_x$ be the family of all $U \in \mathcal{B}'_x$ satisfying the following condition: $\operatorname{bd} U$ contains a continuum F_U such that $X \setminus F_U$ has exactly two components and F_U is an $(n-1, G)$-bubble with respect to the module L_U.

Claim 2. *$\mathcal{B}''_x$ is a local base at x.*

We fix $W_0 \in \mathcal{B}'_x$, and for every $\delta > 0$ denote by $B(x, \delta)$ the open ball in X with center x and radius δ. There exists $\varepsilon_x > 0$ such that $B(x, \delta) \subset W_0$ for all $\delta \leq \varepsilon_x$. Since d is a convex metric, each $B(x, \delta)$ is a connected open set such that $\operatorname{int} \overline{B(x, \delta)} = B(x, \delta)$. Because $\overline{W}_0$ is contractible in W_x, so is $\overline{B(x, \delta)}$. Hence, all $U_\delta = B(x, \delta)$, $\delta \leq \varepsilon_x$, belong to $\mathcal{B}'_x$. Consequently, by Claim 1, the modules $L_\delta = j^{n-1}_{\overline{W}_x \setminus U_\delta, \operatorname{bd} U_\delta}(H^{n-1}(\overline{W}_x \setminus U_\delta; G))$ are finitely generated. Then, by Lemma 2.8, there exists a closed non-empty set $F_\delta \subset \operatorname{bd} U_\delta$ with

F_δ being an $(n-1;G)$-bubble with respect to L_δ. Because F_δ is a carrier for any $\gamma \in L_\delta$, Lemma 2.7 implies that each F_δ is a continuum.

Let us show that the family $\{F_\delta : \delta \le \varepsilon_x\}$ is uncountable. Since the function $f : X \to \mathbb{R}$, $f(y) = d(x,y)$, is continuous and W_0 is connected, $f(W_0)$ is an interval containing $[0,\varepsilon_x]$ and $f^{-1}([0,\varepsilon_x)) = B(x,\varepsilon_x) \subset W_0$. So, $f^{-1}(\delta) = \operatorname{bd} U_\delta \neq \emptyset$ for all $\delta \le \varepsilon_x$. Hence, the family $\{F_\delta : \delta \le \varepsilon_x\}$ is indeed uncountable and consists of disjoint continua.

Moreover, $H^{n-1}(F_\delta;G) \neq 0$ and, according to Lemma 2.4, F_δ separates X. So, each $X \setminus F_\delta$ has at least two components. Then, by [7, Theorem 8], there exists $\delta_0 \le \varepsilon_x$ such that $X \setminus F_{\delta_0}$ has exactly two components. Therefore, $U_{\delta_0} = B(x,\delta_0) \in \mathcal{B}''_x$ and it is contained in W_0. This completes the proof of Claim 2.

Now, let $\mathcal{B}_x$ be the subfamily of all $U \in \mathcal{B}''_x$ such that $H^{n-1}(\operatorname{bd} U;G) \neq 0$ and both U and $X \setminus \overline{U}$ are connected.

Claim 3. *$\mathcal{B}_x$ is a local base at x.*

We take an arbitrary neighborhood U_0 of x such that $\overline{U}_0$ is contractible in W_x and shall construct a member of $\mathcal{B}_x$ contained in U_0. To this end let $\varepsilon = d(x, X \setminus U_0)$. According to Effros' theorem [9], there is $\eta > 0$ such that if $y,z \in X$ with $d(y,z) < \eta$, then $h(y) = z$ for some homeomorphism $h\colon X \to X$, which is $\varepsilon/2$-close to the identity on X. Now, choose a connected neighborhood W of x with $\overline{W} \subset B(x,\varepsilon/2)$ and $\operatorname{diam}(\overline{W}) < \eta$. Finally, take $U \in \mathcal{B}''_x$ such that $\overline{U}$ is contractible in W. There exists a continuum $F_U \subset \operatorname{bd} U$ such that $X \setminus F_U$ has exactly two components and F_U is an $(n-1,G)$-bubble with respect to the module $L_U = j^{n-1}_{\overline{W}_x \setminus U, \operatorname{bd} U}(H^{n-1}(\overline{W}_x \setminus U;G))$ (see Claim 2). If $F_U = \operatorname{bd} U$ we are done, for U is the desired member of $\mathcal{B}_x$.

Suppose that F_U is a proper subset of $\operatorname{bd} U$. Because F_U is an $(n-1,G)$-bubble with respect to L_U, it follows that $j^{n-1}_{\operatorname{bd} U, F_U}(L_U) \neq 0$. Hence, there exists $\gamma \in L_U$ such that $\beta = j^{n-1}_{\operatorname{bd} U, F_U}(\gamma) \neq 0$. Because F_U (as a subset of $\overline{U}$) is contractible in W and $\overline{W}$ (as a subset of $\overline{W}_x$) is contractible in a proper subset of X, we can apply Lemma 2.4 to conclude that F_U separates $\overline{W}$. So, $\overline{W} \setminus F_U = V_1 \cup V_2$ for some open, non-empty disjoint subsets $V_1, V_2 \subset \overline{W}$. Since U is a connected subset of $\overline{W} \setminus F_U$, U is contained in one of the sets V_1, V_2, say $U \subset V_1$. Hence, $F_U \cup \overline{V}_2 \subset \overline{W}_x \setminus U$. Observe that $\gamma \in L_U$ implies γ is extendable over $\overline{W}_x \setminus U$. Consequently, β is also extendable over $\overline{W}_x \setminus U$, in particular β is extendable over $F_U \cup \overline{V}_2$. On the other hand, F_U (as a subset of $\overline{U}$) is contractible in $\overline{W}$, so β is not extendable over $\overline{W}$ (otherwise β would be zero). Thus, since $(F_U \cup \overline{V}_1) \cap (F_U \cup \overline{V}_2) = F_U$, β is not extendable over $F_U \cup \overline{V}_1$. Let $\beta' = j^{n-1}_{F_U,F'}(\beta)$, where $F' = \overline{V}_1 \cap F_U$ (observe that $F' \neq \emptyset$ because $\overline{W}$ is connected).

If F' is a proper subset of F_U, then $\beta' = 0$ (recall that $j^{n-1}_{\mathrm{bd}\,U,F'}(\gamma) = \beta'$ and F_U being a carrier for any non-trivial element of $j^{n-1}_{\mathrm{bd}\,U,F_U}(L_U)$ yields $j^{n-1}_{\mathrm{bd}\,U,Q}(L_U) = 0$ for any proper closed subset Q of F_U). So, β' would be extendable over $\overline{V}_1$, which yields β is extendable over $F_U \cup \overline{V}_1$, a contradiction.

Therefore, $F' = F_U \subset \overline{V}_1$ and β is not extendable over $\overline{V}_1$. Consequently, there exists an $(n-1)$-cohomology membrane $P_\beta \subset \overline{V}_1$ for β spanned on F_U. By Corollary 2.2, $V = P_\beta \setminus F_U$ is a connected open set in X whose boundary, according to Proposition 2.1, is the set $F'' = \overline{X \setminus P_\beta} \cap \overline{P_\beta \setminus F_U} \subset F_U$ (we can apply Proposition 2.1 and Corollary 2.2 because P_β, as a subset of $\overline{W}_x$, is contractible in a proper subset of X). As above, using the fact that β is not extendable over P_β and $j^{n-1}_{\mathrm{bd}\,U,Q}(L_U) = 0$ for any proper closed subset $Q \subset F_U$, we can show that $F'' = F_U$ and $\mathrm{bd}\,\overline{V} = F_U$.

Summarizing the properties of V, we see that $\overline{V}$ is contractible in W_x (because so is $\overline{U}_0$), $V = \mathrm{int}\,\overline{V}$ (because $F_U = \mathrm{bd}\,\overline{V}$) and V is connected. Moreover, since $X \setminus F_U$ is the union of the open disjoint non-empty sets V and $X \setminus P_\beta$ such that V is connected and $X \setminus F_U$ has exactly two components, $X \setminus \overline{V}$ is also connected. Because F_U is an $(n-1, G)$-bubble with respect to the non-trivial module L_U, we have $H^{n-1}(\mathrm{bd}\,V; G) \neq 0$. Thus, if V contains x, then V is the desired member of $\mathcal{B}_x$.

If V does not contain x, we take a point $y \in V$ and a homeomorphism h on X such that $h(y) = x$ and $d(z, h(z)) < \varepsilon$ for all $z \in X$. Such a homeomorphism exists because $\mathrm{diam}(\overline{W}) < \eta$ and $x, y \in \overline{W}$. Then $h(V) \subset U_0$ (from the choice of ε and the fact that h is ε-close to the identity on X). So, $\overline{h(V)}$ is contractible in W_x. Since the remaining properties from the definition of $\mathcal{B}_x$ are invariant under homeomorphisms, $h(V)$ is the desired member of $\mathcal{B}_x$, which completes the proof of Claim 3.

The sets $U \in \mathcal{B}_x$ satisfy condition (1) from Theorem 1.1 (according to the definition of $\mathcal{B}_x$). The next claim completes the proof of Theorem 1.1.

CLAIM 4. *Every $U \in \mathcal{B}_x$ satisfies conditions* (2), (3) *from Theorem* 1.1.

Recall that each $\overline{U}$ is contractible in W_x, and $\overline{W}_x$ is contractible in a proper subset of X. Then, by Lemma 2.4, $H^{n-1}(\overline{U}; G) = 0$ because $\overline{U}$ does not separate X. Therefore, every non-trivial element $\gamma \in H^{n-1}(\mathrm{bd}\,U; G)$ is non-extendable over $\overline{U}$. Consequently, according to Lemma 2.6, $\overline{U}$ is an $(n-1)$-cohomology membrane for γ spanned on $\mathrm{bd}\,U$. So, U satisfies (2).

Since $H^{n-1}(\overline{U}; G) = 0$, the homomorphism $j^{n-1}_{\overline{U},\mathrm{bd}\,U}$ is trivial. Thus, Lemma 2.5 yields $H^{n-1}(\mathrm{bd}\,U; G) = j^{n-1}_{\overline{W}_x \setminus U,\mathrm{bd}\,U}(H^{n-1}(\overline{W}_x \setminus U; G))$ and, by Claim 1(iii), $H^{n-1}(\mathrm{bd}\,U; G)$ is finitely generated. Suppose there exists a proper closed subset $F \subset \mathrm{bd}\,U$ and a non-trivial element $\alpha \in H^{n-1}(F; G)$. Observe that α is not extendable over $\overline{U}$ because $H^{n-1}(\overline{U};G)=0$. Hence, there is an $(n-1)$-cohomology membrane $K_\alpha\subset\overline{U}$ for α spanned on F. Because $\overline{U}\setminus F$ is connected (recall that U is a dense connected subset of $\overline{U}\setminus F$) and $K\setminus F$ is both open and closed in $\overline{U}\setminus F$ (by Corollary 2.2), we have $K_\alpha=\overline{U}$. Finally, according to Proposition 2.1, $(K_\alpha\setminus F)\cap\overline{X\setminus K_\alpha}=\emptyset$. On the other hand, any point from $\operatorname{bd}U\setminus F$ belongs to $(K_\alpha\setminus F)\cap\overline{X\setminus K_\alpha}$, a contradiction. Therefore, $\operatorname{bd}U$ is an $(n-1,G)$-bubble and U satisfies condition (3). ∎

Proof of Theorem 1.2. If $K=X$, the conclusion of Theorem 1.2 is obviously true. Suppose K is a proper closed subset of X, which is an $(n-1)$-cohomology membrane spanned on A for some $\gamma\in H^{n-1}(A;G)$, but there exists a point $a\in(K\setminus A)\cap\overline{X\setminus K}$. Take a neighborhood $U\in\mathcal{B}_a$ such that $\overline{U}\cap A=\emptyset$. Since $K\setminus U$ is a closed proper subset of K containing A, γ is extendable over $K\setminus U$. So, there exists $\beta\in H^{n-1}(K\setminus U;G)$ with $j^{n-1}_{K\setminus U,A}(\beta)=\gamma$. Since $K\setminus A$ is connected (see Corollary 2.2), $\operatorname{bd}U\cap K\neq\emptyset$. Then $\beta_1=j^{n-1}_{K\setminus U,\operatorname{bd}U\cap K}(\beta)$ is a non-zero element of $H^{n-1}(\operatorname{bd}U\cap K;G)$ (otherwise β_1 would be extendable over $\overline{U}\cap K$, and hence, γ would be extendable over K). Since $\dim_G\operatorname{bd}U\le n-1$, β_1 is extendable to an element $\tilde\beta_1\in H^{n-1}(\operatorname{bd}U;G)$. So, $\tilde\beta_1$ is a non-zero element of $H^{n-1}(\operatorname{bd}U;G)$ and, by Theorem 1.1(2), $\overline{U}$ is an $(n-1)$-cohomology membrane for $\tilde\beta_1$ spanned on $\operatorname{bd}U$. Then $\overline{U}\cap K\neq\overline{U}$ would imply that $\tilde\beta_1$ is extendable over $\overline{U}\cap K$. Hence, γ would be extendable over K, a contradiction. Thus, $\overline{U}\subset K\setminus A$, which contradicts $a\in\overline{X\setminus K}$. Therefore, $(K\setminus A)\cap\overline{X\setminus K}=\emptyset$. ∎

Proof of Corollary 1.3. It was shown in [17] and [19] that the cohomology membranes' property from Theorem 1.2 implies the invariance of domain for homogeneous or locally homogeneous ANR spaces X with $\dim X=n$. Similar arguments provide the proof when $\dim_G X=n$. Take a point y in $V=f(U)$ and let $x=f^{-1}(y)$. Choose a connected open set $W\in\mathcal{B}_x$ such that $\overline{W}\subset U$. Then $\overline{W}$ is an $(n-1)$-cohomology membrane for some $\gamma\in H^{n-1}(\operatorname{bd}W;G)$ spanned on $\operatorname{bd}W$. Since $f(\overline{W})$ is homeomorphic to $\overline{W}$, it is an $(n-1)$-cohomology membrane for $(f^*)^{-1}(\gamma)\in H^{n-1}(f(\operatorname{bd}W);G)$ spanned on $f(\operatorname{bd}W)$. Then, by Theorem 1.2, $f(\overline{W})\setminus f(\operatorname{bd}W)$ does not intersect $\overline{X\setminus f(\overline{W})}$. This means that $f(\overline{W})\setminus f(\operatorname{bd}W)$ is an open set in X which contains y and is contained in V. So, V is also open. ∎

4. Proof of Theorem 1.4 and Corollary 1.5. Let $\widehat{H}_*$ be the exact homology (see [18], [20]). It is well known that for locally compact metric spaces the exact homology is isomorphic to the Steenrod homology. For any abelian group G the homological dimension $\operatorname{hdim}_G Y$ of a compactum Y is the greatest integer m such that $\widehat{H}_m(Y,A;G)\neq 0$ for some closed $A\subset Y$ (if there is no such m, then $\operatorname{hdim}_G Y = \infty$). It follows from the exact sequence

$$0 \to \operatorname{Ext}(H^{m+1}(Y,A),G) \to \widehat{H}_m(Y,A;G) \to \operatorname{Hom}(H^m(Y,A),G) \to 0$$

that $\operatorname{hdim}_G Y \le \dim Y$. Moreover, by [21], $\operatorname{hdim}_G X$ is the greatest m such that the local homology group $\widehat{H}_m(X, X\setminus x; G) = \varinjlim_{x\in U} \widehat{H}_m(X, X\setminus U; G)$ is not trivial for some $x \in X$.

Proof of Theorem 1.4. (1)⇒(2). Suppose X is dimensionally full-valued. Then, according to [11], $\operatorname{hdim}_{\mathbb{Z}} X = \dim_{\mathbb{Z}} X = n$. Hence, $\widehat{H}_n(X, X\setminus x) \neq 0$ for some $x \in X$ (the coefficient group $\mathbb{Z}$ in all homology and cohomology groups is suppressed). Because $\dim X = n$, the groups $\widehat{H}_n(X, X\setminus x)$ and $H_n(X, X\setminus x)$ are isomorphic (see [20, Theorem 4]). So, $H_n(X, X\setminus x) \neq 0$.

(2)⇒(3). Let $H_n(X, X\setminus x) \neq 0$ for some $x \in X$. Then $H_n(X, X\setminus U) \neq 0$ for sufficiently small neighborhoods U of x in X. Since by [20, Theorem 4] the groups $H_n(X, X\setminus U)$ and $\widehat{H}_n(X, X\setminus U)$ are isomorphic, $\widehat{H}_n(X, X\setminus V) \neq 0$ for some neighborhood V of x. On the other hand, $\dim X = n$ implies $H^{n+1}(X, X\setminus V) = 0$. Hence, it follows from the exact sequence

$$\operatorname{Ext}(H^{n+1}(X, X\setminus V),\mathbb{Z}) \to \widehat{H}_n(X, X\setminus V) \to \operatorname{Hom}(H^n(X, X\setminus V),\mathbb{Z}) \to 0$$

that there exists a non-trivial homomorphism from $H^n(X, X\setminus V)$ into $\mathbb{Z}$. This implies that $H^n(X, X\setminus V)$ contains elements of infinite order. Thus, we have $H^n(X, X\setminus V) \otimes \mathbb{Q} \neq 0$ and, by the universal coefficients formula, $H^n(X, X\setminus V; \mathbb{Q}) \neq 0$. So, $\dim_{\mathbb{Q}} X = n$. Because X is an ANR, we have $\dim_{\mathbb{Q}} X \le \dim_{\mathbb{S}^1} X \le \dim X$ (see [8, Example 1.3(1) and Theorem 12.3(2)]. Therefore, $\dim_{\mathbb{S}^1} X = n$.

(3)⇒(1). Assume $\dim_{\mathbb{S}^1} X = n$. The exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1 \to 0$$

implies that $\dim_{\mathbb{S}^1} X \le \max\{\dim_{\mathbb{R}} X, \dim X - 1\}$ (see [8]). Hence, $\dim_{\mathbb{R}} X = n$. According to [11], both the homological and the cohomological dimensions with respect to any field coincide, so $\operatorname{hdim}_{\mathbb{R}} X = \dim_{\mathbb{R}} X = n$. Thus, there exist $x \in X$ and a neighborhood U of x in X such that $\widehat{H}_n(X, X\setminus U; \mathbb{R}) \neq 0$. As in the proof of the implication (2)⇒(3), considering the short exact sequence

$$\operatorname{Ext}(H^{n+1}(X, X\setminus U),\mathbb{Z}) \to \widehat{H}_n(X, X\setminus U) \to \operatorname{Hom}(H^n(X, X\setminus U),\mathbb{Z}) \to 0,$$

we can show that $\dim_{\mathbb{Q}} X = n$. This implies that X is dimensionally full-valued. ■

Proof of Corollary 1.5. Let X be a metric homogeneous ANR compactum with $\dim X = 3$. According to [14, Corollary 2.7], we have $\overline{H}_3(X, X\setminus x) \neq 0$, where $\overline{H}_3(X, X\setminus x)$ denotes the singular homology group. On the other hand, by [15, Lemma 4], the groups $\overline{H}_3(X, X\setminus x)$ and $H_3(X, X\setminus x)$ are isomorphic. Then Theorem 1.4 shows that X is dimensionally full-valued. ■

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V. Valov
Department of Computer Science and Mathematics
Nipissing University
100 College Drive, P.O. Box 5002
North Bay, ON, P1B 8L7, Canada
E-mail: veskov@nipissingu.ca