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Noncommutative fractional integrals

by

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Abstract. Let $\mathcal{M}$ be a hyperfinite finite von Nemann algebra and $(\mathcal{M}_k)_{k\geq 1}$ be an increasing filtration of finite-dimensional von Neumann subalgebras of $\mathcal{M}$. We investigate abstract fractional integrals associated to the filtration $(\mathcal{M}_k)_{k\geq 1}$. For a finite noncommutative martingale $x = (x_k)_{1\leq k\leq n} \subseteq L_1(\mathcal{M})$ adapted to $(\mathcal{M}_k)_{k\geq 1}$ and $0 < \alpha < 1$, the fractional integral of x of order α is defined by setting

$$I^\alpha x = \sum_{k=1}^{n} \zeta_k^\alpha dx_k$$

for an appropriate sequence $(\zeta_k)_{k\geq 1}$ of scalars. For the case of a noncommutative dyadic martingale in $L_1(\mathcal{R})$ where $\mathcal{R}$ is the type II_1 hyperfinite factor equipped with its natural increasing filtration, $\zeta_k = 2^{-k}$ for $k \geq 1$.

We prove that I^α is of weak type $(1, 1/(1-\alpha))$. More precisely, there is a constant c depending only on α such that if $x = (x_k)_{k\geq 1}$ is a finite noncommutative martingale in $L_1(\mathcal{M})$ then

$$\|I^\alpha x\|_{L_{1/(1-\alpha),\infty}(\mathcal{M})} \leq \mathrm{c}\|x\|_{L_1(\mathcal{M})}.$$

We also show that I^α is bounded from $L_p(\mathcal{M})$ into $L_q(\mathcal{M})$ where $1 < p < q < \infty$ and $\alpha = 1/p - 1/q$, thus providing a noncommutative analogue of a classical result. Furthermore, we investigate the corresponding result for noncommutative martingale Hardy spaces. Namely, there is a constant c depending only on α such that if $x = (x_k)_{k\geq 1}$ is a finite noncommutative martingale in the martingale Hardy space $\mathcal{H}_1(\mathcal{M})$ then $\|I^\alpha x\|_{\mathcal{H}_{1/(1-\alpha)}(\mathcal{M})} \leq \mathrm{c}\|x\|_{\mathcal{H}_1(\mathcal{M})}$.

0. Introduction. For $n \geq 1$, let $\mathcal{F}_n$ be the σ-algebra generated by dyadic intervals of length 2^{-n} in the unit interval $[0,1]$, $\mathcal{F}$ be the σ-algebra generated by $\bigcup_{n\geq 1}\mathcal{F}_n$, and $\mathbb{P}$ denote the Lebesgue measure on $[0,1]$. A martingale $\{f_n\}_{n\geq 1}$ on the probability space $([0,1],\mathcal{F},\mathbb{P})$ adapted to the increas-

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ing filtration $\{\mathcal{F}_n\}_{n\geq 1}$ is called a *dyadic martingale*. The theory of dyadic martingales has played an important role in the development of classical analysis such as harmonic analysis and Banach space theory. For instance, the connection between the study of Haar basis in rearrangement invariant spaces on $[0,1]$ and dyadic martingales is quite obvious. The monograph by Müller [23] contains a very detailed account of dyadic martingale Hardy spaces and their applications in modern analysis. Dyadic martingales also appear naturally in various Littlewood–Paley type theories. We refer to the books [11, 20] for these historical facts.

Our primary interest in this article is closely related to the so-called *fractional integrals* for dyadic martingales. These are special classes of martingale transforms. Let us review the basic classical setup. Given a dyadic martingale $f = \{f_n\}_{n\geq 1}$ and $0 < \alpha < 1$, the *dyadic fractional integral* (of order α) of f is the sequence $I^\alpha f = \{(I^\alpha f)_n\}_{n\geq 1}$ defined by setting

$$(I^\alpha f)_n = \sum_{k=1}^{n} 2^{-k\alpha} df_k, \qquad n \geq 1, \tag{0.1}$$

where $\{df_k\}_{k\geq 1}$ is the martingale difference sequence of f. Dyadic fractional integrals are closely related to some particular types of Walsh–Fourier series. They also appear in various forms in function theory which goes back to Hardy and Littewood. In [4], Chao and Ombe proved the boundedness of fractional integrals between various L_p-spaces depending on the size of α. Their results can be summarized as follows:

THEOREM 0.1 ([4]).

(1) *For $1 < p < q < \infty$ and $\alpha = 1/p - 1/q$, there exists a constant $C_{p,q}$ depending only on p and q such that*

$$\Big\| \sup_{n\geq 1} |(I^\alpha f)_n| \Big\|_q \leq C_{p,q} \|f\|_p, \qquad f \in L_p[0,1].$$

(2) *For $0 < \alpha < 1$, there exists a constant C_α depending only on α such that for every $f \in L_1[0,1]$,*

$$\mathbb{P}\Big[\sup_{n\geq 1} |(I^\alpha f)_n| \geq \lambda\Big] \leq C_\alpha \left(\frac{\|f\|_1}{\lambda}\right)^{1/(1-\alpha)}, \qquad \forall \lambda > 0.$$

Recall that martingale transforms are of strong type (p,p) for $1 < p < \infty$ and of weak type $(1,1)$. The emphasis here is that the special nature of the coefficients in the fractional integrals provides these L_p-L_q type boundedness results as opposed to just the familiar L_p-boundedness of martingale transforms.

Our primary objective in this article is to investigate possible generalizations of fractional integrals in the general framework of noncommutative martingales. This is of course part of the general development of noncommutative martingale theory; we refer the reader to [27, 15, 17, 29] for recent history and results. We will work with a general hyperfinite finite von Neumann algebra $\mathcal{M}$ with increasing filtration $(\mathcal{M}_n)_{n\geq 1}$ of finite-dimensional subalgebras. We consider a unified approach to fractional integrals for noncommutative martingales adapted to $(\mathcal{M}_n)_{n\geq 1}$. These abstract fractional integrals are of course closely connected to the size of the filtration $(\mathcal{M}_n)_{n\geq 1}$. For the case of noncommutative dyadic martingales, i.e., when the von Neumann algebra is the hyperfinite type II_1 factor $\mathcal{R}$ equipped with its natural increasing filtration, these fractional integrals turn out to be exactly as in (0.1) (we refer the reader to Section 2 below for details).

The paper is organized as follows. In the next section, we collect notions and notation from noncommutative symmetric spaces and noncommutative martingale theory necessary for our presentation. In Section 2, we formulate the general fractional integrals and provide systematic studies of their actions on various spaces. In particular, we prove results that mirror those from classical settings. Our first result can be roughly stated as fractional integrals of order α being of weak type $(1, 1/(1-\alpha))$. Using duality and interpolation, we also obtain boundedness between various noncommutative Lorentz spaces. Moreover, they can be strengthened using the noncommutative maximal functions developed by Junge [15] (see Theorems 2.2 and 2.9). These results go beyond Theorem 0.1 in two ways: they provide a unified approach to fractional integrals that are not restricted to dyadic martingales, and also the method we use is general enough to include martingales that are not necessarily regular. We also investigate fractional integrals acting between noncommutative Hardy spaces. More precisely, we obtain $\mathcal{H}_1$-$\mathcal{H}_p$ boundedness of the fractional integral I^α where $p = 1/(1-\alpha)$. This is formulated in Theorem 2.11 below. In the last section, we explore when the various results obtained in the previous section can be extended to include the case $0 < p < 1$. This is accomplished through the use of noncommutative atomic decompositions and noncommutative atomic Hardy spaces for martingales.

1. Preliminaries and notation. In this preliminary section we introduce some basic definitions and well-known results concerning noncommutative L_p-spaces and noncommutative martingales. We use standard notation for operator algebras as may be found in the books [19, 30].

1.1. Noncommutative symmetric spaces. In this subsection we will review the general construction of noncommutative spaces. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a distinguished faithful normal semifinite trace τ. Assume that $\mathcal{M}$ is acting on a Hilbert space H.

A closed densely defined operator x on H is said to be *affiliated with* $\mathcal{M}$ if x commutes with every unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If a is a densely defined self-adjoint operator on H and $a = \int_{\mathbb{R}} s\, de_s^a$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(a)$ the corresponding spectral projection $\int_{\mathbb{R}} \chi_B(s)\, de_s^a$. An operator x affiliated with $\mathcal{M}$ is called *τ-measurable* if there exists $s > 0$ such that $\tau(\chi_{(s,\infty)}(|x|)) < \infty$.

Let $\widetilde{\mathcal{M}}$ denote the topological $*$-algebra of all τ-measurable operators. For $x \in \widetilde{\mathcal{M}}$,

$$\mu_t(x) = \inf\{s > 0 : \tau(\chi_{(s,\infty)}(|x|)) \leq t\}, \qquad t > 0.$$

The function $t \mapsto \mu_t(x)$ from the interval $[0, \tau(\mathbf{1}))$ to $[0, \infty]$ is called the *generalized singular value function* of x. Note that $\mu_t(x) < \infty$ for all $t > 0$, and $t \mapsto \mu_t(x)$ is a decreasing function. We observe that if $\mathcal{M} = L^\infty(\mathbb{R}_+)$ then $\widetilde{\mathcal{M}}$ is the space of Lebesgue measurable functions on $\mathbb{R}_+$, and for any given $f \in \widetilde{\mathcal{M}}$, $\mu(f)$ is precisely the classical decreasing rearrangement of the function $|f|$ commonly used in the theory of rearrangement invariant function spaces as described in [2, 20]. We refer the reader to [12] for a more in-depth study of $\mu(\cdot)$.

For $0 < p < \infty$, we recall that the *noncommutative L_p-space* associated with $(\mathcal{M}, \tau)$ is defined by $L_p(\mathcal{M}, \tau) = \{x \in \widetilde{\mathcal{M}} : \tau(|x|^p) < \infty\}$ with

$$\|x\|_p = \tau(|x|^p)^{1/p} = \Big(\int_0^\infty \mu_t(x)^p\, dt\Big)^{1/p}.$$

More generally, one can extend the preceding definition to more general function spaces which we now summarize. We first recall some basic definitions from the general theory of rearrangement invariant spaces. We denote by $L_0(\mathbb{R}_+)$ the space of all $\mathbb{C}$-valued Lebesgue measurable functions defined on $\mathbb{R}_+$.

A quasi-Banach space $(E, \|\cdot\|_E)$, where $E \subset L_0(\mathbb{R}_+)$, is called a *rearrangement invariant quasi-Banach function space* if it follows from $f \in E$, $g \in L^0(\mathbb{R}_+)$, and $\mu(g) \leq \mu(f)$ that $g \in E$ and $\|g\|_E \leq \|f\|_E$. Furthermore, $(E, \|\cdot\|_E)$ is called *symmetric Banach function space* if it has the additional property that $f, g \in E$ and $g \prec\prec f$ imply that $\|g\|_E \leq \|f\|_E$. Here $g \prec\prec f$ denotes submajorization in the sense of Hardy–Littlewood–Pólya:

$$\int_0^t \mu_s(g)\, ds \leq \int_0^t \mu_s(f)\, ds \qquad \text{for all } t > 0.$$

We refer the reader to [20] for any unexplained terminology from the general theory of rearrangement invariant function spaces and symmetric spaces.

Given a semifinite von Neumann algebra $(\mathcal{M},\tau)$ and a symmetric quasi-Banach function space $(E,\|\cdot\|_E)$ on the interval $[0,\infty)$, we define the corresponding noncommutative space by setting:

$$E(\mathcal{M},\tau)=\{x\in\widetilde{\mathcal{M}}:\mu(x)\in E\}.$$

Equipped with the quasi-norm $\|x\|_{E(\mathcal{M},\tau)}:=\|\mu(x)\|_E$, the space $E(\mathcal{M},\tau)$ (or simply $E(\mathcal{M})$) is a complex quasi-Banach space and is generally referred to as the *noncommutative symmetric space* associated with $(\mathcal{M},\tau)$ corresponding to $(E,\|\cdot\|_E)$. Extensive discussions of various properties of such spaces can be found in [5, 9, 31].

In this article, we will be mainly working with Lorentz spaces. For $0<p,q\le\infty$, the *Lorentz space* $L_{p,q}$ is the subspace of all $f\in L_0(\mathbb{R}_+)$ such that

$$\|f\|_{p,q}=\begin{cases}\Big(\int_0^\infty (t^{1/p}\mu_t(f))^q\,\frac{dt}{t}\Big)^{1/q} & \text{if } 0<q<\infty,\\ \sup_{t>0} t^{1/p}\mu_t(f) & \text{if } q=\infty.\end{cases}$$

is finite. Clearly, $L_{p,p}(\mathbb{R}_+)=L_p(\mathbb{R}_+)$. If $1\le q\le p<\infty$ or $p=q=\infty$, then $L_{p,q}(\mathbb{R}_+)$ is a symmetric Banach function space. If $1<p<\infty$ and $p\le q\le\infty$, then $L_{p,q}(\mathbb{R}_+)$ can be equivalently renormed to become a symmetric Banach function space. In general, $L_{p,q}(\mathbb{R}_+)$ is only a symmetric quasi-Banach function space. Basic properties of Lorentz spaces may be found in [2, 20]. Through the general construction of noncommutative spaces described above we may define the noncommutative Lorentz space $L_{p,q}(\mathcal{M},\tau)$ associated with $(\mathcal{M},\tau)$ corresponding to $L_{p,q}(\mathbb{R}_+)$.

We now review some properties of noncommutative Lorentz spaces that we will need throughout. We will make use of the well-known fact that for $1\le p<\infty$ and $x\in L_{p,\infty}(\mathcal{M},\tau)$,

$$\|x\|_{p,\infty}=\sup_{\lambda>0}\lambda(\tau(\chi_{(\lambda,\infty)}(|x|)))^{1/p}.$$

The following quasi-triangle inequality is a very simple but useful fact. We refer to [29] for a short proof.

Lemma 1.1. *If $x_1,x_2\in L_{1,\infty}(\mathcal{M},\tau)$ and $\lambda>0$,*

$$\lambda\tau(\chi_{(\lambda,\infty)}(|x_1+x_2|))\le 2\lambda\tau(\chi_{(\lambda/2,\infty)}(|x_1|))+2\lambda\tau(\chi_{(\lambda/2,\infty)}(|x_2|)).$$

From the general duality theory for noncommutative spaces developed by Dodds et al. [10], we may also state that for $1<p,q<\infty$,

$$(L_{p,q}(\mathcal{M},\tau))^*=L_{p',q'}(\mathcal{M},\tau), \tag{1.1}$$

where p' and q' denote the conjugate indices of p and q, respectively. Noncommutative Lorentz spaces behave well with respect to real interpolation.

Indeed, we may deduce from [3, Theorem 5.3.1, p. 113] and [28, Corollary 2.2] that if $0 < \theta < 1$, $0 < p_j, q_j \leq \infty$ for $j \in \{0,1\}$, and $p_0 \neq p_1$, then

$$L_{p,q}(\mathcal{M},\tau) = [L_{p_0,q_0}(\mathcal{M},\tau), L_{p_1,q_1}(\mathcal{M},\tau)]_{\theta,q} \tag{1.2}$$

(with equivalent quasi-norms), where $1/p = (1-\theta)/p_0 + \theta/p_1$. All these basic facts will be used in what follows.

1.2. Noncommutative martingales. In this subsection, we recall some background from the theory of noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumann subalgebras of a von Neumann algebra $\mathcal{M}$ such that the union of the $\mathcal{M}_n$'s is w^*-dense in $\mathcal{M}$. Assume that there exists a trace preserving conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ (this is always the case if $\mathcal{M}$ is a finite von Neumann algebra). It is well-known that $\mathcal{E}_n$ extends to a bounded projection from $L_1(\mathcal{M})+\mathcal{M}$ onto $L_1(\mathcal{M}_n)+\mathcal{M}_n$ and consequently, by interpolation, from $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_n)$ for all $1 \leq p \leq \infty$.

A sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M}) + \mathcal{M}$ is called a *noncommutative martingale* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if

$$\mathcal{E}_n(x_{n+1}) = x_n, \quad \forall n \geq 1.$$

If additionally $x \subseteq L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, then x is called an $L_p(\mathcal{M})$-*martingale*. In this case, we set

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, then x is called an L_p-*bounded martingale*. Similarly, we may also consider martingales that are bounded in $L_{p,q}(\mathcal{M})$ when $1 < p \leq \infty$, $0 < q \leq \infty$ and set

$$\|x\|_{p,q} = \sup_{n\geq 1} \|x_n\|_{p,q}.$$

We refer to [14] for more information on $L_{p,q}$-bounded martingales.

For a given martingale $x = (x_n)_{n\geq 1}$, we assume the usual convention that $x_0 = 0$. The martingale difference sequence $dx = (dx_k)_{k\geq 1}$ of x is defined by

$$dx_k = x_k - x_{k-1}, \quad k \geq 1.$$

Let us now recall the definitions of the square functions and Hardy spaces for noncommutative martingales. Following [27], we introduce the column and row versions of square functions relative to a martingale $x = (x_n)_{n\geq 1}$:

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2\Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2\Big)^{1/2};$$

and

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2\Big)^{1/2}.$$

Let $0 < p \leq \infty$. Define $\mathcal{H}_p^c(\mathcal{M})$ (resp., $\mathcal{H}_p^r(\mathcal{M})$) as the completion of the space of all finite martingales in $\mathcal{M} \cap L_p(\mathcal{M})$ under the (quasi) norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp., $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$). When $1 \leq p \leq \infty$, $\mathcal{H}_p^c(\mathcal{M})$ and $\mathcal{H}_p^r(\mathcal{M})$ are Banach spaces, while for $0 < p < 1$ they are only p-Banach spaces. The *Hardy space of noncommutative martingales* is defined as follows: if $0 < p < 2$, then

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}),$$

equipped with the (quasi) norm

$$\|x\|_{\mathcal{H}_p} = \inf\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\},$$

where the infimum is taken over all decompositions $x = y + z$ with $y \in \mathcal{H}_p^c(\mathcal{M})$ and $z \in \mathcal{H}_p^r(\mathcal{M})$. For $2 \leq p \leq \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

We also need $\ell_p(L_p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\geq 1} \|a_n\|_p^p\Big)^{1/p} < \infty.$$

For a martingale x, set

$$s_d(x) = \Big(\sum_{n\geq 1} |dx_n|_p^p\Big)^{1/p}.$$

It is clear that for such x, we have

$$\|s_d(x)\|_p = \|dx\|_{\ell_p(L_p(\mathcal{M}))}.$$

We define $\mathsf{h}_p^d(\mathcal{M})$ to be the set of all martingales x such that $dx \in \ell_p(L_p(\mathcal{M}))$, equipped with the (quasi) norm

$$\|x\|_{\mathsf{h}_p^d} = \|s_d(x)\|_p.$$

We mention that there are also other Hardy spaces such as the noncommutative conditioned Hardy spaces in the literature, but they will not be used in this paper.

Our primary examples are noncommutative bounded martingales in various Lorentz spaces associated with the type II_1-hyperfinite factor $\mathcal{R}$. Let $\mathbb{M}_2$ be the algebra of 2×2 matrices with the usual normalized trace tr_2. Recall that

$$(\mathcal{R}, \tau) = \overline{\bigotimes_{i\geq 1}}(\mathbb{M}_2, \mathrm{tr}_2).$$

For $n \geq 1$, we denote by $\mathcal{R}_n$ the finite-dimensional von Neumann subalgebra given by the finite tensor product $\bigotimes_{1\leq i\leq n}(\mathbb{M}_2, \mathrm{tr}_2)$ of $\mathcal{R}$. It is customary to identify $\mathcal{R}_n$ with $\mathbb{M}_{2^n}$, where $\mathbb{M}_{2^n}$ is the algebra of $2^n \times 2^n$ matrices equipped with the normalized trace tr_{2^n}. Moreover, we view $\mathcal{R}_n$ as a von Neumann subalgebra of $\mathcal{R}_{n+1}$ via the inclusion

$$x \in \mathcal{R}_n \mapsto x \otimes \mathbf{1}_{\mathbb{M}_2} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \mathcal{R}_{n+1},$$

where $\mathbf{1}_{\mathbb{M}_2}$ is the identity of $\mathbb{M}_2$. With these inclusions, it is clear that $(\mathcal{R}_n)_{n\geq 1}$ forms an increasing filtration of von Neumann subalgebras whose union is weak*-dense in $\mathcal{R}$. Martingales corresponding to the filtration $(\mathcal{R}_n)_{n\geq 1}$ are called *"noncommutative" dyadic martingales*. They are indeed generalizations of dyadic martingales from classical probability theory.

We conclude this subsection with the statement of the noncommutative Gundy decomposition from [25] which will be crucial in what follows. Below, $\mathrm{supp}(a)$ denotes the support projection of the measurable operator a in the sense of [30].

THEOREM 1.2 ([25]). *If $x = (x_n)_{n\geq 1}$ is an L_1-bounded noncommutative martingale and λ is a positive real number, then there exist four martingales φ, ψ, η, and υ satisfying the following properties for some absolute constant* c*:*

(i) $x = \varphi + \psi + \eta + \upsilon$;

(ii) *the martingale φ satisfies*

$$\|\varphi\|_1 \leq \mathrm{c}\|x\|_1, \qquad \|\varphi\|_2^2 \leq \mathrm{c}\lambda\|x\|_1, \qquad \|\varphi\|_\infty \leq \mathrm{c}\lambda;$$

(iii) *the martingale ψ satisfies*

$$\sum_{k=1}^{\infty} \|d\psi_k\|_1 \leq \mathrm{c}\|x\|_1;$$

(iv) *γ and υ are L_1-martingales with*

$$\max\Big\{\lambda\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}\,|d\eta_k|\Big),\, \lambda\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}\,|d\upsilon_k^*|\Big)\Big\} \leq \mathrm{c}\|x\|_1.$$

In the following, the letters $C_p, \kappa_p, \ldots$ will denote positive constants depending only on the subscripts indicated, and $C, \kappa, \ldots$ are absolute constants. All these constants can change from line to line.

2. Noncommutative fractional integrals. In this section, we define fractional integrals for noncommutative martingales. For the remainder of the paper, we assume that $\mathcal{M}$ is a hyperfinite and finite von Neumann algebra and the filtration $(\mathcal{M}_k)_{k\geq 1}$ consists of finite-dimensional von Neumann subalgebras of $\mathcal{M}$.

For $k \geq 1$, we define the difference operator $\mathcal{D}_k = \mathcal{E}_k - \mathcal{E}_{k-1}$ where $\mathcal{E}_0 = 0$. Let

$$\mathcal{D}_{k,p} := \mathcal{D}_k(L_p(\mathcal{M})) = \{x \in L_p(\mathcal{M}_k) : \mathcal{E}_{k-1}(x) = 0\}.$$

Since $\dim(\mathcal{M}_k) < \infty$, the $\mathcal{D}_{k,p}$'s are finite-dimensional subspaces of $L_p(\mathcal{M})$ for all $1 \leq p \leq \infty$. Moreover, for $p \neq q$, the spaces $\mathcal{D}_{n,p}$ and $\mathcal{D}_{n,q}$ coincide as sets. In particular, the formal identity $\iota_k : \mathcal{D}_{k,\infty} \to \mathcal{D}_{k,2}$ is a natural isomorphism.

For $k \geq 1$, set

$$\zeta_k := 1/\|\iota_k^{-1}\|^2. \tag{2.1}$$

Clearly, $0 < \zeta_k \leq 1$ for all $k \geq 1$ and $\lim_{k\to\infty} \zeta_k = 0$. Moreover, for every $x \in \mathcal{D}_{k,2}$, we have

$$\|x\|_\infty \leq \zeta_k^{-1/2}\|x\|_2. \tag{2.2}$$

Furthermore, if we denote by j_k the inclusion map from $\mathcal{D}_{k,\infty}$ into $\mathcal{M}_k$, then one can easily verify that for every $x \in L_1(\mathcal{M}_k)$, $(j_k\iota_k^{-1}\mathcal{D}_k)^*(x) = \mathcal{E}_k(x) - \mathcal{E}_{k-1}(x) \in L_2(\mathcal{M}_k)$ (here $\mathcal{D}_k : L_2(\mathcal{M}_k) \to \mathcal{D}_{k,2}$). In particular, for every $x \in \mathcal{D}_{k,1}$,

$$\|x\|_2 \leq 2\zeta_k^{-1/2}\|x\|_1. \tag{2.3}$$

The following definition constitutes the main topic of this paper. This was primarily inspired by a similar notion used by Chao and Ombe [4] for classical dyadic martingales in $L_1[0,1]$ described in the introduction. We propose a setup that goes beyond the dyadic situation.

Definition 2.1. For a given noncommutative martingale $x = (x_n)_{n\geq 1}$ and $0 < \alpha < 1$, we define the *fractional integral of order* α of x to be the sequence $I^\alpha x = \{(I^\alpha x)_n\}_{n\geq 1}$ where for every $n \geq 1$,

$$(I^\alpha x)_n = \sum_{k=1}^{n} \zeta_k^\alpha dx_k$$

with $(\zeta_k)_{k\geq 1}$ from (2.1).

Since $\alpha > 0$, the operation I^α is a martingale transform with bounded coefficients, and thus, according to [27, 29], I^α of strong type (p,p) for $1 < p < \infty$ and is of weak type $(1,1)$. In particular, if x is an L_1-bounded martingale then $\{(I^\alpha x)_n\}_{n\geq 1}$ is a martingale (adapted to the same filtration) that is bounded in $L_{1,\infty}(\mathcal{M})$. At the end of this section we will briefly discuss the motivation for our choice of the scalar coefficients $(\zeta_k)_{k\geq 1}$ as defined in (2.1) and point out that it is the optimal choice for all the results in this section to hold. We should also emphasize here that for the case of "noncommutative" dyadic filtration on $\mathcal{R}$, one can easily verify that $\zeta_k = 2^{-k}$ for $k \geq 1$, and therefore our definition is indeed a proper generalization of the classical dyadic fractional integrals described in the introduction.

Our goal is to strengthen the above facts about martingale transforms. More precisely, we aim to generalize Theorem 0.1 to our abstract noncommutative settings. In particular, we show that I^α is of weak type $(1, 1/(1-\alpha))$. This specific result leads to various weak-type inequalities and to boundedness of fractional integrals between different Lorentz spaces.

2.1. Weak-type boundedness and consequences. The following weak-type estimate is the main result of this subsection.

THEOREM 2.2. *Let $0 < \alpha < 1$. There exists a constant c_α such that if x is an L_1-bounded dyadic martingale then*

$$\|I^\alpha x\|_{L_{1/(1-\alpha),\infty}(\mathcal{M})} \leq c_\alpha \|x\|_1.$$

In preparation for the proof of Theorem 2.2, we first establish various preliminary lemmas.

LEMMA 2.3. *Let $k \geq 1$ and $a \in \mathcal{D}_{k,\infty}$. Then:*

(i) *For any given $0 < \alpha < 1$,*

$$\zeta_k^\alpha \|a\|_{1/(1-\alpha)} \leq 2^\alpha \|a\|_1.$$

(ii) *For $1 < p < 2$ and $\alpha = 1/p - 1/2$,*

$$\zeta_k^\alpha \|a\|_2 \leq \|a\|_p.$$

Proof. For (i), we have

$$\begin{aligned}\zeta_k^\alpha \|a\|_{1/(1-\alpha)} &= \zeta_k^\alpha \tau(|a|^{1/(1-\alpha)})^{1-\alpha} = \zeta_k^\alpha \tau(|a|^{\alpha/(1-\alpha)}|a|)^{1-\alpha} \\ &\leq \zeta_k^\alpha \|a\|_\infty^\alpha \|a\|_1^{1-\alpha}.\end{aligned}$$

By (2.2) and (2.3), we have

$$\|a\|_\infty \leq \zeta_k^{-1/2} \|a\|_2 \leq 2\zeta_k^{-1} \|a\|_1.$$

When combined with the above estimate, this leads to

$$\zeta_k^\alpha \|a\|_{1/(1-\alpha)} \leq 2^\alpha \zeta_k^\alpha \zeta_k^{-\alpha} \|a\|_1^\alpha \|a\|_1^{1-\alpha} = 2^\alpha \|a\|_1.$$

The argument for (ii) is similar. Assume $\alpha = 1/p - 1/2$ and $a \in \mathcal{D}_{k,\infty}$. Then

$$\begin{aligned}\|a\|_2 &= \tau(|a|^2)^{1/2} = \tau(|a|^{2-p}|a|^p)^{1/2} \leq \|a\|_\infty^{(2-p)/2} \|a\|_p^{p/2} \\ &\leq (\zeta_k^{-1/2})^{(2-p)/2} \|a\|_2^{(2-p)/2} \|a\|_p^{p/2} \leq \zeta_k^{-\alpha p/2} \|a\|_2^{(2-p)/2} \|a\|_p^{p/2}.\end{aligned}$$

This implies that $\zeta_k^{\alpha p/2} \|a\|_2^{p/2} \leq \|a\|_p^{p/2}$, which after raising to the power $2/p$ gives the stated inequality. ∎

As immediate consequences of Lemma 2.3, we obtain

LEMMA 2.4.

(i) *For* $0 < \alpha < 1$, *the operator* I^α *is bounded from* $\mathsf{h}_1^d(\mathcal{M})$ *into* $L_{1/(1-\alpha)}(\mathcal{M})$.

(ii) *If* $1 < p < 2$ *and* $\alpha_0 = 1/p - 1/2$, *then there exists a constant* c_p *such that for every* $z \in L_p(\mathcal{M})$,

$$\|(I^{\alpha_0}z)_n\|_2 \le \mathrm{c}_p\|z\|_p.$$

That is, I^{α_0} *is bounded from* $L_p(\mathcal{M})$ *into* $L_2(\mathcal{M})$.

Proof. The first item is immediate from Lemma 2.3(i). For the second item, fix $z \in L_p(\mathcal{M})$ and $n \ge 1$. Then, since for every $k \ge 1$, $dz_k \in \mathcal{D}_{k,\infty}$, we may deduce from Lemma 2.3(ii) that

$$\|(I^{\alpha_0}z)_n\|_2^2 = \sum_{k=1}^n \zeta_k^{2\alpha_0}\|dz_k\|_2^2 \le \sum_{k=1}^n \|dz_k\|_p^2.$$

Using the fact that $L_p(\mathcal{M})$ is of cotype 2 ([28]), it follows that there is a constant κ_p such that

$$\|(I^{\alpha_0}z)_n\|_2^2 \le \kappa_p^2\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k dz_k\Big\|_p^2$$

where $(\varepsilon_k)_k$ is a Rademacher sequence and $\mathbb{E}$ denotes the expectation on the ε_k's. Furthermore, by the L_p-boundedness of martingale transforms (see [27]), there is another constant β_p such that

$$\|(I^{\alpha_0}z)_n\|_2^2 \le \kappa_p^2\beta_p^2\|z\|_p^2,$$

which proves (ii). ∎

LEMMA 2.5. *Let* $1 < p < 2$, $1/p + 1/p' = 1$, *and* $\alpha = 1/p - 1/p'$. *Then* I^α *is bounded from* $L_p(\mathcal{M})$ *into* $L_{p'}(\mathcal{M})$. *More precisely, for every* $z \in L_p(\mathcal{M})$ *and* $n \ge 1$,

$$\|(I^\alpha z)_n\|_{p'} \le \mathrm{c}_p^2\|z\|_p,$$

where c_p *is the constant from Lemma* 2.3(ii).

Proof. Note first that $\alpha = 2\alpha_0$ where α_0 is from Lemma 2.4(ii). Fix $y \in L_p(\mathcal{M})$ with $\|y\|_p = 1$. Then

$$\begin{aligned}
|\langle I^\alpha z, y\rangle| = |\tau((I^\alpha z)y^*)| &= \Big|\tau\Big(\Big(\sum_k \zeta_k^\alpha dz_k\Big)\Big(\sum_k dy_k^*\Big)\Big)\Big| \\
&= \Big|\tau\Big(\sum_k \zeta_k^\alpha dz_k dy_k^*\Big)\Big| = \Big|\tau\Big(\sum_k \zeta_k^{2\alpha_0} dz_k dy_k^*\Big)\Big| \\
&= \Big|\tau\Big(\Big(\sum_k \zeta_k^{\alpha_0} dz_k\Big)\Big(\sum_k \zeta_k^{\alpha_0} dy_k^*\Big)\Big)\Big| \\
&\le \|I^{\alpha_0}z\|_2\,\|I^{\alpha_0}y\|_2\,.
\end{aligned}$$

It then follows from Lemma 2.4(ii) that $|\langle I^\alpha z, y\rangle| \le c_p^2 \|z\|_p$. Since y is arbitrary, the desired inequality follows. ■

Proof of Theorem 2.2. We have to prove the existence of a constant c_α such that for any fixed $n \ge 1$ and every $s > 0$, we have

$$\tau\big(\chi_{(s,\infty)}(|(I^\alpha x)_n|)\big) \le c_\alpha(\|x\|_1/s)^{1/(1-\alpha)}. \tag{2.4}$$

By linearity and homogeneity, we may assume without loss of generality that $x \ge 0$ with $\|x\|_1 = 1$. Since the trace τ is normalized, it is enough to consider only the case $s > 1$. Let $\lambda = s^{1/(1-\alpha)}$.

We apply the noncommutative Gundy decomposition stated in Theorem 1.2 to the martingale x and $\lambda > 1$. There exist four martingales φ, ψ, η, and υ with $x = \varphi + \psi + \eta + \upsilon$ that satisfy the properties enumerated in Theorem 1.2. Clearly, for any given $n \ge 1$ we have

$$(I^\alpha x)_n = (I^\alpha \varphi)_n + (I^\alpha \psi)_n + (I^\alpha \eta)_n + (I^\alpha \upsilon)_n.$$

Using the elementary inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$ for operators, we obtain

$$|(I^\alpha x)_n|^2 \le 4|(I^\alpha \varphi)_n|^2 + 4|(I^\alpha \psi)_n|^2 + 4|(I^\alpha \eta)_n|^2 + 4|(I^\alpha \upsilon)_n|^2.$$

Now, according to Lemma 1.1, we have

$$\begin{aligned}
\tau\big(\chi_{(s,\infty)}(|(I^\alpha x)_n|)\big) &= \tau\big(\chi_{(s^2,\infty)}(|(I^\alpha x)_n|^2)\big)\\
&\le 4\tau\big(\chi_{(s^2/4,\infty)}(4|(I^\alpha \varphi)_n|^2)\big) + 4\tau\big(\chi_{(s^2/4,\infty)}(4|(I^\alpha \psi)_n|^2)\big)\\
&\quad + 4\tau\big(\chi_{(s^2/4,\infty)}(4|(I^\alpha \eta)_n|^2)\big) + 4\tau\big(\chi_{(s^2/4,\infty)}(4|(I^\alpha \upsilon)_n|^2)\big)\\
&=: I + II + III + IV.
\end{aligned}$$

It suffices to estimate I, II, III, and IV separately.

For I, let $1 < p < 2$ be such that $\alpha = 1/p - 1/p'$ with $1/p + 1/p' = 1$. Then using Chebyshev's inequality and the result already established in Lemma 2.5 that I^α is bounded from $L_p(\mathcal{M})$ into $L_{p'}(\mathcal{M})$, we get

$$\begin{aligned}
I &= 4\tau\big(\chi_{(s/2,\infty)}(2|(I^\alpha \varphi)_n|)\big) \le 4^{p'+1}s^{-p'}\|(I^\alpha\varphi)_n\|_{p'}^{p'}\\
&\le 4^{p'+1}c_p^2 s^{-p'}\|\varphi\|_p^{p'} = 4^{p'+1}c_p^2 s^{-p'}\tau(|\varphi|^p)^{p'/p}\\
&= 4^{p'+1}c_p^2 s^{-p'}\tau(|\varphi|^{p-1}|\varphi|)^{p'/p} \le 4^{p'+1}c_p^2 s^{-p'}\|\varphi\|_\infty^{(p-1)p'/p}\|\varphi\|_1^{p'/p}.
\end{aligned}$$

Since $\|\varphi\|_1 \le c$ and $\|\varphi\|_\infty \le c\lambda$, we deduce that

$$I \le 4^{p'+1}c_p^2 c^{p'} s^{-p'}\lambda^{(p-1)p'/p} \le 4^{(2p-1)/(p-1)}c_p^2 c^{p/(p-1)}\lambda^{-1},$$

which shows the existence of a constant c_α such that

$$I \le c_\alpha \lambda^{-1}.$$

For II, we first apply Chebyshev's inequality as above to get

$$\begin{aligned} II &= 4\tau\big(\chi_{(s/2,\infty)}(2|(I^\alpha\psi)_n|)\big) \\ &\le 4^{(2-\alpha)/(1-\alpha)}s^{-1/(1-\alpha)}\|(I^\alpha\psi)_n\|_{1/(1-\alpha)}^{1/(1-\alpha)}. \end{aligned}$$

According to Lemma 2.4(i), I^α is bounded from $\mathsf{h}_1^d(\mathcal{M})$ into $L_{1/(1-\alpha)}(\mathcal{M})$. Therefore, we may deduce that

$$II \le \mathrm{c}_\alpha \lambda^{-1}\|\psi\|_{\mathsf{h}_1^d}^{1/(1-\alpha)} \tag{2.5}$$

for $\mathrm{c}_\alpha = 4^{(2-\alpha)/(1-\alpha)}$. Combining (2.5) with Theorem 1.2(iii) provides the desired estimate for II.

To estimate III, we note that using polar decompositions of the $d\eta_k$'s, the operator $(I^\alpha\eta)_n$ is right-supported by the projection $\bigvee_{k\ge1}\operatorname{supp}|d\eta_k|$. Consequently, the operator $|(I^\alpha\eta)_n|$ is supported by $\bigvee_{k\ge1}\operatorname{supp}|d\eta_k|$ and thus we may conclude from Theorem 1.2(iv) that

$$III \le 4\tau\Big(\bigvee_{k\ge1}\operatorname{supp}|d\eta_k|\Big) \le 4\mathrm{c}\lambda^{-1}.$$

For IV, we observe that $(I^\alpha\upsilon)^* = I^\alpha\upsilon^*$. Arguing as in the case of III, we find that $|(I^\alpha\upsilon)_n^*|$ is supported by the projection $\bigvee_{k\ge1}\operatorname{supp}|d\upsilon_k|$. Similarly, we may deduce from Theorem 1.2(iv) that

$$IV = 4\tau\big(\chi_{(s/2,\infty)}(2|(I^\alpha\upsilon)_n^*|)\big) \le 4\tau\Big(\bigvee_{k\ge1}\operatorname{supp}|d\upsilon_k|\Big) \le 4\mathrm{c}\lambda^{-1}.$$

As noted above, combining the estimates on I–IV proves (2.4). ∎

We now consider some applications of Theorem 2.2 to strong type boundedness of fractional integrals. Given $0 < \alpha < 1$, we observe from (1.1) that the noncommutative Lorentz space $L_{1/\alpha,1}(\mathcal{M})$ is the Köthe dual of the noncommutative symmetric space $L_{1/(1-\alpha),\infty}(\mathcal{M})$ in the sense of [10]. Thus, it immediately follows from Theorem 2.2 that restricting the adjoint of I^α to the Köthe dual implies that $(I^\alpha)^* : L_{1/\alpha,1}(\mathcal{M}) \to \mathcal{M}$ is bounded. On the other hand, it can be easily verified from the definition that the adjoint $(I^\alpha)^*$ is formally equal to the fractional integral I^α itself and thus we may state:

Corollary 2.6. *Let $0 < \alpha < 1$. Then I^α is bounded from $L_{1/\alpha,1}(\mathcal{M},\tau)$ into $\mathcal{M}$.*

Using interpolation, we also get:

Corollary 2.7. *Let $1 < p < q < \infty$, $0 < r \le \infty$, and $\alpha = 1/p - 1/q$. The mapping I^α is bounded from $L_{p,r}(\mathcal{M})$ into $L_{q,r}(\mathcal{M})$. In particular, I^α is bounded from $L_p(\mathcal{M})$ into $L_q(\mathcal{M})$.*

Proof. Interpolating Theorem 2.2 and Corollary 2.6, we deduce that for $0<\theta<1$ and $0<r\le\infty$,

$$I^\alpha : [L_1(\mathcal{M}), L_{1/\alpha,1}(\mathcal{M})]_{\theta,r} \to [L_{1/(1-\alpha),\infty}(\mathcal{M}), L_\infty(\mathcal{M})]_{\theta,r}$$

is bounded. Choosing θ so that $1/p = 1+(\alpha-1)\theta$ and $1/q=(1-\theta)(1-\alpha)$, the interpolation result in (1.2) yields the desired conclusion. ∎

Next, we consider improvements of Theorem 2.2 and Corollary 2.7 using maximal functions. For this, let us recall the noncommutative ℓ_∞-valued spaces considered first in [26, 15] for the noncommutative L_p-spaces and in [7] for the more general case of noncommutative symmetric spaces.

Let E be a symmetric Banach function space on $\mathbb{R}_+$ and $\mathcal{N}$ be a semifinite von Neumann algebra equipped with a semifinite trace σ. We set $E(\mathcal{N},\ell_\infty)$ to be the space of all sequences $x=(x_k)_{k\ge1}$ in $E(\mathcal{N},\sigma)$ for which there exist $a,b\in E^{(2)}(\mathcal{N},\sigma)$ and a bounded sequence $y=(y_k)_{k\ge1}$ in $\mathcal{N}$ such that for every $k\ge1$,

$$x_k = a y_k b,$$

where $E^{(2)}(\mathcal{N},\sigma) = \{a\in\widetilde{\mathcal{N}} : |a|^2\in E(\mathcal{N},\sigma)\}$ equipped with the norm $\|a\|_{E^{(2)}(\mathcal{N})} = \||a|^2\|_{E(\mathcal{N})}^{1/2}$.

For $x\in E(\mathcal{N};\ell_\infty)$, we define

$$\|x\|_{E(\mathcal{N};\ell_\infty)} := \inf\Big\{\|a\|_{E^{(2)}(\mathcal{N})}\sup_{k\ge1}\|y_k\|_\infty\|b\|_{E^{(2)}(\mathcal{N})}\Big\},$$

where the infimum is taken over all possible factorizations of x as described above. We point out that whenever $(|x_k|)_{k\ge1}$ is a commuting sequence and thus the maximal function $Mx=\sup_{k\ge1}|x_k|$ is well-defined, the value of $\|(x_k)_{k\ge1}\|_{E(\mathcal{N};\ell_\infty)}$ is precisely the norm of Mx in $E(\mathcal{N},\sigma)$. This justifies the use of the space $E(\mathcal{N};\ell_\infty)$ as a substitute for the lack of supremum or maximum for sets of noncommuting operators. This remarkable discovery was made by Junge [15] who applied this analogy to formulate the noncommutative Doob maximal inequalities.

Before proceeding, we also need to recall the notion of Boyd indices. Let E be a symmetric Banach space on $(0,\infty)$. For $s>0$, the dilation operator $D_s : E\to E$ is defined by setting

$$D_s f(t) = f(t/s),\qquad t>0,\ f\in E.$$

The *lower* and *upper Boyd indices* of E are defined by

$$p_E := \lim_{s\to\infty}\frac{\log s}{\log\|D_s\|}\quad\text{and}\quad q_E := \lim_{s\to0^+}\frac{\log s}{\log\|D_s\|}.$$

It is a well known fact that $1\le p_E\le q_E\le\infty$. Moreover, if $E=L_p$ for $1\le p\le\infty$ then $p_E=q_E=p$.

The key tool we use is provided by a recent generalization of Junge's noncommutative Doob maximal inequality due to Dirksen which we now state:

THEOREM 2.8 ([8, Corollary 5.4]). *Let $(\mathcal{E}_n)_{n\geq 1}$ be an increasing sequence of conditional expectations in $(\mathcal{N},\sigma)$. If E is a symmetric Banach space on $\mathbb{R}_+$ with $p_E > 1$, then there is a constant c_E depending only on E such that*

$$\|(\mathcal{E}_n(x))_{n\geq 1}\|_{E(\mathcal{N};\ell_\infty)} \leq \mathrm{c}_E \|x\|_{E(\mathcal{N})}, \qquad x \in E(\mathcal{N},\sigma).$$

Our next result follows by combining Theorem 2.2, Corollary 2.7, Theorem 2.8, and the fact that if $1 < p < \infty$ and $1 \leq q \leq \infty$ then the upper and lower Boyd indices of $L_{p,q}(\mathbb{R}_+)$ are both equal to the first index p (see for instance [2, Theorem 4.6]). It provides noncommutative generalizations of classical results stated in Theorem 0.1.

THEOREM 2.9.

(1) *Let $0 < \alpha < 1$. There exists a constant c_α such that if x is an L_1-bounded noncommutative martingale then*

$$\|I^\alpha x\|_{L_{1/(1-\alpha),\infty}(\mathcal{M};\ell_\infty)} \leq \mathrm{c}_\alpha \|x\|_1.$$

(2) *Let $1 < p < q < \infty$, $0 < r \leq \infty$, and $\alpha = 1/p - 1/q$. There exists a constant $\kappa_{\alpha,r}$ such that if x is a noncommutative martingale that is bounded in $L_{p,r}(\mathcal{M})$, then*

$$\|I^\alpha x\|_{L_{q,r}(\mathcal{M};\ell_\infty)} \leq \kappa_{\alpha,r} \|x\|_{p,r}.$$

In particular, there exists a constant κ_α such that if x is a martingale that is bounded in $L_p(\mathcal{M})$, then

$$\|I^\alpha x\|_{L_q(\mathcal{M};\ell_\infty)} \leq \kappa_\alpha \|x\|_p.$$

We conclude this subsection with a note that all the indices involved in Theorem 2.9 are optimal. In fact, they cannot be improved even for classical dyadic martingales. We assume that these facts are known but we could not find any specific reference in the literature. For completeness, we include a simple example to support these claims.

EXAMPLE 2.10. We consider classical dyadic martingales and fractional integrals as defined in (0.1). Fix $N \geq 1$ and set $f_N := 2^N \chi_{[0,2^{-N})}$. Then we have the following properties:

(i) $\|f_N\|_1 = 1$.
(ii) For every $0 < \varepsilon < 1$, $\|f_N\|_{(4-\varepsilon)/3} = 2^{\frac{1-\varepsilon}{4-\varepsilon}N}$.
(iii) $\|I^{1/2} f_N\|_2 = \sqrt{N/2}$.
(iv) $\|I^{1/4} f_N\|_2 \sim 2^{N/4}$.

Consequently, we may deduce that

$$\lim_{N\to\infty} \|I^{1/2} f_N\|_2/\|f_N\|_1 = \infty \quad \text{and} \quad \lim_{N\to\infty} \|I^{1/4} f_N\|_2/\|f_N\|_{(4-\varepsilon)/3} = \infty.$$

The first two items can be easily verified. For the last two, we note first that for every $1 \le k \le N$, $f_k = \mathbb{E}(f_N/\mathcal{F}_k) = 2^k \chi_{[0,2^{-k})}$ and therefore $df_k = 2^{k-1}\chi_{[0,2^{-k})} - 2^{k-1}\chi_{[2^{-k},2^{-(k-1)})}$. One can then easily see that $\|df_k\|_2^2 = 2^{k-1}$ for $1 \le k \le N$ and thus

$$\|I^{1/2} f_N\|_2^2 = \sum_{k=1}^{N} 2^{-k}\|df_k\|_2^2 = N/2.$$

Moreover, for $I^{1/4}$ we have the identities

$$\begin{aligned}\|I^{1/4} f_N\|_2^2 &= \sum_{k=1}^{N} 2^{-k/2}\|df_k\|_2^2 \\ &= \sum_{k=1}^{N} 2^{-k/2}2^{k-1} = (2^{N/2}-1)/(2-\sqrt{2}).\end{aligned}$$

This verifies the last item. The statements about the two limits clearly follow from the four items listed. ■

Example 2.10 shows that $I^{1/2}$ is not bounded from $L_1[0,1]$ into $L_2[0,1]$. In particular, the weak-type $(1, 1/(1-\alpha))$ boundedness of I^α in Theorem 2.2 cannot be improved to strong type. Moreover, the index $1/(1-\alpha)$ is the best possible, as $I^{1/2}$ cannot be bounded from $L_1[0,1]$ into $L_{q,\infty}[0,1]$ for any $q > 2$ since the formal inclusion is bounded from $L_{q,\infty}[0,1]$ into $L_2[0,1]$. On the other hand, Example 2.10 also shows that for any given $0 < \varepsilon < 1$, $I^{1/4}$ is not bounded from $L_{(4-\varepsilon)/3}[0,1]$ into $L_2[0,1]$. Taking the adjoint and setting $\delta = 3\varepsilon/(1-\varepsilon)$, we find that $I^{1/4}$ is not bounded from $L_2[0,1]$ into $L_{4+\delta}[0,1]$ for any $\delta > 0$. This reveals that the indices in Corollary 2.7 are the best possible in the sense that if $\alpha = 1/p - 1/q$ with $1 < p < q < \infty$ then L_p-L_q boundedness of I^α is optimal.

2.2. Fractional integrals and Hardy spaces. In this subsection, we will examine boundedness of fractional integrals with respect to martingale Hardy space norms. The following theorem is our primary result in this subsection. It may be viewed as the Hardy-space version of Theorem 2.2.

THEOREM 2.11. *Let $0 < \alpha < 1$. There exists a constant c_α such that for every $x \in \mathcal{H}_1^c(\mathcal{M})$,*

$$\|I^\alpha x\|_{\mathcal{H}^c_{1/(1-\alpha)}} \le \mathrm{c}_\alpha \|x\|_{\mathcal{H}_1^c}.$$

For the proof, we first establish the following lemma. It relates the fractional integrals I^α and $I^{2\alpha}$ when $0 < \alpha < 1/2$.

LEMMA 2.12. *Assume that* $0 < \alpha < 1/2$. *For every* $a \in L_1(\mathcal{M})$ *and* $n \geq 1$,

$$\mu_t(S_{c,n}(I^\alpha a)) \leq \mu_{t/2}(S_{c,n}(I^{2\alpha}a))^{1/2}\mu_{t/2}(S_{c,n}(a))^{1/2}, \quad t > 0. \tag{2.6}$$

Proof. Denote by (e_{ij}) the canonical matrix of $\mathbb{M}_n$. Then

$$\begin{aligned} S^2_{c,n}(I^\alpha a) \otimes e_{11} &= \sum_{k=1}^n \zeta_k^{2\alpha}|da_k|^2 \otimes e_{11} \\ &= \Big(\sum_{k=1}^n \zeta_k^{2\alpha}|da_k| \otimes e_{1n}\Big)\Big(\sum_{k=1}^n |da_k| \otimes e_{n1}\Big) = A.B. \end{aligned}$$

Taking singular values relative to $\mathcal{M}_n \otimes \mathbb{M}_n$, we have

$$\mu_t(S^2_{c,n}(I^\alpha a) \otimes e_{11}) \leq \mu_{t/2}(A)\mu_{t/2}(B) = \mu_{t/2}(AA^*)^{1/2}\mu_{t/2}(B^*B)^{1/2}.$$

Since $AA^* = \sum_{k=1}^n \zeta_k^{4\alpha}|da_k|^2 \otimes e_{11} = S^2_{c,n}(I^{2\alpha}a) \otimes e_{11}$ and $B^*B = S^2_{c,n}(a) \otimes e_{11}$, the above inequality translates into

$$\mu_t(S^2_{c,n}(I^\alpha a) \otimes e_{11}) \leq \mu_{t/2}(S^2_{c,n}(I^{2\alpha}a) \otimes e_{11})^{1/2}\mu_{t/2}(S^2_{c,n}(a) \otimes e_{11})^{1/2},$$

which is equivalent to (2.6). ■

As a consequence of Lemma 2.12, we may deduce the next statement.

LEMMA 2.13. *Assume that* $0 < \alpha < 1/2$. *For every* $a \in \mathcal{H}_1^c(\mathcal{M})$ *and* $n \geq 1$,

$$\|(I^\alpha a)_n\|_{\mathcal{H}^c_{1/1-\alpha}} \leq 2^{(1-\alpha)}\|(I^{2\alpha}a)_n\|^{1/2}_{\mathcal{H}^c_{1/(1-2\alpha)}}\|a\|^{1/2}_{\mathcal{H}^c_1}.$$

Proof. Let $u = 1/(1-\alpha)$ and $s = 2/(1-2\alpha)$. Then $1/u = 1/s + 1/2$. Using Hölder's inequality on (2.6), we have

$$\|\mu_t(S_{c,n}(I^\alpha a))\|_u \leq \|\mu_{t/2}(S_{c,n}(I^{2\alpha}a))^{1/2}\|_s \cdot \|\mu_{t/2}(S_{c,n}(a))^{1/2}\|_2.$$

This can be easily verified to be equivalent to the statement of the lemma. ■

We are now ready to provide the proof of Theorem 2.11.

Proof of Theorem 2.11. Let $0 < \alpha < 1$ and let $\nu \in \mathbb{N}$ be such that

$$1/2^{\nu+1} \leq \alpha < 1/2^\nu.$$

The proof is done by induction on ν.

• $\nu = 0$, i.e, $1/2 \leq \alpha < 1$. Let $u = 1/(1-\alpha)$. Then $u \geq 2$. By the noncommutative Khintchine inequalities [21, 22], for every $n \geq 1$ we have

$$\|(I^\alpha x)_n\|^2_{\mathcal{H}^c_u} \leq \mathbb{E}\Big\|\sum_{k=1}^n \zeta_k^\alpha \varepsilon_k dx_k\Big\|_u^2$$

where $(\varepsilon_k)_k$ is a Rademacher sequence and $\mathbb{E}$ denotes the expectation on the ε_k's. From the fact that $L_u(\mathcal{M})$ is of type 2 (see [28]), there is a constant η_u

such that

$$\|(I^\alpha x)_n\|_{\mathcal{H}_u^c}^2 \le \eta_u^2 \sum_{k=1}^n \zeta_k^{2\alpha} \|dx_k\|_u^2.$$

Now we apply Lemma 2.3(i) to find that since $dx_k \in \mathcal{D}_{k,\infty}$ for every $k \ge 1$, we have

$$\|(I^\alpha x)_n\|_{\mathcal{H}_u^c}^2 \le \eta_u^2 \sum_{k=1}^n \|dx_k\|_1^2.$$

Using the fact that $L_1(\mathcal{M})$ is of cotype 2, and the noncommutative Khintchine inequality once again, we deduce that there is a constant κ such that

$$\|(I^\alpha x)_n\|_{\mathcal{H}_u^c} \le \kappa\eta_u \inf\left\{\Big\|\Big(\sum_{k=1}^n a_k^* a_k\Big)^{1/2}\Big\|_1 + \Big\|\Big(\sum_{k=1}^n b_k b_k^*\Big)^{1/2}\Big\|_1\right\}$$

where the infimum is taken over all decompositions $dx_k = a_k + b_k$ for $k \ge 1$. A fortiori, we obtain

$$\|(I^\alpha x)_n\|_{\mathcal{H}_u^c} \le \kappa\eta_u\|x\|_{\mathcal{H}_1} \le \kappa\gamma_u\|x\|_{\mathcal{H}_1^c}.$$

Taking the limit in n proves the case $\nu = 0$.

• Assume that the assertion is true for $\nu \ge 0$ and fix $\alpha \in [2^{-(\nu+2)}, 2^{-(\nu+1)})$. Note that in this case, necessarily $0 < \alpha < 1/2$ and therefore Lemma 2.13 applies. We then have $2^{-(\nu+1)} \le 2\alpha < 2^{-\nu}$ and thus by assumption there exists a constant $\mathrm{c}_{2\alpha}$ such that

$$\|(I^{2\alpha} x)_n\|_{\mathcal{H}_{1/(1-2\alpha)}^c} \le \mathrm{c}_{2\alpha}\|x\|_{\mathcal{H}_1^c}$$

for all $x \in \mathcal{H}_1^c(\mathcal{M})$ and all $n \ge 1$. Combining the latter inequality with Lemma 2.13, we deduce that

$$\|(I^\alpha x)_n\|_{\mathcal{H}_{1/(1-\alpha)}^c} \le 2^{(1-\alpha)}\sqrt{\mathrm{c}_{2\alpha}}\,\|x\|_{\mathcal{H}_1^c},$$

which proves that the assertion is true for $\nu + 1$. ∎

REMARKS 2.14. (a) Working with adjoints, we also have the row version of Theorem 2.11.

(b) For $1/2 \le \alpha < 1$, the argument above provides the stronger statement that for every $x \in \mathcal{H}_1(\mathcal{M})$,

$$\|I^\alpha x\|_{\mathcal{H}_{1/(1-\alpha)}} \le \mathrm{c}_\alpha\|x\|_{\mathcal{H}_1}. \tag{2.7}$$

The next result extends (2.7) to the full range $0 < \alpha < 1$.

COROLLARY 2.15. *Let $0 < \alpha < 1$. There exists a constant c_α such that for every $x \in \mathcal{H}_1(\mathcal{M})$,*

$$\|I^\alpha x\|_{\mathcal{H}_{1/(1-\alpha)}} \le \mathrm{c}_\alpha\|x\|_{\mathcal{H}_1}.$$

Proof. As noted in Remark 2.14(b), we only need to consider the case where $0 < \alpha < 1/2$. Then $1 < 1/(1-\alpha) < 2$. Let $x \in \mathcal{H}_1(\mathcal{M})$ and $\varepsilon > 0$. Fix $a \in \mathcal{H}_1^c(\mathcal{M})$ and $b \in \mathcal{H}_1^r(\mathcal{M})$ so that:

(1) $x = a + b$;
(2) $\|a\|_{\mathcal{H}_1^c} + \|b\|_{\mathcal{H}_1^r} \leq \|x\|_{\mathcal{H}_1} + \varepsilon$.

From Theorem 2.11 and Remark 2.14(a), we have

$$\|I^\alpha a\|_{\mathcal{H}^c_{1/(1-\alpha)}} \leq c_\alpha \|a\|_{\mathcal{H}_1^c} \quad \text{and} \quad \|I^\alpha b\|_{\mathcal{H}^r_{1/(1-\alpha)}} \leq c_\alpha \|b\|_{\mathcal{H}_1^r}.$$

Adding the two inequalities gives

$$\|I^\alpha a\|_{\mathcal{H}^c_{1/(1-\alpha)}} + \|I^\alpha b\|_{\mathcal{H}^r_{1/(1-\alpha)}} \leq c_\alpha (\|x\|_{\mathcal{H}_1} + \varepsilon).$$

The left hand side of the last inequality is clearly larger than $\|I^\alpha x\|_{\mathcal{H}_{1/(1-\alpha)}}$. Since ε is arbitrary, we obtain the desired statement. ∎

REMARK 2.16. Using the noncommutative Burkholder–Gundy inequalities from [17, 27], one can show that Corollary 2.15 is equivalent to the statement that for any given $0 < \alpha < 1$, I^α is bounded from $\mathcal{H}_1(\mathcal{M})$ into $L_{1/(1-\alpha)}(\mathcal{M})$. This is often easier to apply when dealing with dualities.

Before stating the next result, let us recall the notion of $\mathcal{BMO}$-spaces for noncommutative martingales introduced in [27]. Let

$$\mathcal{BMO}_C(\mathcal{M}) := \Big\{ a \in L^2(\mathcal{M}) : \sup_{n\geq 1} \big\| \mathcal{E}_n |a - \mathcal{E}_{n-1} a|^2 \big\|_\infty < \infty \Big\}.$$

Then $\mathcal{BMO}_C(\mathcal{M})$ becomes a Banach space when equipped with the norm

$$\|a\|_{\mathcal{BMO}_C} = \Big(\sup_{n\geq 1} \|\mathcal{E}_n |a - \mathcal{E}_{n-1} a|^2\|_\infty \Big)^{1/2}.$$

Similarly, we define $\mathcal{BMO}_R(\mathcal{M})$ as the space of all a with $a^* \in \mathcal{BMO}_C(\mathcal{M})$ equipped with the norm $\|a\|_{\mathcal{BMO}_R} = \|a^*\|_{\mathcal{BMO}_C}$. The space $\mathcal{BMO}(\mathcal{M})$ is the intersection of these two spaces:

$$\mathcal{BMO}(\mathcal{M}) := \mathcal{BMO}_C(\mathcal{M}) \cap \mathcal{BMO}_R(\mathcal{M})$$

with the intersection norm

$$\|a\|_{\mathcal{BMO}} = \max\{\|a\|_{\mathcal{BMO}_C}, \|a\|_{\mathcal{BMO}_R}\}.$$

We recall that as in the classical case, for $1 \leq p < \infty$,

$$\mathcal{M} \subset \mathcal{BMO}(\mathcal{M}) \subset L_p(\mathcal{M}).$$

For more information on noncommutative martingale BMO-spaces, we refer to [27, 17, 24, 16]. It was shown in [27] that the classical Feffermann duality is still valid in the noncommutative settings. That is, we have

$$(\mathcal{H}_1(\mathcal{M}))^* = \mathcal{BMO}(\mathcal{M}), \qquad \text{with equivalent norms.}$$

Our next result should be compared with Corollary 2.6 above. It is a direct consequence of Corollary 2.15 and duality. Note that it is the noncommutative analogue of [4, Theorem 3(v)]. See also the next section for more discussion of other items of [4, Theorem 3].

Corollary 2.17. *For $0 < \alpha < 1$, the operator I^α is bounded from $L_{1/\alpha}(\mathcal{M})$ into $\mathcal{BMO}(\mathcal{M})$.*

We remark that as shown in Example 2.10, $I^{1/2}$ is not bounded from $L_2[0,1]$ into $L_\infty[0,1]$ for the case of a dyadic filtration. Thus, in general $\mathcal{BMO}(\mathcal{M})$ in Corollary 2.17 cannot be replaced by the von Neumann algebra $\mathcal{M}$.

We also have a closely related result which follows from Corollary 2.17 but cannot be directly formulated in the language of fractional integrals since we only defined the latter with $0 < \alpha < 1$. See the Appendix below for a more detailed discussion of this.

Proposition 2.18. *There is an absolute constant κ such that for any (finite) martingale difference sequence $dx = (dx_k)_{k=1}^n$ in $\mathcal{H}_1(\mathcal{M})$,*

$$\Big\|\sum_{k=1}^n \zeta_k dx_k\Big\|_{\mathcal{BMO}} \le \kappa \Big\|\sum_{k=1}^n dx_k\Big\|_{\mathcal{H}_1}.$$

Proof. Apply the fact that $I^{1/2}$ is bounded simultaneously from $\mathcal{H}_1(\mathcal{M})$ into $L_2(\mathcal{M})$ and from $L_2(\mathcal{M})$ into $\mathcal{BMO}(\mathcal{M})$, and then use composition. ■

We end this section with a short discussion of our choice of the scalar sequence $(\zeta_k)_{k\ge1}$ introduced in (2.1) and used in Definition 2.1. Fix an arbitrary sequence $\nu = (\nu_k)_{k\ge1}$ of nonnegative scalars and consider fractional integrals using ν. We denote them by I_ν^α. That is, $I_\nu^\alpha x = \sum_{k\ge1} \nu_k^\alpha dx_k$ for every finite martingale x.

For fixed $k \ge 1$ and $a \in \mathcal{D}_{k,\infty}$, let $dx = (\delta_{j,k} a)_{j\ge1}$ where $\delta_{j,k} = 0$ for $j \ne k$ and $\delta_{k,k} = 1$. Then dx is a martingale difference sequence. If x is the corresponding martingale then it is easy to check that $\|x\|_{\mathcal{BMO}^c} = \|a\|_\infty$ and $\|x\|_2 = \|a\|_2$. Similarly, $\|I_\nu^\alpha x\|_{\mathcal{BMO}^c} = \nu_k\|a\|_\infty$. If $I_\nu^{1/2}$ satisfies Corollary 2.17, that is, if $I_\nu^{1/2}$ is bounded from $L_2(\mathcal{M})$ into $\mathcal{BMO}(\mathcal{M})$, then there is an absolute constant c such that

$$\nu_k^{1/2}\|a\|_\infty \le \mathrm{c}\|a\|_2.$$

Since this is valid for all $a \in \mathcal{D}_{k,\infty}$, we deduce from (2.1) that $\zeta_k^{-1/2} \le \mathrm{c}\nu_k^{-1/2}$. This yields $\mathrm{c}^{-2}\nu_k \le \zeta_k$ for all $k \ge 1$. In particular, it shows that (modulo some constants) our initial choice of $(\zeta_k)_{k\ge1}$ in Definition 2.1 is the best possible.

Appendix. The case $0 < p < 1$ and open problems. In this section, we explore the boundedness of fractional integrals when the domain spaces are Hardy spaces indexed by $p \in (0,1)$. Our primary tool is the atomic decomposition for martingales. We begin by recalling the concept of noncommutative atoms introduced in [1] for general noncommutative martingales.

DEFINITION A.1. Let $0 < p < 2$. An operator $a \in L_2(\mathcal{M})$ is said to be a *$(p,2)_c$-atom* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that:

(i) $\mathcal{E}_n(a) = 0$;
(ii) $ae = a$;
(iii) $\|a\|_2 \leq \tau(e)^{1/2-1/p}$.

Replacing (ii) by (ii)$'$ $ea = a$, we have the notion of *$(p,2)_r$-atoms*.

Clearly, $(p,2)_c$-atoms and $(p,2)_r$-atoms are natural noncommutative analogues of the concept of $(p,2)$-atoms from classical martingales.

Let us now recall the atomic Hardy spaces for $0 < p < 2$.

DEFINITION A.2. We define the *atomic column martingale Hardy space* $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ as the space of all $x \in L_p(\mathcal{M})$ which admit a decomposition

$$x = \sum_k \lambda_k a_k,$$

where for each k, a_k is a $(p,2)_c$-atom or an element of the unit ball of $L_p(\mathcal{M}_1)$, and $(\lambda_k) \subset \mathbb{C}$ satisfies $\sum_k |\lambda_k|^p < \infty$.

We equip $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ with the (quasi) norm

$$\|x\|_{\mathsf{h}_p^{c,\mathrm{at}}} = \inf \Big(\sum_k |\lambda_k|^p\Big)^{1/p},$$

where the infimum is taken over all decompositions of x as above.

Similarly, we define the row version $\mathsf{h}_p^{r,\mathrm{at}}(\mathcal{M})$ as the space of all $x \in L_p(\mathcal{M})$ for which $x^* \in \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$. The space $\mathsf{h}_p^{r,\mathrm{at}}(\mathcal{M})$ is equipped with the (quasi) norm $\|x\|_{\mathsf{h}_p^{r,\mathrm{at}}} = \|x^*\|_{\mathsf{h}_p^{c,\mathrm{at}}}$.

The *atomic Hardy space of noncommutative martingales* is defined as follows:

$$\mathsf{h}_p^{\mathrm{at}}(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}) + \mathsf{h}_p^{r,\mathrm{at}}(\mathcal{M}),$$

equipped with the (quasi) norm

$$\|x\|_{\mathsf{h}_p^{\mathrm{at}}} = \inf \big\{\|w\|_{\mathsf{h}_p^d} + \|y\|_{\mathsf{h}_p^{c,at}} + \|z\|_{\mathsf{h}_p^{r,\mathrm{at}}}\big\},$$

where the infimum is taken over all $w \in \mathsf{h}_p^d(\mathcal{M})$, $y \in \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$, and $z \in \mathsf{h}_p^{r,\mathrm{at}}(\mathcal{M})$ such that $x = w + y + z$. We refer to [1, 13] for more details on the concept of atomic decomposition for noncommutative martingales.

One can describe the dual space of $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ as a Lipschitz space. For $\beta \geq 0$, we set

$$\Lambda_\beta^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda_\beta^c} < \infty\}$$

where

$$\|x\|_{\Lambda_\beta^c} = \max\left\{\|\mathcal{E}_1(x)\|_\infty, \sup_{n\geq 1} \sup_{e\in\mathcal{M}_n,\,\text{projection}} \frac{\|(x-\mathcal{E}_n(x))e\|_2}{\tau(e)^{\beta+1/2}}\right\}.$$

The space $\Lambda_\beta^c(\mathcal{M})$ is called the *Lipschitz space* of order β. For $0 < p \leq 1$ and $\beta = 1/p - 1$, it was shown in [1] that

$$(\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}))^* = \Lambda_\beta^c(\mathcal{M}), \quad \text{with equivalent norms.} \tag{A.1}$$

Our first result shows that fractional integrals essentially transform $(p,2)_c$-atoms into $(q,2)_c$-atoms for appropriate values of p and q. This could be of independent interest.

Proposition A.3. *Assume that* $0 < p < 1$, $p < q < 2$, *and* $\alpha = 1/p - 1/q \in (0,1)$. *There exists a constant* C_α *such that if* a *is a* $(p,2)_c$*-atom then* $C_\alpha^{-1} I^\alpha a$ *is a* $(q,2)_c$*-atom. In particular,*

$$\|I^\alpha a\|_{\mathsf{h}_q^{c,\mathrm{at}}} \leq C_\alpha.$$

Proof. Let a be a $(p,2)_c$-atom. There exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that

(i) $\mathcal{E}_n(a) = 0$;
(ii) $ae = a$;
(iii) $\|a\|_2 \leq \tau(e)^{1/2-1/p}$.

Clearly, $\mathcal{E}_n(I^\alpha a) = 0$ and $(I^\alpha a)e = I^\alpha a$. We treat the cases $0 < \alpha < 1/2$ and $1/2 \leq \alpha < 1$ separately.

Case 1: $1/2 \leq \alpha < 1$. First we note that since $\alpha^{-1} \leq 2$, from Hölder's inequality we have

$$\|a\|_{\alpha^{-1}} = \|ae\|_{\alpha^{-1}} \leq \|a\|_2 \tau(e)^{\alpha-1/2}.$$

By Corollary 2.17, there exists a constant C_α such that

$$\begin{aligned} C_\alpha^{-1}\|I^\alpha a\|_{\mathcal{BMO}^c} &\leq C_\alpha^{-1}\|I^\alpha a\|_{\mathcal{BMO}} \\ &\leq \|a\|_{\alpha^{-1}} \leq \|a\|_2\tau(e)^{\alpha-1/2} \leq \tau(e)^{-1/q}. \end{aligned} \tag{A.2}$$

Next, we estimate the L_2-norm of $I^\alpha a$. Since $\mathcal{E}_n(I^\alpha a) = 0$ and $e \in \mathcal{M}_n$,

$$S_c^2(I^\alpha a) = \sum_{k\geq n} |d_k(I^\alpha a)|^2 = e\Big(\sum_{k\geq n} |d_k(I^\alpha a)|^2\Big)e.$$

We have

$$\begin{aligned}\|I^\alpha a\|_2^2 &= \tau\Big(\sum_{k\ge n} |d_k(I^\alpha a)|^2 e\Big) = \tau\Big(\mathcal{E}_n\Big[\sum_{k\ge n} |d_k(I^\alpha a)|^2\Big] e\Big)\\ &\le \Big\|\mathcal{E}_n\Big[\sum_{k\ge n} |d_k(I^\alpha a)|^2\Big]\Big\|_\infty \tau(e).\end{aligned}$$

Since $\mathcal{E}_n(|I^\alpha a - \mathcal{E}_{n-1}(I^\alpha a)|^2) = \mathcal{E}_n(\sum_{k\ge n} |d_k(I^\alpha a)|^2)$, we deduce that

$$\|I^\alpha a\|_2 \le \|I^\alpha a\|_{\mathcal{BMO}^c} \tau(e)^{1/2}.$$

Combining the preceding inequality with (A.2), we conclude that

$$C_\alpha^{-1}\|I^\alpha a\|_2 \le \tau(e)^{1/2-1/q},$$

which is equivalent to $C_\alpha^{-1} I^\alpha a$ being a $(q,2)_c$-atom.

Case 2: $0 < \alpha < 1/2$. Fix $1 < r < 2$ such that $\alpha = 1/r - 1/2$. By Corollary 2.7, I^α is bounded from $L_r(\mathcal{M})$ into $L_2(\mathcal{M})$. Taking adjoint, $I^\alpha : L_2(\mathcal{M}) \to L_{r'}(\mathcal{M})$ is bounded where $1/r + 1/r' = 1$. There exists a constant C_α such that

$$C_\alpha^{-1}\|I^\alpha a\|_{r'} \le \|a\|_2 \le \tau(e)^{1/2-1/p}. \tag{A.3}$$

On the other hand, from Hölder's inequality, we have

$$\|I^\alpha a\|_2^2 = \tau(|I^\alpha a|^2 e) \le \tau(|I^\alpha a|^{r'})^{2/r'} \tau(e)^{1-2/r'}.$$

Therefore,

$$\|I^\alpha a\|_2 \le \|I^\alpha a\|_{r'} \tau(e)^{1/2-1/r'}. \tag{A.4}$$

Combining (A.3) and (A.4), we get

$$C_\alpha^{-1}\|I^\alpha a\|_2 \le \tau(e)^{1-1/p-1/r'} = \tau(e)^{1/r-1/p}.$$

From the choice of r above, we have $1/r - 1/p = 1/2 - 1/q$. That is, $C_\alpha^{-1}\|I^\alpha a\|_2 \le \tau(e)^{1/2-1/q}$, which again shows that $C_\alpha^{-1} I^\alpha a$ is a $(q,2)_c$-atom. ∎

The next theorem is an immediate consequence of Proposition A.1 and duality. We leave the details of its proof to the reader.

Theorem A.4.

(a) *Assume that* $0 < p < 1$, $p < q < 2$, *and* $\alpha = 1/p - 1/q \in (0,1)$. *Then* I^α *is bounded from* $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ *into* $\mathsf{h}_q^{c,\mathrm{at}}(\mathcal{M})$.
(b) *For* $\beta > 0$ *and* $0 < \alpha < 1$, I^α *is bounded from* $\Lambda_\beta^c(\mathcal{M})$ *into* $\Lambda_{\beta+\alpha}^c(\mathcal{M})$.

So far we have considered only fractional integrals of order α under the assumption that $0 < \alpha < 1$. Indeed, all the results stated in Section 2 do require this assumption. In fact, both the statements and techniques of proofs used in Theorems 2.2, 2.9, and 2.11 highlighted the need for $1 - \alpha$ to be nonnegative. However, when considering the case $0 < p < 1$, the situation is different. For instance, both statements in Theorem A.4 still make sense without the assumption $0 < \alpha < 1$.

To avoid any potential confusion, we will introduce different notation for the general case. Let $\gamma > 0$. We denote by $\tilde{I}^\gamma$ the transformation defined by setting, for any martingale $x = (x_n)_{n\geq 1}$, $\tilde{I}^\gamma x = \{(\tilde{I}^\gamma x)_n\}_{n\geq 1}$ where for $n \geq 1$,

$$(\tilde{I}^\gamma x)_n = \sum_{k=1}^{n} \zeta_k^\gamma dx_k.$$

Clearly, $\tilde{I}^\gamma$ is simply I^γ when $0 < \gamma < 1$. Moreover, when $\gamma \geq 1$, set $n(\gamma) := \lfloor \gamma \rfloor + 1$ where $\lfloor \cdot \rfloor$ denotes the greatest integer function and $\alpha(\gamma) := \gamma/n(\gamma)$. Then clearly $0 < \alpha(\gamma) < 1$ and we may view $\tilde{I}^\gamma$ as the composition of the fractional integral $I^{\alpha(\gamma)}$ with itself $n(\gamma)$ times.

We now consider boundedness properties of $\tilde{I}^\gamma$ as a linear transformation. The following theorem should be compared with [4, Theorem 3].

Theorem A.5.

(i) *If $\beta > 0$ and $0 < \gamma$, then $\tilde{I}^\gamma$ is bounded from $\Lambda_\beta^c(\mathcal{M})$ into $\Lambda_{\beta+\gamma}^c(\mathcal{M})$.*

(ii) *If $0 < p < q < \infty$ and $\gamma = 1/p - 1/q$, then $\tilde{I}^\gamma$ is bounded from $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ into $\mathcal{H}_q^c(\mathcal{M})$.*

(iii) *For every $\gamma > 0$, $\tilde{I}^\gamma$ is bounded from $\mathcal{BMO}^c(\mathcal{M})$ into $\Lambda_\gamma^c(\mathcal{M})$.*

(iv) *If $1 < p < \infty$ and $\gamma > 1/p$, then $\tilde{I}^\gamma$ is bounded from $L_p(\mathcal{M})$ into $\Lambda_{\gamma-1/p}^c(\mathcal{M})$.*

(v) *If $\gamma > 1$, then $\tilde{I}^\gamma$ is bounded from $\mathcal{H}_1(\mathcal{M})$ into $\Lambda_{\gamma-1}^c(\mathcal{M})$.*

Proof. Item (i) is already the second part of Theorem A.4 if $0 < \gamma < 1$. Assume that $\gamma \geq 1$ and let $\alpha(\gamma)$ and $n(\gamma)$ be as described above. From the second part of Theorem A.4, $I^{\alpha(\gamma)}$ is bounded from $\Lambda_{\beta+(k-1)\alpha(\gamma)}^c(\mathcal{M})$ into $\Lambda_{\beta+k\alpha(\gamma)}^c(\mathcal{M})$ for all integers $k \in [1, n(\gamma)]$. We apply $I^{\alpha(\gamma)}$ successively $n(\gamma)$ times and get $\tilde{I}^\gamma$ as the composition

$$\tilde{I}^\gamma : \Lambda_\beta^c(\mathcal{M}) \xrightarrow{I^{\alpha(\gamma)}} \Lambda_{\beta+\alpha(\gamma)}^c(\mathcal{M}) \xrightarrow{I^{\alpha(\gamma)}} \Lambda_{\beta+2\alpha(\gamma)}^c(\mathcal{M}) \to \cdots \to \Lambda_{\beta+\gamma-\alpha(\gamma)}^c(\mathcal{M}) \xrightarrow{I^{\alpha(\gamma)}} \Lambda_{\beta+\gamma}^c(\mathcal{M}).$$

This shows that $\tilde{I}^\gamma : \Lambda_\beta^c(\mathcal{M}) \to \Lambda_{\beta+\gamma}^c(\mathcal{M})$ is bounded.

(ii) follows directly from (i) by duality when $0 < p < q < 1$. So we assume that $q \geq 1$. Fix $\varepsilon > 0$ such that $0 < p < 1 - \varepsilon$. Let $\gamma_1 := 1/p - 1/(1-\varepsilon)$, $\gamma_2 := 1/(1-\varepsilon) - 1$, and $\gamma_3 := 1 - 1/q$. Then $\tilde{I}^{\gamma_1} : \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}) \to \mathsf{h}_{1-\varepsilon}^{c,\mathrm{at}}(\mathcal{M})$ is bounded. Also since $0 < \gamma_2 < 1$, Theorem A.4(a) shows that $I^{\gamma_2} : \mathsf{h}_{1-\varepsilon}^{c,\mathrm{at}}(\mathcal{M}) \to \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ is bounded. From the inclusion $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) \subset \mathcal{H}_1^c(\mathcal{M})$ [1, Proposition 2.2], also $I^{\gamma_2} : \mathsf{h}_{1-\varepsilon}^{c,\mathrm{at}}(\mathcal{M}) \to \mathcal{H}_1^c(\mathcal{M})$

is bounded. Furthermore, since $0 < \gamma_3 < 1$, Theorem 2.11 shows that $I^{\gamma_3} : \mathcal{H}_1^c(\mathcal{M}) \to \mathcal{H}_q^c(\mathcal{M})$ is bounded. We conclude that $\tilde{I}^\gamma = I^{\gamma_3} I^{\gamma_2} \tilde{I}^{\gamma_1}$ is bounded from $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ into $\mathcal{H}_q^c(\mathcal{M})$. For $q = 1$, we only need to consider γ_1 and γ_2.

(iii) is an immediate consequence of (ii) by duality and $q = 1$.

For (iv), we observe from Corollary 2.17 that if $\alpha = 1/p$ then $I^\alpha : L_p(\mathcal{M}) \to \mathcal{BMO}(\mathcal{M})$ is bounded. A fortiori, $I^\alpha : L_p(\mathcal{M}) \to \mathcal{BMO}^c(\mathcal{M})$ is bounded. On the other hand, by (iii), so is $\tilde{I}^{\gamma-\alpha} : \mathcal{BMO}^c(\mathcal{M}) \to \Lambda_{\gamma-\alpha}^c(\mathcal{M})$. Thus, (iv) follows by taking composition.

(v) follows by combining Proposition 2.18 and (iii). Indeed, from Proposition 2.18, $\tilde{I}^1 : \mathcal{H}_1(\mathcal{M}) \to \mathcal{BMO}^c(\mathcal{M})$ is bounded, and from (iii), $\tilde{I}^{\gamma-1} : \mathcal{BMO}^c(\mathcal{M}) \to \Lambda_{\gamma-1}^c(\mathcal{M})$ is bounded. ∎

Remark A.6. All the results in Theorems A.3 and A.5 are valid for the corresponding row versions.

For the case of mixed Hardy spaces, we may also state:

Corollary A.7. *If $0 < p < q < \infty$ and $\gamma = 1/p - 1/q$, then $\tilde{I}^\gamma$ is bounded from $\mathsf{h}_p^{\mathrm{at}}(\mathcal{M})$ into $\mathcal{H}_q(\mathcal{M})$.*

Proof. It is enough to prove that $\tilde{I}^\gamma$ is bounded from $\mathsf{h}_p^d(\mathcal{M})$ into $\mathsf{h}_q^d(\mathcal{M})$. In view of the proof of Theorem A.5(ii), it suffices to verify the special case $0 < p < q \le 1$ and $\gamma \in (0, 1)$. This can be deduced from the following claim: for every $k \ge 1$,

$$\zeta_k^{\gamma q} \|a\|_q^q \le \|a\|_p^p \quad \text{for all } a \in \mathcal{M}_k.$$

To prove this claim, we apply Lemma 2.3(i) to $|a|^p$ and any $0 < \beta < 1$ to get

$$\zeta_k^{\beta/(1-\beta)} \tau(|a|^{p/(1-\beta)}) \le \tau(|a|^p).$$

Take β such that $p/(1-\beta) = q$. One can easily verify that $\beta/(1-\beta) = \gamma q$. This proves the claim. ∎

In [4, Theorem 3], the classical dyadic filtration was handled without specifically referring to atomic decompositions or atomic Hardy spaces. We do not know if the use of atoms and more specifically of Proposition A.3 can be avoided. More precisely, we do not know if the atomic Hardy spaces in the statements of Theorems A.4 and A.5 can be replaced with the usual Hardy spaces. This of course is closely connected with the problem of atomic decomposition for noncommutative martingales. We leave this as an open question.

Problem 1. Is $\tilde{I}^\gamma$ bounded from $\mathcal{H}_p^c(\mathcal{M})$ into $\mathcal{H}_q^c(\mathcal{M})$ when $0 < p < q \le 1$ and $\gamma = 1/p - 1/q$?

To complete this circle of ideas, we consider maximal Hardy spaces. Let us recall the classical Davis theorem [6] that for every commutative martingale $x \in \mathcal{H}_1$ we have $\|x\|_{\mathcal{H}_1} \sim \big\|\sup_k |\mathcal{E}_k(x)|\big\|_1$. Following the ideas described in Subsection 2.1, one can define the maximal Hardy space of noncommutative martingales $\mathcal{H}_1^{\max}(\mathcal{M})$ as the space of all martingales $x \in L_1(\mathcal{M})$ for which $\|x\|_{\mathcal{H}_1^{\max}} = \|(\mathcal{E}_k(x))_k\|_{L_1(\mathcal{M};\ell_\infty)}$ is finite. The Davis theorem stated above is equivalent to saying that for the commutative case, the Hardy spaces $\mathcal{H}^1$ and $\mathcal{H}_1^{\max}$ coincide. Unfortunately, the Davis theorem does not extend to the noncommutative case. Indeed, it was shown in [18, Corollary 14] that $\mathcal{H}_1$ and $\mathcal{H}_1^{\max}$ do not coincide in general. The following problem was motivated by Theorem 2.11:

Problem 2. Is I^α bounded from $\mathcal{H}_1^{\max}(\mathcal{M})$ into $\mathcal{H}_{1/(1-\alpha)}(\mathcal{M})$ for $0 < \alpha < 1$?

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