ANNALES
POLONICI MATHEMATICI
116.3 (2016)

Positive solution for a quasilinear equation with critical growth in $\mathbb{R}^N$

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Abstract. We study the existence of positive solutions of the quasilinear problem

$$\begin{cases} -\Delta_N u + V(x)|u|^{N-2}u = f(u, |\nabla u|^{N-2}\nabla u), & x \in \mathbb{R}^N, \\ u(x) > 0, \quad x \in \mathbb{R}^N, \end{cases}$$

where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential, $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function. The main result follows from an iterative method based on Mountain Pass techniques.

1. Introduction and main result. In this paper, we study the existence of positive solutions of the quasilinear problem

$$\begin{cases} -\Delta_N u + V(x)|u|^{N-2}u = f(u, |\nabla u|^{N-2}\nabla u), & x \in \mathbb{R}^N, \\ u(x) > 0, \quad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential, and $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function.

In recent years, quasilinear problems with a gradient term have been subject to deep investigations: see for example [C, ZW, CW, FQ, DS] and the references therein. This kind of problem arises in numerous physical models: the turbulent flow of a gas in a porous medium, generalized reaction-diffusion theory etc. (see [A, CH, DI, CS, MH]).

We study in particular the so-called p-Laplacian equations, which are usually seen as the simplest generalizations of the Laplacian equation to the quasilinear context. The p-Laplacian is the second order nonlinear differen-

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2010 *Mathematics Subject Classification*: Primary 35J20; Secondary 35J62.
Key words and phrases: quasilinear problem, N-Laplacian equation, mountain pass theorem, iterative method.
Received 23 March 2015; revised 7 September 2015.
Published online 4 January 2016.

DOI: 10.4064/ap3664-1-2016

tial operator defined as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Here $p > 1$, and for $p = 2$ the p-Laplacian is the usual Laplacian. Quasilinear equations involving the p-Laplacian operator are widely used in physical models, for example, pseudo-plastic fluids correspond to $1 < p < 2$, dilatant fluids correspond to $p > 2$, and Newtonian fluids correspond to $p = 2$ [AE].

It is well known that the classical variational methods are not directly applicable to equations involving the derivatives of the solution in the nonlinear term. In [FG], the authors developed an iterative method based on Mountain Pass techniques to overcome this difficulty. A method inspired by this technique is applied in the present paper.

The motivation for our investigation is the case $1 < p < N$, which was studied by G. M. Figueiredo [F]. Using Mountain Pass techniques in $\mathbb{R}^N$, he proved the existence of a positive solution in $W^{1,p}(\mathbb{R}^N)$. In this paper, we are interested in the case $p = N$. We will use an iterative method to prove the existence of a positive solution of problem (1.1).

In order to state our main result, we make the following assumptions:

(A_1) $f(s, |\xi|^{N-2}\xi) = 0$ in $(-\infty, 0) \times \mathbb{R}^N$.

Since we are looking for a positive solution, assumption (A_1) is reasonable.

(A_2) The function f has exponential critical growth at the origin and at infinity (see [D]), that is,

$$\lim_{|s|\to 0} \frac{|f(s, |\xi|^{N-2}\xi)|}{|s|^{N-1}} = 0 \quad \text{for all } \xi \in \mathbb{R}^N,$$

and there exists $d_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(s, |\xi|^{N-2}\xi)|}{e^{d|s|^{N/(N-1)}}} = \begin{cases} 0 & \text{if } d > d_0, \\ \infty & \text{if } d < d_0, \end{cases}$$

for all $\xi \in \mathbb{R}^N$.

This assumption is motivated by the Trudinger–Moser inequality for bounded domains [T, M].

(A_3) (see [AF]) There exist constants $p > N$ and $\theta > 0$ such that

$$f(s, |\xi|^{N-2}\xi) \geq \theta s^{p-1}, \quad \forall s \geq 0, \forall \xi \in \mathbb{R}^N,$$

where

$$\theta > \left(\frac{8^N \nu(p-N)}{p(\nu - N)}\right)^{(p-N)/N} S^{p/N},$$

$$S = \inf_{u \in W^{1,N}\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N)\, dx}{(\int_{\mathbb{R}^N} |u|^p\, dx)^{N/p}}.$$

(A_4) There exists $C > 0$ such that
$$\left|\frac{\partial}{\partial s} f(s, |\xi|^{N-2}\xi)\right| \le C\exp(d_N s^{N/(N-1)}), \qquad \forall s \ge 0,\ \forall \xi \in \mathbb{R}^N,$$
where $d_N = N\omega_{N-1}^{1/(N-1)} > 0$ and ω_{N-1} is the $(N-1)$-dimensional measure of the $(N-1)$-sphere.

(A_5) For all $|s| > 0$ and $\xi \in \mathbb{R}^N$, there exists $\nu > N$ such that
$$0 < \nu F(s, |\xi|^{N-2}\xi) \le s f(s, |\xi|^{N-2}\xi),$$
where $F(s, |\xi|^{N-2}\xi) = \int_0^s f(t, |\xi|^{N-2}\xi)\,dt$.

(A_6) For each $\xi \in \mathbb{R}^N$, the function $g(s) =: f(s, |\xi|^{N-2}\xi)/s^{N-1}$ is nondecreasing for $s > 0$.

(A_7) The function f satisfies the following conditions:
$$|f(s_1, |\xi|^{N-2}\xi)| \le L_1|s_1 - s_2|^{N-1}$$
for all $s_1, s_2 \in [0, \rho_1]$ and all $|\xi| \le \rho_2$, and
$$|f(s, |\xi_1|^{N-2}\xi_1) - f(s, |\xi_2|^{N-2}\xi_2)| \le L_2|\xi_1 - \xi_2|^{N-1}$$
for all $s \in [0, \rho_1]$ and all $|\xi_1|, |\xi|_2 \le \rho_2$, where ρ_1 and ρ_2 depend on N and ν given in the previous assumptions.

The following inequality in $\mathbb{R}^N$ [DI] plays an important role in our proof:
$$\langle |\xi|^{N-2}\xi - |\eta|^{N-2}\eta, \xi - \eta\rangle \ge C_N|\xi - \eta|^N, \tag{1.2}$$
where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^N$.

Consider the following conditions on the potential:

(V_1) $V(x) \ge V_0 > 0$ for all $x \in \mathbb{R}^N$;

(V_2) $V(x)$ is a continuous 1-periodic function, that is, $V(x+y) = V(x)$ for all $y \in \mathbb{Z}^N$ and all $x \in \mathbb{R}^N$ (see [AF]).

REMARK 1.1. Condition (V_1) ensures that X below is a reflexive Banach space for the norm $\|u\|$.

In this paper, we always assume $V(x)$ satisfies (V_1) and (V_2). Before stating our main results, we give some notation.

For $1 \le p < \infty$, $L^p(\mathbb{R}^N)$ denotes the Lebesgue space with the norm
$$\|u\|_p = \Big(\int_{\mathbb{R}^N} |u|^p\,dx\Big)^{1/p}.$$

Define the function space
$$W^{1,N}(\mathbb{R}^N) = \{u \in L^N(\mathbb{R}^N) : |\nabla u| \in L^N(\mathbb{R}^N)\}$$
with the usual norm
$$\|u\|_{1,N} = \Big(\int_{\mathbb{R}^N} (|\nabla u|^N + |u|^N)\,dx\Big)^{1/N}.$$

Let

$$X = \Big\{ u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N)\, dx < \infty \Big\}.$$

Then X is a reflexive Banach space with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N)\, dx \Big)^{1/N},$$

and for all $N \le q < \infty$,

$$X \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \tag{1.3}$$

with continuous embeddings (see [DS]).

The main result in this paper is as follows.

THEOREM 1.2. *Assume* (A_1)–(A_7) *hold. Then problem* (1.1) *admits a positive solution in* $W^{1,N}(\mathbb{R}^N)$ *provided*

$$\frac{C_N - L_1}{C_N} > 0 \quad \textit{and} \quad \Big(\frac{L_2}{C_N - L_1}\Big)^{1/(N-1)} < 1.$$

2. Preliminary results. The following lemma is a version of the Trudinger–Moser inequality for $\mathbb{R}^N$.

LEMMA 2.1 (Trudinger–Moser inequality for unbounded domains; see also [BJ, Lemma 1]). *Given any* $u \in W^{1,N}(\mathbb{R}^N)$ *with* $N \ge 2$, *we have*

$$\int_{\mathbb{R}^N} \big(e^{d|u|^{N/(N-1)}} - S_{N-2}(d,u)\big)\, dx < \infty \quad \textit{for every } d > 0.$$

Moreover, if $\|\nabla u\|_N^N \le 1$, $\|u\|_N \le M < \infty$ *and* $d < d_N$, *then there exists a positive constant* $C = C(N, M, d)$ *such that*

$$\int_{\mathbb{R}^N} \big(e^{d|u|^{N/(N-1)}} - S_{N-2}(d,u)\big)\, dx < C,$$

where $d_N = N\omega_{N-1}^{1/(N-1)} > 0$ *and* ω_{N-1} *is the* $(N-1)$*-dimensional measure of the* $(N-1)$*-sphere, and*

$$S_{N-2}(d,u) = \sum_{k=0}^{N-2} \frac{d^k}{k!} |u|^{Nk/(N-1)}.$$

First, we consider the problem

$$\begin{cases} -\Delta_N u + V(x)|u|^{N-2}u = f(u, |\nabla v|^{N-2}\nabla v), & x \in \mathbb{R}^N, \\ u(x) > 0, \quad x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

for $v \in X \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ with $0 < \alpha < 1$.

Definition 2.2. A function $u \in X$ is said to be a (*weak*) *solution* of (2.1) if for any $\varphi \in X$,

$$\int_{\mathbb{R}^N} \big(|\nabla u|^{N-2}\nabla u \nabla\varphi + V(x)|u|^{N-2}u\varphi\big)\,dx = \int_{\mathbb{R}^N} f(u, |\nabla v|^{N-2}\nabla v)\varphi\,dx.$$

It is clear that problem (2.1) has a variational structure. The Euler functional associated with (2.1) is

$$J_v(u) = \frac{1}{N}\|u\|^N - \int_{\mathbb{R}^N} F(u, |\nabla v|^{N-2}\nabla v)\,dx.$$

We say that $J_v \in C^1(X, \mathbb{R})$ and its Gateaux derivative is given by

$$\begin{aligned} J_v'(u)\varphi = &\int_{\mathbb{R}^N} (|\nabla u|^{N-2}\nabla u\nabla\varphi + V(x)|u|^{N-2}u\varphi)\,dx \\ &- \int_{\mathbb{R}^N} f(u, |\nabla v|^{N-2}\nabla v)\varphi\,dx. \end{aligned}$$

It is well known that the weak solutions of (2.1) are the critical points of the energy functional $J_v(u)$.

Lemma 2.3. *Suppose that* (A_2) *holds. Let* $v \in X \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $0 < \alpha < 1$. *Then there exist* $\beta, \rho > 0$ *such that* $J_v(u) \ge \beta > 0$ *for all* $u \in X$ *with* $\|u\| = \rho$.

Proof. By (A_2), given $\varepsilon > 0$ and $s \ge 1$, there exists $C_\varepsilon = C(\varepsilon, s) > 0$ such that, for every $d > d_0$,

$$|F(t, |\xi|^{N-2}\xi)| \le \frac{\varepsilon}{N}|t|^N + C_\varepsilon |t|^s\big(e^{d|t|^{N/(N-1)}} - S_{N-2}(d, t)\big)$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

The following inequality can be found in [DJ, DS]:

$$\int_{\mathbb{R}^N} |u|^s\big(e^{d|u|^{N/(N-1)}} - S_{N-2}(d, u)\big)\,dx \le C(d, N)\|u\|^s,$$

provided that $\|u\| \le \delta$, where the positive constant δ is sufficiently small.

Using the Hölder inequality, we get

$$\begin{aligned} J_v(u) &\ge \frac{1}{N}\|u\|^N - \frac{\varepsilon}{N}\|u\|_N^N - C_\varepsilon \int_{\mathbb{R}^N} |u|^s\big(e^{d|u|^{N/(N-1)}} - S_{N-2}(d, u)\big)\,dx \\ &\ge \frac{1}{N}\|u\|^N - \frac{\varepsilon}{N}C_1\|u\|^N - C_2\|u\|^s \end{aligned}$$

with small $\|u\|$. Taking $\varepsilon = 1/(2C_1)$, we deduce that

$$J_v(u) \ge \frac{1}{2N}\|u\|^N - C_2\|u\|^s.$$

Choosing $s > N$, we can consider $\rho > 0$ sufficiently small satisfying

$$\beta := \frac{1}{2N}\rho^N - C_2\rho^s > 0.$$

For $\|u\| = \rho$, we have

$$J_v(u) \geq \frac{1}{2N}\rho^N - C_2\rho^s = \beta > 0. \ \blacksquare$$

LEMMA 2.4. *Suppose that* (A_1)–(A_5) *hold. Let* $v \in X \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $0 < \alpha < 1$ *and* $w_0 \in C_0^\infty(\mathbb{R}^N)$ *with* $\|w_0\|_X = 1$. *Then there exists* $T > 0$, *independent of* v, *such that*

$$J_v(tw_0) \leq 0 \quad \textit{for all } t \geq T. \tag{2.2}$$

Proof. For $s \in \mathbb{R}$, we define

$$L(t) = t^{-\nu}F(ts, |\xi|^{N-2}\xi) - F(s, |\xi|^{N-2}\xi), \quad t \geq 1.$$

Then it follows from (A_5) that

$$L'(t) = t^{-\nu-1}\big(stf(ts, |\xi|^{N-2}\xi) - \nu F(ts, |\xi|^{N-2}\xi)\big) \geq 0$$

for all $t \geq 1$. Hence, $L(t) \geq L(1) = 0$ for all $t \geq 1$ and then

$$F(ts, |\xi|^{N-2}\xi) \geq t^\nu F(s, |\xi|^{N-2}\xi).$$

Using this it is easy to check that

$$J_v(tw_0) \leq \frac{1}{N}t^N\|w_0\| - t^\nu \int_{\mathbb{R}^N} F(w_0, |\nabla v|^{N-2}\nabla v)\, dx.$$

Since $\nu > N$, we have $J_v(tw_0) \to -\infty$ as $t \to \infty$. Hence, there exists a constant $T > 0$ such that (2.2) holds. $\blacksquare$

LEMMA 2.5. *Under assumptions* (A_1)–(A_6), *problem* (2.1) *has a positive solution* $u_v \in C^{1,\alpha}_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^N)$ *with* $0 < \alpha < 1$ *for any* $v \in X \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. *Moreover, there exist constants* $\rho_1, \rho_2 > 0$, *independent of* v, *such that* $\|u_v\|_{C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \leq \rho_1$ *and* $\|\nabla u_v\|_{C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \leq \rho_2$.

Proof. By Lemmas 2.3 and 2.4, the functional J_v satisfies the geometric conditions of the Mountain Pass Theorem. Hence, by a version of the Mountain Pass Theorem without the (PS) condition [W], there exists a sequence $\{u_n\} \subset X$ satisfying

$$J_v(u_n) \to c_v \quad \text{and} \quad J'_v(u_n) \to 0, \quad \text{as } n \to \infty,$$

where

$$c_v = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_v(\gamma(t)) > 0,$$

with

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = Tw_0\},$$

where w_0 and T are as in Lemma 2.4.

By virtue of (A_5), we have

$$c_v + \|u_n\| + o_n(1) \ge J_v(u_n) + \frac{1}{\nu} J_v'(u_n)u_n \ge \left(\frac{1}{N} - \frac{1}{\nu}\right)\|u_n\|^N.$$

When n is sufficiently large, we get

$$c_v + \|u_n\| \ge \left(\frac{1}{N} - \frac{1}{\nu}\right)\|u_n\|^N.$$

Denoting $C_3 = 1/N - 1/\nu$, we obtain

$$C_3\|u_n\|^N \le c_v + \|u_n\|.$$

Thus, $\{u_n\}$ is bounded in X. Hence, there exist $u_v \in X$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u_v \quad \text{in } X, \tag{2.3}$$

$$u_n \to u_v \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } N \le s, \tag{2.4}$$

$$u_n(x) \to u_v(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{2.5}$$

Also, as proved in [D], we get

$$\frac{\partial u_n}{\partial x_i}(x) \to \frac{\partial u_v}{\partial x_i}(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Passing to a subsequence if necessary, we can deduce that

$$\nabla u_n(x) \to \nabla u_v(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{2.6}$$

Thanks to (2.6), we obtain

$$|\nabla u_n|^{N-2}\nabla u_n \to |\nabla u_v|^{N-2}\nabla u_v \quad \text{a.e. in } \mathbb{R}^N.$$

Since $\{|\nabla u_n|^{N-2}\nabla u_n\}$ is bounded in $L^{N/(N-1)}(\mathbb{R}^N)$, we conclude that

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u_v|^{N-2}\nabla u_v \quad \text{in } L^{N/(N-1)}(\mathbb{R}^N).$$

Therefore

$$\int_{\mathbb{R}^N} |\nabla u_n|^{N-2}\nabla u_n \nabla\varphi\, dx \to \int_{\mathbb{R}^N} |\nabla u_v|^{N-2}\nabla u_v \nabla\varphi\, dx$$

for all $\varphi \in X$.

Similarly, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^{N-2}u_n\varphi\, dx \to \int_{\mathbb{R}^N} |u_v|^{N-2}u_v\varphi\, dx$$

for all $\varphi \in X$.

By using assumptions (A_2) and (A_4), given $\varepsilon > 0$, $q \ge 0$ and $\beta_0 > 1$ there exists $C_\varepsilon > 0$ such that

$$f(s, |\xi|^{N-2}\xi) \le \varepsilon s^{N-1} + C_\varepsilon s^q\big(\exp(\beta_0 d_N s^{N/(N-1)}) - S_{N-2}(\beta_0 d_N, s)\big) \tag{2.7}$$

for all $s \ge 0$ and $\xi \in \mathbb{R}^N$.

Thanks to the proof of Lemma 3 in [AF] we conclude that

$$c_v < \frac{\nu - N}{8^N N\nu}.$$

From (2.3)–(2.5) and (A_5) we obtain

$$\begin{aligned} c_v = \lim_{n\to\infty} J_v(u_n) &= \lim_{n\to\infty}\left(J_v(u_n) - \frac{1}{\nu}J_v'(u_n)u_n\right) \\ &\geq \frac{\nu - N}{N\nu}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}(|\nabla u_n|^N + V_0|u_n|^N)\,dx. \end{aligned}$$

It follows that

$$\limsup_{n\to\infty}\|\nabla u\|_N^N \leq \frac{N\nu c_v}{\nu - N} < \frac{1}{8^N} < 1$$

and

$$\limsup_{n\to\infty}\|u_n\|_N^N \leq \frac{N\nu c_v}{V_0(\nu - N)}.$$

Then, based on Lemma 2.1, we conclude that there exist $C > 0$, and $\beta_0, r > 1$ close to 1, such that the sequence $\{G_n\}$ given by

$$G_n(x) = \exp(\beta_0 d_N|u_n|^{N/(N-1)}) - S_{N-2}(\beta_0 d_N, u_n)$$

belongs to $L^r(\mathbb{R}^N)$ and $\|G_n\|_r \leq C$ for all $n \in \mathbb{N}$.

Applying (2.7) and the Dominated Convergence Theorem [B], we have

$$\int\limits_{\mathbb{R}^N} f(u_n, |\nabla v|^{N-2}\nabla v)\varphi\,dx \to \int\limits_{\mathbb{R}^N} f(u_v, |\nabla v|^{N-2}\nabla v)\varphi\,dx \tag{2.8}$$

for all $\varphi \in X$.

So, we obtain $J_v'(u_v)\varphi = 0$ for all $\varphi \in X$.

Suppose $u_v \not\equiv 0$. By (A_1), we get $u_v \geq 0$ and $u_v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $0 < \alpha < 1$. By the Harnack inequality, $u_v > 0$ for all $x \in \mathbb{R}^N$. Moreover, similar to the proof in [BE], there exist constants $\rho_1, \rho_2 > 0$, independent of v, such that $\|u_v\|_{C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \leq \rho_1$ and $\|\nabla u_v\|_{C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \leq \rho_2$.

If $u_v \equiv 0$, we first prove that there exist a sequence $\{x_n\} \subset \mathbb{R}^N$ and $\alpha_1, R > 0$ such that

$$\int\limits_{B_R(x_n)} |u_n|^N\,dx \geq \alpha_1. \tag{2.9}$$

Supposing the contrary, we have

$$\limsup_{\substack{n\to\infty \\ y\in\mathbb{R}^N}} \int\limits_{B_R(y)} |u_n|^N\,dx = 0.$$

Applying [L, Lemma 8.4], we obtain

$$u_n \to 0 \quad \text{in } L^t(\mathbb{R}^N) \text{ for all } t \in (N, \infty),$$

which implies that

$$J_v(u_n) \to 0 \quad \text{as } n \to \infty.$$

This is absurd because it implies $c_v = 0$. Let $w_n(x) = u_n(x + x_n)$. Since $V(x)$ is a 1-periodic function, we can use the invariance of $\mathbb{R}^N$ under translations to conclude that $J_v(w_n) \to c_v$ and $J_v'(w_n) \to 0$. Moreover, up to a subsequence, $w_n \rightharpoonup w_v$ in X and $w_n \to w_v$ in $L^N(B_R(0))$ with w_v being a critical point of J_v and $w_v \neq 0$. In the similar manner to the proof of the case $u_v \not\equiv 0$, we conclude that w_v is a nontrivial solution of (1.1), and the lemma is proved. ∎

LEMMA 2.6. *Let $v \in X \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ with $0 < \alpha < 1$. Then there exists a constant $K > 0$, independent of v, such that $\|u_v\| \le K$ for all solutions u_v obtained in Lemma* 2.5.

Proof. Using (A_6), we obtain

$$c_v = \inf_{u \in X \setminus \{0\}} \sup_{t \ge 0} J_v(tu).$$

By (A_5), there exist constants $a, b > 0$ such that

$$F(s, |\xi|^{N-2}\xi) \ge a|s|^\nu - b$$

for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

Choosing w_0 in Lemma 2.4 and by (A_5), we obtain

$$\begin{aligned} J_v(tw_0) &\le \frac{1}{N} \int_{\mathbb{R}^N} |\nabla(tw_0)|^N \, dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x)|tw_0|^N \, dx \\ &\quad - \int_{\operatorname{supp} w_0} (at^\nu |w_0|^\nu - b) \, dx \\ &\le \frac{1}{N} t^N - C_3 t^\nu + b|\operatorname{supp} w_0|. \end{aligned} \tag{2.10}$$

Denote

$$\max_{t \ge 0} \left(\frac{t^N}{N} - C_3 t^\nu + b|\operatorname{supp} w_0| \right) =: k.$$

Then $c_v \le k$. By (A_6), we obtain

$$J_v(u_v) - \frac{1}{\nu} J_v'(u_v) u_v \ge \left(\frac{1}{N} - \frac{1}{\nu} \right) \|u_v\|^N.$$

A simple computation yields

$$\|u_v\| \le \left(k \left(\frac{1}{N} - \frac{1}{\nu} \right)^{-1} \right)^{1/N} =: K. \ ∎$$

3. Proof of Theorem 1.2. Thanks to Lemma 2.5, we construct a sequence $\{u_n\} \subset X \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ with $0 < \alpha < 1$ as solutions of

$$(P_n) \qquad -\Delta_N u_n + V(x)|u_n|^{N-2}u = f(u, |\nabla u_{n-1}|^{N-2}\nabla u_{n-1}), \qquad x \in \mathbb{R}^N,$$

starting with an arbitrary $u_0 \in X \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, with

$$\|u_n\|_{C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \le \rho_1 \quad \text{and} \quad \|\nabla u_n\|_{C^{0,\alpha}_{\mathrm{loc}}}(\mathbb{R}^N) \le \rho_2.$$

Since u_{n+1} is the solution of (P_{n+1}), we have

$$\begin{aligned}(3.1) \qquad & \int_{\mathbb{R}^N} \big(|\nabla u_{n+1}|^{N-2}\nabla u_{n+1} \cdot \nabla u_{n+1} + V(x)|u_{n+1}|^{N-2}u_{n+1} \cdot u_{n+1}\big)\,dx \\ &= \int_{\mathbb{R}^N} f(u_{n+1}, |\nabla u_n|^{N-2}\nabla u_n)u_{n+1}\,dx\end{aligned}$$

and

$$\begin{aligned}(3.2) \qquad & \int_{\mathbb{R}^N} \big(|\nabla u_{n+1}|^{N-2}\nabla u_{n+1} \cdot \nabla u_n + V(x)|u_{n+1}|^{N-2}u_{n+1}u_n\big)\,dx \\ &= \int_{\mathbb{R}^N} f(u_{n+1}, |\nabla u_n|^{N-2}\nabla u_n)u_n\,dx.\end{aligned}$$

Applying (3.1) and (3.2), we see that

$$\begin{aligned}(3.3) \qquad & \int_{\mathbb{R}^N} |\nabla u_{n+1}|^{N-2}\nabla u_{n+1}(\nabla u_{n+1} - \nabla u_n)\,dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|u_{n+1}|^{N-2}u_{n+1}(u_{n+1} - u_n)\,dx \\ &= \int_{\mathbb{R}^N} f(u_{n+1}, |\nabla u_n|^{N-2}\nabla u_n)(u_{n+1} - u_n)\,dx.\end{aligned}$$

Similarly, since u_n is the solution of (P_n), we have

$$\begin{aligned}(3.4) \qquad & \int_{\mathbb{R}^N} |\nabla u_n|^{N-2}\nabla u_n(\nabla u_{n+1} - \nabla u_n)\,dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|u_n|^{N-2}u_n(u_{n+1} - u_n)\,dx \\ &= \int_{\mathbb{R}^N} f(u_n, |\nabla u_{n-1}|^{N-2}\nabla u_{n-1})(u_{n+1} - u_n)\,dx.\end{aligned}$$

By (1.2), (3.3) and (3.4), we obtain

$$\|u_{n+1}-u_n\|^N \leq \frac{1}{C_N}\int_{\mathbb{R}^N}\big(f(u_{n+1},|\nabla u_n|^{N-2}\nabla u_n)-f(u_n,|u_n|^{N-2}\nabla u_n)\big)\delta(u_n)\,dx + \frac{1}{C_N}\int_{\mathbb{R}^N}\big(f(u_n,|\nabla u_n|^{N-2}\nabla u_n)-f(u_n,|\nabla u_{n-1}|^{N-2}\nabla u_{n-1})\big)\delta(u_n)\,dx,$$

where $\delta(u_n)=u_{n+1}-u_n$.

Applying (A_7), we obtain

$$\frac{C_N-L_1}{C_N}\|u_{n+1}-u_n\|^N \leq \frac{L_2}{C_N}\int_{\mathbb{R}^N}|\nabla u_n-\nabla u_{n-1}|^{N-1}|u_{n+1}-u_n|\,dx.$$

Then, by Hölder's inequality, it follows that

$$\|u_{n+1}-u_n\| \leq \left(\frac{L_2}{C_N-L_1}\right)^{1/(N-1)}\|u_n-u_{n-1}\| =: \tilde{k}\|u_n-u_{n-1}\|,$$

where $\tilde{k}=\big(\frac{L_2}{C_N-L_1}\big)^{1/(N-1)}$. Since the coefficient $\tilde{k}$ is less than 1, the sequence $\{u_n\}$ strongly converges in X to some function $u\in X$. Furthermore, by Lemma 2.3, we know that $u>0$ in $\mathbb{R}^N$. Theorem 1.2 is proved.

Acknowledgements. This research was partly supported by YSYB (grant no. 201519), NSFC (grant no. 11461075) and NSFC (grant no. 11461016). The authors would like to thank the referee and the editor for valuable comments and suggestions.

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