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Arhangel'skiĭ sheaf amalgamations in topological groups

by

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Abstract. We consider amalgamation properties of convergent sequences in topological groups and topological vector spaces. The main result of this paper is that, for arbitrary topological groups, Nyikos's property $\alpha_{1.5}$ is equivalent to Arhangel'skiĭ's formally stronger property α_1. This result solves a problem of Shakhmatov (2002), and its proof uses a new perturbation argument. We also prove that there is a topological space X such that the space $\mathrm{C_p}(X)$ of continuous real-valued functions on X with the topology of pointwise convergence has Arhangel'skiĭ's property α_1 but is not countably tight. This follows from results of Arhangel'skiĭ–Pytkeev, Moore and Todorčević, and provides a new solution, with stronger properties than the earlier solution, of a problem of Averbukh and Smolyanov (1968) concerning topological vector spaces.

1. Sheaf amalgamations in topological groups. To avoid trivialities, by *convergent sequence* $x_n \to x$ we mean a proper one, that is, such that $x \neq x_n$ for all n. This way, convergence is a property of countably infinite sets: a countably infinite set A converges to x if all (equivalently, some) bijective enumerations of A converge to x. Thus, in the following definition, by *sequence* we always mean a countably infinite set. The following concepts are due to Arhangel'skiĭ [1, 2], except for $\alpha_{1.5}$ which is due to Nyikos [14].

Definition 1.1. A topological space X is α_i, for $i = 1, 1.5, 2, 3, 4$, if for each $x \in X$ and all pairwise disjoint sequences $S_1, S_2, \ldots \subset X$, each converging to x, there is a sequence $S \subset \bigcup_n S_n$ such that S converges to x and, respectively:

(α_1) $S_n \setminus S$ is finite for all n.
($\alpha_{1.5}$) $S_n \setminus S$ is finite for infinitely many n.

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(α_2) $S_n \cap S$ is infinite for all n.
(α_3) $S_n \cap S$ is infinite for infinitely many n.
(α_4) $S_n \cap S$ is nonempty for infinitely many n.

A survey of these properties is available in [23]. In the integer-indexed properties α_i, we may remove the requirement that the sequences $S_1, S_2, \ldots$ are pairwise disjoint [14]. Indeed, we can move to subsequences $S'_n = S_n \setminus \bigcup_{k<n} S_k$ of S_n for $n \in \mathbb{N}$. If S'_n is infinite for infinitely many n, we can dispose of the other ones. And if not, then the sequence $S := \bigcup_{k<n} S_k$ for any n with S'_n finite would be as required in α_1. However, removing the disjointness requirement in the property $\alpha_{1.5}$ renders it superfluous: Applying it to the modified sequence $\bigcup_{k\le n} S_k$ for $n \in \mathbb{N}$ would result in a sequence S as required in α_1.

Each of the properties in Definition 1.1 implies the subsequent one. To see that $\alpha_{1.5}$ implies α_2, for each n decompose $S_n = \bigcup_k S_{nk}$, and take $S'_n = \bigcup_{m\le n} S_{mn}$ [14].

None of the implications

$$\alpha_1 \Rightarrow \alpha_{1.5} \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_4$$

can be (provably) reversed, not even in the class of Fréchet–Urysohn spaces [23]. Recall that a topological space X is *Fréchet–Urysohn* if each point in the closure of a set is in fact a limit of a sequence in that set.

In the present paper, we consider these properties in the context of *topological groups*. This direction was pioneered by Nyikos in his 1981 paper [13]. He proved that Fréchet–Urysohn groups are α_4, and that sequential α_2 groups are Fréchet–Urysohn. Shakhmatov [22] constructed, in the Cohen reals model, an example of a Fréchet–Urysohn group which is not α_3, and a Fréchet–Urysohn α_2 group which is not $\alpha_{1.5}$. In particular, none of the implications

$$\alpha_1 \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_4$$

is provably reversible in the realm of topological groups. The question whether $\alpha_{1.5}$ groups are α_1 is implicit in Shakhmatov's paper. The problem whether *Fréchet–Urysohn* $\alpha_{1.5}$ groups are α_1 is stated there. This variant of the problem was settled in the positive by Shibakov, in his 1999 paper [24].

In his 2002 chapter for *Recent Progress in General Topology* [23], Shakhmatov cites Shibakov's solution, and writes: "It seems unclear if $\alpha_{1.5}$ and α_1 are equivalent for all (i.e., not necessarily Fréchet–Urysohn) topological groups". For groups of the form $\mathrm{C_p}(X)$, the continuous real-valued functions on a space X with the topology of pointwise convergence, Sakai solved this problem in the positive [18]. One step in his solution involves a pullback method which was used earlier by Scheepers [20] to show that for $\mathrm{C_p}(X)$ spaces, we have $\alpha_2 = \alpha_3 = \alpha_4$: Replace the nth sequence $\{f_{nm} : m \in \mathbb{N}\}$ by

$\{|f_{1m}| + \cdots + |f_{nm}| : m \in \mathbb{N}\}$. This approach is not applicable to arbitrary topological groups. Indeed, Sakai proves some of his lemmata in the context of general topological groups, but his main theorems are proved only for $\mathrm{C_p}(X)$. The following theorem answers Shakhmatov's question.

THEOREM 1.2. *A topological group is $\alpha_{1.5}$ if, and only if, it is α_1.*

Proof. Let G be a topological group, and $S_1, S_2, \ldots \subset G$ be sequences converging to e. Let T be any sequence converging to e (e.g., let $T := S_1$). For each n, fix a bijective enumeration $S_n = \{g_{nm} : m \in \mathbb{N}\}$.

Let $\{(n_k, m_k) : k \in \mathbb{N}\}$ be an enumeration of the set $\mathbb{N} \times \mathbb{N}$ where each pair (n, m) appears infinitely often. For each k, as the set $(T \setminus \{t_1, \ldots, t_{k-1}\}) \cdot g_{n_k m_k}$ is infinite, we can pick

$$t_k \in T \setminus \{t_1, \ldots, t_{k-1}\}$$

such that

$$t_k \cdot g_{n_k m_k} \notin \{t_1 \cdot g_{n_1 m_1}, \ldots, t_{k-1} \cdot g_{n_{k-1} m_{k-1}}\}.$$

For each pair (n, m), let $\{k(n, m, i) : i \in \mathbb{N}\}$ be an increasing enumeration of the set $\{k : (n_k, m_k) = (n, m)\}$. Note that the function $(n, m, i) \mapsto k(n, m, i)$ is injective. For each i, define the following perturbation of S_n:

$$S_n^{(i)} = \{t_{k(n,1,i)} \cdot g_{n1}, t_{k(n,2,i)} \cdot g_{n2}, t_{k(n,3,i)} \cdot g_{n3}, \ldots\}.$$

The sequence $S_n^{(i)}$ converges to e. By the construction, the sets $S_n^{(i)}$ for $n, i \in \mathbb{N}$ are pairwise disjoint, and therefore so are

$$S'_n = S_1^{(n)} \cup \cdots \cup S_n^{(n)}$$

for $n \in \mathbb{N}$. Being finite unions of sequences converging to e, the sequences $S'_1, S'_2, \ldots$ converge to e, too.

Apply $\alpha_{1.5}$ to $S'_1, S'_2, \ldots$ to find a sequence S' converging to e such that $S'_n \setminus S'$ is finite for each n in an infinite set $I \subset \mathbb{N}$. Define

$$S := \bigcup_{n \in I} \bigcup_{j=1}^{n} \{g_{jm} : m \in \mathbb{N},\ t_{k(j,m,n)} \cdot g_{jm} \in S'\}.$$

Since for each $n \in I$ and each $j = 1, \ldots, n$ we have $t_{k(j,m,n)} \cdot g_{jm} \in S'$ for all but finitely many m, the set $S_j \setminus S$ is finite for all j.

Finally, note that S is obtained by taking a subsequence of S' and multiplying its elements by distinct elements $t_{k(j,m,n)}^{-1}$, that is, elements of a subsequence of $\{t^{-1} : t \in T\}$, which also converges to e. Thus, S converges to e, too. ∎

We obtain a short proof of a result of Nogura and Shakhmatov.

DEFINITION 1.3 (Nogura–Shakhmatov [12]). A topological space X is *Ramsey* if, whenever $\lim_n \lim_m x_{nm} = x$, there is an infinite $I \subset \mathbb{N}$ such that for each neighborhood U of x, there is k such that $\{x_{nm} : k < n < m, n, m \in I\} \subset U$.

In general, α_1 topological spaces need not be Ramsey. In the context of topological groups, the above definition simplifies to the following one.

LEMMA 1.4 (Sakai [18]). *A topological group G is Ramsey if, and only if, whenever* $\lim_m g_{nm} = e$ *for all n, there is an infinite set $I \subset \mathbb{N}$ such that the sequence $\{g_{nm} : n, m \in I,\ n < m\}$ converges to e.*

Proof. Assume that $\lim_m g_{nm} = g_n$ and $\lim_n g_n = e$. For each n, define $g'_{nm} = g_n^{-1} g_{nm}$. Then $\lim_m g_{nm} = e$ for all n. ∎

THEOREM 1.5 (Nogura–Shakhmatov [12]). *Every $\alpha_{1.5}$ topological group is Ramsey.*

Proof. Let G be an $\alpha_{1.5}$ topological group. We establish the property stated in Lemma 1.4. Assume that $\lim_m g_{nm} = e$ for all n. By Theorem 1.2, G is α_1, and thus there is an increasing function $f \colon \mathbb{N} \to \mathbb{N}$ such that the sequence $\{g_{nm} : m \geq f(n)\}$ converges to e. Take I to be the image of f. ∎

2. New amalgamations. Using the above-mentioned pullback method of Scheepers, Sakai [18] proved that for groups of the form $\mathrm{C_p}(X)$, Ramsey is equivalent to α_2. The following problem, though, remains open.

PROBLEM 2.1 (Shakhmatov [23]).

(1) *Is every (Fréchet–Urysohn) α_2 topological group Ramsey?*
(2) *Is every (Fréchet–Urysohn) Ramsey topological group α_2?*

In Definitions 2.2, 2.3, 2.5, and 2.7 below, we introduce several new local properties related to Ramsey and α_2, and prove implications among them. The exact relations among these new properties and among them and the classical ones remain unknown. Some of the most interesting problems that remain open are summarized in Section 4.

DEFINITION 2.2. A topological space X is *locally Ramsey* if, for each $x \in X$, whenever $\lim_m x_{nm} = x$ for all n, there is an infinite set $I \subset \mathbb{N}$ such that the sequence $\{x_{nm} : n, m \in I, n < m\}$ converges to x.

Locally Ramsey spaces are α_3. By Lemma 1.4, a topological group is Ramsey if, and only if, it is locally Ramsey.

DEFINITION 2.3. A topological space X is α_{2^-} if, for each $x \in X$, whenever $\lim_m x_{nm} = x$ for all $n \in \mathbb{N}$, there are natural numbers $m_1 < m_2 < \cdots$ such that the sequence $\bigcup_n \{x_{1m_n}, \ldots, x_{nm_n}\}$ converges to x.

Thus, every α_{2^-} topological space is α_2.

PROPOSITION 2.4.

(1) *Every α_{2^-} topological space is locally Ramsey.*
(2) *Every α_{2^-} topological group is Ramsey.*

Proof. (1) Take $m_1 < m_2 < \cdots$ as in the definition of α_{2^-}, and set $I := \{m_n : n \in \mathbb{N}\}$.

(2) Use (1) and Lemma 1.4. ∎

DEFINITION 2.5. A topological space X is α_{3^-} if, for each $x \in X$, whenever $\lim_m x_{nm} = x$ for all n, there are infinite sets $I, J \subset \mathbb{N}$ such that the sequence $\{x_{nm} : n \in I,\, m \in J,\, n < m\}$ converges to x.

Thus, every locally Ramsey space is α_{3^-}, and every α_{3^-} space is α_3. The above-mentioned results of Scheepers and Sakai follow.

PROPOSITION 2.6. *For topological groups of the form* $\mathrm{C_p}(X)$, *the properties*

$$\alpha_{2^-},\ \alpha_2,\ \alpha_{3^-},\ \alpha_3,\ \alpha_4,\ \textit{locally Ramsey, and Ramsey}$$

are equivalent.

Proof. By the above observations, it suffices to show that α_4 implies α_{2^-} for such spaces. This follows from Scheepers's pullback method: Given sequences $S_n = \{f_{nm} : m \in \mathbb{N}\}$, each converging to 0, replace each sequence S_n with

$$S'_n = \{|f_{1m}| + \cdots + |f_{nm}| : m \geq n\}.$$

Applying α_4 and thinning out, we obtain an increasing sequence of indices $m_1 < m_2 < \cdots$ such that the sequence

$$|f_{1m_n}| + \cdots + |f_{nm_n}| \quad (n \in \mathbb{N})$$

converges to 0. Then the sequence $\bigcup_n \{f_{1m_n}, \ldots, f_{nm_n}\}$ converges to 0. ∎

DEFINITION 2.7. Let X be a topological space, and $x \in X$. The game $\alpha_2^{\mathrm{game}}(X, x)$ is played by two players, ONE and TWO, and has an inning per each natural number. On the nth inning, ONE chooses a sequence S_n converging to x, and TWO responds by choosing a subsequence $T_n \subset S_n$. TWO wins if the sequence $\bigcup_n T_n$ converges to x. Otherwise, ONE wins.

PROPOSITION 2.8. *Assume that for each $x \in X$, ONE does not have a winning strategy in $\alpha_2^{\mathrm{game}}(X, x)$. Then the space X is α_{2^-} (and thus locally Ramsey).*

Proof. Assume that $\lim_m x_{nm} = x$ for all n. Consider the following strategy for ONE: In the first inning, ONE plays the sequence $\{x_{1m} : m \in \mathbb{N}\}$. If TWO plays the subsequence

$$\{x_{1m} : m \in I_1\},$$

then ONE responds by playing

$$\{x_{2m} : m \in I_1 \setminus \{\min I_1\}\}.$$

In general, if in the nth inning TWO chooses a subsequence

$$\{x_{nm} : m \in I_n\},$$

then ONE plays

$$\{x_{n+1,m} : m \in I_n \setminus \{\min I_n\}\}.$$

Since this strategy is not winning for ONE, there is a play lost by ONE. Let $I_1, I_2, \dots$ be the infinite sets of sequence indices which correspond to the moves of TWO in this play. Define $m_n := \min I_n$ for each n. Then for each n,

$$\bigcup_{n\in\mathbb{N}} \{x_{1m_n}, \dots, x_{nm_n}\} \subset \bigcup_{n\in\mathbb{N}} T_n,$$

and the latter sequence converges to x. ■

Corollary 2.9. *Let G be a topological group. If ONE does not have a winning strategy in the game $\alpha_2^{\mathrm{game}}(G,e)$, then G is Ramsey (indeed, α_{2^-}).* ■

Proposition 2.10. *Let X be an α_1 space. For each $x \in X$, ONE does not have a winning strategy in the game $\alpha_2^{\mathrm{game}}(X,x)$.*

Proof. Define the game $\alpha_1^{\mathrm{game}}(X,x)$ corresponding to the property α_1 (at x). This game is similar to $\alpha_2^{\mathrm{game}}(X,x)$, with the only difference that here TWO must choose a *cofinite* subset of each sequence provided by ONE.

Lemma 2.11. *A topological space X is α_1 if, and only if, for each point $x \in X$, ONE does not have a winning strategy in the game $\alpha_1^{\mathrm{game}}(X,x)$.*

Proof. ($\Leftarrow$) Immediate.

($\Rightarrow$) The following method was used by Scheepers [19] to prove similar results for games involving open covers.

Fix a strategy for ONE in $\alpha_1^{\mathrm{game}}(X,x)$. For each sequence played by ONE, there are only countably many possible legal responses by TWO. Let $\mathcal{F}$ be the family of all possible sequences which ONE may play according to the fixed strategy. As $\mathcal{F}$ is countable, we can apply α_1 to $\mathcal{F}$, and find for each sequence $S \in \mathcal{F}$ a cofinite subset $S' \subset S$ such that $\bigcup_{S\in\mathcal{F}} S'$ converges to x.

Consider a play where TWO responds to each given sequence S_n by S'_n. This play is lost by ONE. ■

If ONE does not have a winning strategy in $\alpha_1^{\mathrm{game}}(X,x)$, then ONE does not have one in $\alpha_2^{\mathrm{game}}(X,x)$, where the moves of TWO are less restricted. ■

3. Sheaf amalgamations in topological vector spaces. In their 1968 paper [6], Averbukh and Smolyanov asked whether every α_1 topological vector space is Fréchet–Urysohn. The problem was only settled in Plichko's 2008 paper [16], using Banach spaces with certain weak topologies. Knowledge that was available in the field of selection principles, even before its solidification in 1996 [19, 9], was enough to have a consistent counterexample for the Averbukh–Smolyanov problem: Assume the Continuum Hypothesis, and let $S \subset \mathbb{R}$ be a *Sierpiński set*, that is, a set of size continuum whose intersection with every Lebesgue-null set is countable. It is known that every Borel image of a Sierpiński set in $\mathbb{N}^{\mathbb{N}}$ is bounded, and consequently the space $\mathrm{C_p}(S)$ is α_1. On the other hand, $\mathrm{C_p}(S)$ cannot be Fréchet–Urysohn since S is not Lebesgue-null [8]. Moreover, there is an example based solely on cardinality: It is known that the combinatorial cardinal $\mathfrak{p}$ (respectively, $\mathfrak{b}$) is the minimal cardinality of a set $X \subset \mathbb{R}$ such that $\mathrm{C_p}(X)$ is not Fréchet–Urysohn (respectively, α_1). Thus, the consistent assumption $\mathfrak{p} < \mathfrak{b}$ provides a counterexample in a trivial manner. We show in Theorem 3.3 that this approach provides a counterexample within ZFC. Moreover, this example has the following remarkable properties: Every separable subspace is metrizable, but the topological vector space is not even countably tight.

In the proof of Theorem 3.3, we will use several known facts, for which we provide proofs for completeness.

General versions of the following fact were proved in the 1970's (see, e.g., [10] and references therein). Recall that the *Σ-product* of spaces X_i, for $i \in I$, with respect to a point $x \in \prod_{i\in I} X_i$ is the subspace $\sum_{i\in I} X_i$ of the product space $\prod_{i\in I} X_i$ consisting of all $y \in \prod_{i\in I} X_i$ such that $y_i = x_i$ for all but countably many $i \in I$.

Proposition 3.1. *Let X be a Σ-product of a family of first countable spaces. Then:*

(1) *Every countable subspace of X is first countable.*
(2) *The space X is α_1.*
(3) *The space X has countable tightness.*
(4) *The space X is Fréchet–Urysohn.*

Proof. (1) Countable subspaces of X are supported on a countable set of indices.

(2) follows from (1).

(3) Let $X = \sum_{i\in I} X_i$, $A \subset X$ and $y \in \overline{A}$. For each $i \in I$, let $\mathcal{B}_i$ be a countable base at y_i. For a finite set $F \subset I$ and an element $U \in \prod_{i\in F} \mathcal{B}_i$, let

$$[U] := \{x \in X : \forall i \in F,\ x_i \in U_i\}.$$

Fix an arbitrary, countably infinite set $I_1 \subset I$. Continue by induction on n. Let $A_n \subset A$ be a countable set intersecting $[U]$ for all finite $F \subset I_n$

and all $U \in \prod_{i\in F} \mathcal{B}_i$. Let I_{n+1} be the union of I_n and the supports of the elements of A_n.

The point y is in the closure of the countable set $\bigcup_n A_n$. Indeed, let F be a finite subset of I, and $U \in \prod_{i\in F} \mathcal{B}_i$. Let $F_1 = F\cap\bigcup_n I_n$ and $F_2 = F\setminus\bigcup_n I_n$. As F is finite, there is n such that $F_1 \subset I_n$. Let $V = (U_i : i \in F_1)$. Then there is an $a \in A_n \cap [V]$. As the support of a is contained in I_{n+1}, $a_i = y_i$ for all $i \in F_2$. Thus, $a \in [U]$.

(4) follows from (3) and (1). ∎

The following result, brought to our attention by J. Moore, is proved for S in [27, Theorem 7.10], where it is pointed out that the L case is analogous. For completeness, we provide a proof for the L case, which is the one needed here.

Lemma 3.2. *Assume that Y is a regular topological space with all finite powers Lindelöf and countably tight, and X is a nonseparable subspace of Y. There exists a c.c.c. poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, the space X has an uncountable discrete subspace.*

Proof. It suffices to show that there are a c.c.c. poset $\mathbb{P}$ and a family $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{P}$ such that:

For each ZFC model $V' \supseteq V$ with $\omega_1^{V'} = \omega_1^V$, if there is in V' a filter $G \subset \mathbb{P}$ meeting each $D \in \mathcal{D}$, then the space X has an uncountable discrete subspace in V'.

Passing to a subset of X, if necessary, we may assume that

$$X = \{x_\xi : \xi < \omega_1\} \quad \text{and} \quad \overline{\{x_\xi : \xi < \alpha\}}^Y \cap \{x_\eta : \eta \ge \alpha\} = \emptyset$$

for every $\alpha < \omega_1$. There are two cases to consider.

Case 1: $\overline{\{x_\xi : \xi < \alpha\}}^Y \cap \overline{\{x_\eta : \eta \ge \alpha\}}^Y = \emptyset$ *for all* $\alpha < \omega_1$; in other words, X is a free sequence in Y. Since Y has countable tightness, $\overline{X}^Y = \bigcup_{\alpha<\omega_1} \overline{\{x_\beta : \beta < \alpha\}}^Y$. The space $\overline{X}^Y$ is closed in Y, and thus Lindelöf. On the other hand, the family

$$\{\overline{X}^Y \setminus \overline{\{x_\beta : \beta \ge \alpha\}}^Y : \alpha < \omega_1\}$$

is an open cover of $\overline{X}^Y$ without a countable subcover, a contradiction.

Case 2: $\overline{\{x_\xi : \xi < \alpha\}}^Y \cap \overline{\{x_\eta : \eta \ge \alpha\}}^Y \neq \emptyset$ *for some* α. In particular, the set $\{x_\eta : \eta \ge \alpha\}$ is not compact. We may assume that X is not compact. Let $\mathcal{U}$ be an ultrafilter on X whose elements are uncountable. If there exists some α such that $\mathcal{U}$ contains all open neighborhoods in X of x_α, then the Hausdorff property implies that every x_β for $\beta \neq \alpha$ has a neighborhood in X which is not in $\mathcal{U}$. By removing a point from X if needed, we may assume that every element of X has a neighborhood in X that is not in $\mathcal{U}$.

For each α, pick neighborhoods U_α, V_α of x_α in Y such that $\overline{V_\alpha} \subset U_\alpha$, $\overline{U_\alpha} \cap \overline{\{x_\xi : \xi < \alpha\}}^Y = \emptyset$ and $\{U_\alpha \cap X : \alpha < \omega_1\} \subset P(X) \setminus \mathcal{U}$. Then finitely many sets U_α cannot cover a cocountable subset of X. Let $\mathbb{P}$ be the poset consisting of all finite sets $\{\alpha_0, \dots, \alpha_{n-1}\} \subset \omega_1$, $\alpha_0 < \dots < \alpha_{n-1}$, such that $x_{\alpha_j} \notin V_{\alpha_i}$ whenever $i < j$. A condition H is stronger than F, written $H \leq F$, if $F \subset H$.

Assume, towards a contradiction, that there is an uncountable antichain $\{F_\alpha : \alpha < \omega_1\}$ in $\mathbb{P}$. For incompatible elements $F, H \in \mathbb{P}$, the elements $F \setminus H$ and $H \setminus F$ are also incompatible. By the Δ-System Lemma, we may assume that the sets F_α are pairwise disjoint, $\min F_\alpha > \max F_\beta$ for all $\beta < \alpha$, and $|F_\alpha| = n$ for all α. For each α, let $\{\xi_\alpha^0, \dots, \xi_\alpha^{n-1}\}$ be the increasing enumeration of F_α. Set

$$W_\alpha^0 := \{(x_0, \dots, x_{n-1}) \in X^n : \forall i, j < n,\, x_i \notin U_{\xi_\alpha^j}\},$$
$$W_\alpha^1 := \{(x_0, \dots, x_{n-1}) \in X^n : \exists i, j < n,\, x_i \in \overline{V_{\xi_\alpha^j}}\}.$$

Then $W_\alpha^0 \cap W_\alpha^1 = \emptyset$, and W_α^0, W_α^1 are closed. Moreover, by our choice of U_δ, we have $(x_{\xi_\beta^i})_{i<n} \in W_\alpha^0$ for all $\beta < \alpha$. By the definition of $\mathbb{P}$ and the incompatibility of F_α and F_β, we see that $(x_{\xi_\beta^i})_{i<n} \in W_\alpha^1$ for all $\beta > \alpha$. Thus, the subset $A := \{\vec{x}_\alpha := (x_{\xi_\alpha^i})_{i<n} : \alpha < \omega_1\}$ of X^n satisfies

$$\overline{\{\vec{x}_\beta : \beta < \alpha\}}^{Y^n} \cap \overline{\{\vec{x}_\beta : \beta \geq \alpha\}}^{Y^n} = \emptyset$$

for all α, a contradiction.

Thus, the forcing notion $\mathbb{P}$ is c.c.c. For each $\alpha < \omega_1$, let $D_\alpha := \{F \in \mathbb{P} : \max F > \alpha\}$. Since no finite subfamily of $\{U_\alpha : \alpha < \omega_1\}$ covers a cocountable subset of X, each set D_α is dense in $\mathbb{P}$. Assume that G is a subfilter of $\mathbb{P}$ (possibly, in some extension $V' \supseteq V$) which intersects every D_α. Then $x_\beta \notin V_\alpha$ for all $\beta, \alpha \in \bigcup G$: if $\beta < \alpha$ this follows from the choice of V_α, and if $\beta > \alpha$ this follows from the existence of an element $F \in G$ containing both α and β. Thus, G gives rise to the discrete subspace $\{x_\alpha : \alpha \in \bigcup G\}$ of X, which is uncountable if $\omega_1^{V'} = \omega_1^V$. ∎

We are ready for the main result of this section. An *L-space* is a hereditarily Lindelöf nonseparable topological space. The existence of L-spaces was established by Moore [11]. A classical result of Arhangel'skiĭ, and independently Pytkeev, asserts that a function space $\mathrm{C_p}(X)$ has countable tightness if and only if all finite powers of the space X are Lindelöf.

THEOREM 3.3. *There is a hereditarily Lindelöf nonseparable Fréchet–Urysohn space L such that:*

(1) $\mathrm{C_p}(L)$ *is α_1; moreover, every separable subspace of* $\mathrm{C_p}(L)$ *is metrizable.*
(2) $\mathrm{C_p}(L)$ *is not Fréchet–Urysohn; moreover, it is not countably tight.*

Proof. Let L be an L-space of the kind constructed by Moore [11]. Following Todorčević [26], Moore considered a function

$$\mathrm{osc}\colon \{(\alpha,\beta)\in\omega_1^2 : \alpha<\beta\}\to\omega$$

with strong combinatorial properties. Let $(z_\alpha)_{\alpha<\omega_1}$ be a sequence of rationally independent points on the multiplicative circle group $\mathbb{T}=\{z\in\mathbb{C} : |z|=1\}$. For each $\beta<\omega_1$, define an element $w_\beta\in\mathbb{T}^{\omega_1}$ by

$$w_\beta(\alpha)=\begin{cases} z_\alpha^{\mathrm{osc}(\alpha,\beta)+1} & \text{if } \alpha<\beta,\\ 1 & \text{otherwise.}\end{cases}$$

By [11, Theorem 7.11], the set $L=\{w_\beta : \beta<\omega_1\}$ is an L-space. By [11, Theorem 7.8], or directly by Proposition 3.1, the space L is Fréchet–Urysohn.

(1) Let D be a countable subset of $\mathrm{C_p}(L)$. Since L is a hereditarily Lindelöf subspace of a product space and $\mathbb{R}$ is a second countable Hausdorff space, every continuous function $f\colon L\to\mathbb{R}$ is determined by countably many coordinates; equivalently, there are $\alpha<\omega_1$ and a continuous function $g_\alpha\colon \mathrm{pr}_\alpha[L]\to\mathbb{R}$ such that $f=g_\alpha\circ\mathrm{pr}_\alpha$.

Lemma 3.4. *For each $\alpha<\omega_1$, the set $\mathrm{pr}_\alpha[L]$ is countable.*

Proof. By [11, Proposition 7.13], the subtree $\{\mathrm{osc}(\cdot,\delta)\restriction\alpha : \delta\ge\alpha\}$ of $\omega^{<\omega_1}$ is Aronszajn, where $\mathrm{osc}(\cdot,\delta)\colon \xi\mapsto\mathrm{osc}(\xi,\delta)$ for $\xi<\delta$. By the definition of Aronszajn tree, the set

$$\{\mathrm{osc}(\cdot,\delta)\restriction\alpha : \alpha<\delta<\omega_1\}$$

is countable for each $\alpha<\omega_1$. Thus, $\{w_\delta\restriction\alpha : \alpha<\delta<\omega_1\}$ is countable, and hence so is $\mathrm{pr}_\alpha[L]$. ∎

As D is countable, there is $\alpha<\omega_1$ such that every function $f\in D$ is determined by a continuous function on the first α coordinates. Thus,

$$\mathrm{pr}_\alpha^*\colon \mathrm{C_p}(\mathrm{pr}_\alpha[L])\to\mathrm{C_p}(L),\qquad g\mapsto g\circ\mathrm{pr}_\alpha,$$

is an embedding (see, e.g., [3, Proposition 0.4.6]). As $\mathrm{pr}_\alpha[L]$ is countable, the space $\mathrm{C_p}(\mathrm{pr}_\alpha[L])$ is metrizable, and therefore so is its image, which contains D.

(2) By Proposition 3.1, every finite power of $\sum_{\alpha<\omega_1}\mathbb{T}$ is countably tight. As countable tightness is hereditary, all finite powers of L are countably tight. By Lemma 3.2 with $X=Y=L$, we see that if all finite powers of L are Lindelöf, then there is a c.c.c. poset $\mathbb{P}$ such that L has an uncountable discrete subspace in $V^{\mathbb{P}}$. But in the proof of [11, Theorem 7.17] it is pointed out that the space L remains an L-space in c.c.c. forcing extensions. In fact, c.c.c. is not necessary, as the following lemma shows.

Lemma 3.5. *Moore's L-space remains an L-space in every forcing extension that does not collapse $\aleph_1$.*

Proof sketch. In accordance with [11, Definition 2.1], the construction of L is based on a C-sequence

$$\bar{C} = \langle C_\alpha : \alpha < \omega_1,\ \alpha \text{ limit}\rangle.$$

The function osc is constructed from $\bar{C}$ in such a way that, for each poset $\mathbb{P}$ preserving ω_1, the constructions of osc in V and in $V^{\mathbb{P}}$ give the same function, and hence give rise to the same subspace of the Σ-product of circles ([1]). By the same proof carried out in $V^{\mathbb{P}}$, this space is an L-space in $V^{\mathbb{P}}$. ■

It follows that some finite power of L is not Lindelöf. By the Arhangel'skiĭ–Pytkeev Theorem, $\mathrm{C_p}(X)$ is not countably tight, which completes the proof of Theorem 3.3. ■

4. Open problems and closing remarks. By Section 2 we know that, for topological groups,

$$\alpha_1 \Leftrightarrow \alpha_{1.5} \Rightarrow \text{ONE} \not\uparrow \alpha_2^{\text{game}}(G,e) \Rightarrow \alpha_{2^-} \Rightarrow \text{Ramsey} \Rightarrow \alpha_{3^-} \Rightarrow \alpha_3 \Rightarrow \alpha_4$$

and $\alpha_{2^-} \Rightarrow \alpha_2 \Rightarrow \alpha_3$.

Problem 4.1. *Are there, in ZFC or consistently, topological groups G that are:*

(1) α_3 *but not* α_{3^-}*?*
(2) α_{3^-} *but not Ramsey?*
(3) *Ramsey but not* α_{2^-}*?*
(4) α_{2^-} *but ONE has a winning strategy in* $\alpha_2^{\text{game}}(G,e)$*?*
(5) *not* α_1 *and ONE has no winning strategy in* $\alpha_2^{\text{game}}(G,e)$*?*
(6) α_2 *but not* α_{2^-}*?*
(7) α_{3^-} *but not* α_2*?*

Problem 4.2. *Let X be a Tychonoff space such that $\mathrm{C_p}(X)$ satisfies α_2. Does it follow that ONE does not have a winning strategy in $\alpha_2^{\text{game}}(\mathrm{C_p}(X),0)$?*

The results and methods of Section 3 have already been used in a number of papers, including [4, 5, 15, 21, 25]. The direct union of an L-space and the Sorgenfrey line is an L-space with non-Lindelöf square. However, such a space does not enjoy the properties described in Section 3. Answering a question from an earlier version of this paper, Yinhe Peng proved that the square of Moore's original L-space is also non-Lindelöf [15]. Peng's arguments are highly nontrivial, and use the fine details of Moore's construction. Our proof in item (2) of Theorem 3.3 is potentially more general, as it applies to all absolute modifications of Moore's construction where Lemma 3.5

([1]) This can be checked by going through the relevant definitions in Section 4 of [11] in a quite straightforward manner, i.e., without involving any deep absoluteness arguments.

holds. We do not know of any modification of Moore's construction where Lemma 3.5 fails.

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