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Where is pointwise multiplication in real CK-spaces locally open?

by

Ehrhard Behrends (Berlin)

Abstract. Let K be a compact Hausdorff space, and $B(f,r)$ the closed ball with center f and radius r in the space CK of real-valued continuous functions on K.

Given $f,g \in CK$ we say that *multiplication is locally open at* (f,g) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $B(fg,\delta) \subset B(f,\varepsilon)B(g,\varepsilon)$; here fg is the pointwise product of f and g, and $B(f,\varepsilon)B(g,\varepsilon) := \{\tilde{f}\tilde{g} \mid \tilde{f} \in B(f,\varepsilon), \tilde{g} \in B(g,\varepsilon)\}$. For $K = [0,1]$ a characterization of the pairs (f,g) at which multiplication is locally open is known. We extend it to arbitrary K.

1. Introduction. Let X,Y,Z be Banach spaces and $T : X \times Y \to Z$ a bilinear mapping. For certain rather special situations one can show that T is *locally open* at every pair (x,y): for all $x \in X$, $y \in Y$ and every $\varepsilon > 0$ one can find a positive δ such that

$$B_Z(T(x,y),\delta) \subset \{T(\tilde{x},\tilde{y}) \mid \tilde{x} \in B_X(x,\varepsilon),\ \tilde{y} \in B_Y(y,\varepsilon)\}.$$

(Closed balls in X,Y,Z are denoted by B_X, B_Y, B_Z, respectively.) It might even happen that, given ε, the same δ can be chosen for all x,y; then T is called *uniformly open* (cf. [1]). On the other hand, there are simple examples where T is *not* locally open at some (x,y), and by now it is not well understood what gives rise to such phenomena.

A natural case to be considered here is $X = Y = Z = A$, where A is a Banach algebra and T is multiplication in A. Even for this situation surprisingly little is known. We mention here the investigations of Komisarski [7] (who studies, in the case $A = CK$ for compact K, the connection between local openness and the dimension of K) and the author [4] (about multiplication on $C[0,1]$). For further results concerning local openness of multilinear mappings see [2], [5] and [8].

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The aim of the present paper is a characterization of local openness for general CK-spaces: multiplication turns out to be locally open at f, g if and only if K "splits" suitably in those regions where both f and g are close to zero.

In order to make this precise we need some notation. We fix a nonvoid compact Hausdorff space K and two continuous functions $f, g : K \to \mathbb{R}$. The map $\phi_{f,g} : K \to \mathbb{R}^2$ will be defined by $k \mapsto (f(k), g(k))$. We will also deal with some special subsets of $\mathbb{R}^2$. For positive η, ε we define

$$\begin{aligned}
M_{\eta,1} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \geq \eta\},\\
M_{\eta,2} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \leq \eta\},\\
M_{\eta,\varepsilon,1} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \leq \eta,\ x \geq -\varepsilon, y \leq \varepsilon\},\\
M_{\eta,\varepsilon,2} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \leq \eta,\ x \leq \varepsilon, y \geq -\varepsilon\},\\
M_{-\eta,1} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \leq -\eta\},\\
M_{-\eta,2} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \geq -\eta\},\\
M_{-\eta,\varepsilon,1} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \geq -\eta,\ x, y \leq \varepsilon\},\\
M_{-\eta,\varepsilon,2} &:= \{(x, y) \mid x, y \in \mathbb{R},\ xy \geq -\eta,\ x, y \geq -\varepsilon\}.
\end{aligned}$$

These sets are sketched in the following pictures:

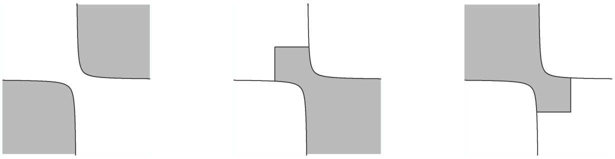

Fig. 1. $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$; $M_{\eta,2} = M_{\eta,\varepsilon,1} \cup M_{\eta,\varepsilon,2}$.

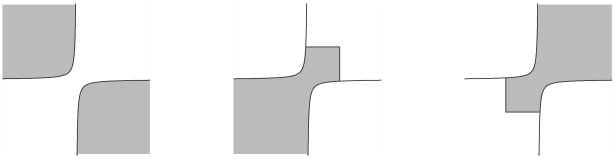

Fig. 2. $M_{-\eta,1}, M_{-\eta,\varepsilon,1}$ and $M_{-\eta,\varepsilon,2}$; $M_{-\eta,2} = M_{-\eta,\varepsilon,1} \cup M_{-\eta,\varepsilon,2}$.

Since the union of $M_{\eta,1}, M_{\eta,\varepsilon,1}, M_{\eta,\varepsilon,2}$ (resp. of $M_{-\eta,1}, M_{-\eta,\varepsilon,1}, M_{-\eta,\varepsilon,2}$) is all of $\mathbb{R}^2$, the union of the preimages $\phi^{-1}_{f,g}(M_{\eta,1}), \phi^{-1}_{f,g}(M_{\eta,\varepsilon,1}), \phi^{-1}_{f,g}(M_{\eta,\varepsilon,2})$ (resp. the union of $\phi^{-1}_{f,g}(M_{-\eta,1}), \phi^{-1}_{f,g}(M_{-\eta,\varepsilon,1}), \phi^{-1}_{f,g}(M_{-\eta,\varepsilon,2})$) will exhaust K.

In particular, $\phi^{-1}_{f,g}(M_{\eta,2})$ (resp. $\phi^{-1}_{f,g}(M_{-\eta,2})$) is the union of $\phi^{-1}_{f,g}(M_{\eta,\varepsilon,1})$ and $\phi^{-1}_{f,g}(M_{\eta,\varepsilon,2})$ (resp. of $\phi^{-1}_{f,g}(M_{-\eta,\varepsilon,1})$ and $\phi^{-1}_{f,g}(M_{-\eta,\varepsilon,2})$). Our characterization states that multiplication is locally open at (f,g) if and only if this union can be replaced by the union of suitable disjoint closed sets:

Theorem 1.1. *Let K be a nonvoid compact Hausdorff space. For $f,g \in CK$, the space of real-valued continuous functions on K, provided with the supremum norm, the following are equivalent:*

- (i) *Multiplication is locally open at (f,g).*
- (ii) *The following two conditions are satisfied:*
  - (C_1) *For every $\varepsilon_0 > 0$ there is an $\eta > 0$ with the following property: $\phi_{f,g}^{-1}(M_{\eta,2})$ can be written as the disjoint union of two closed subsets K_1, K_2 such that $\phi_{f,g}(K_1) \subset M_{\eta,\varepsilon_0,1}$ and $\phi_{f,g}(K_2) \subset M_{\eta,\varepsilon_0,2}$.*
  - (C_2) *For every $\varepsilon_0 > 0$ there is an $\eta > 0$ with the following property: $\phi_{f,g}^{-1}(M_{-\eta,2})$ can be written as the disjoint union of two closed subsets K_1, K_2 such that $\phi_{f,g}(K_1) \subset M_{-\eta,\varepsilon_0,1}$ and $\phi_{f,g}(K_2) \subset M_{-\eta,\varepsilon_0,2}$.*

The proof will be given in Section 3; some necessary preparations are provided in Section 2. In Section 4, we discuss some consequences, and finally, in Section 5, we collect some invitations for further research.

2. Some preparations

Positive and negative constant differences will suffice. Fix a positive δ and let α, β, γ be real numbers such that the polynomial $P : \lambda \mapsto \alpha + \beta\lambda + \gamma\lambda^2$ satisfies $P(0) = -\delta$ and $P(1) = \delta$. Then P has in fact only one free parameter: if γ is selected arbitrarily then $P(\lambda) = -\delta + (2\delta - \gamma)\lambda + \gamma\lambda^2 =: P_\gamma(\lambda)$. Figure 3 shows sketches of some P_γ where γ varies from negative values (light gray) to positive values (black).

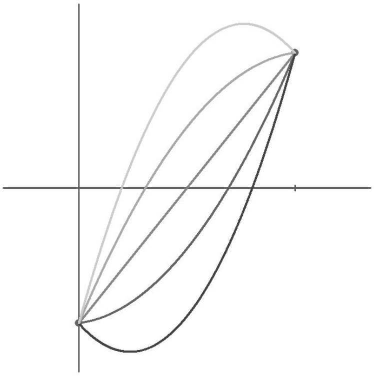

Fig. 3

By the intermediate value theorem, for every $a \in [-\delta, \delta]$ there exists a $\lambda \in [0,1]$ such that $P_\gamma(\lambda) = a$. The numbers λ and a are sketched in Figure 4.

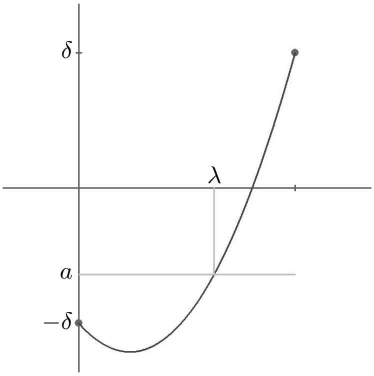

Fig. 4

One sees that, for certain γ, there might be two solutions λ of $P_\gamma(\lambda) = \delta$ or $P_\gamma(\lambda) = -\delta$. We claim that nevertheless one can choose λ as a continuous function of a and γ:

Lemma 2.1. *There is a continuous function* $\Lambda : [-\delta, \delta] \times \mathbb{R} \to [0,1]$ *such that* $P_\gamma(\Lambda(a,\gamma)) = a$ *for all* $a \in [-\delta, \delta]$ *and* $\gamma \in \mathbb{R}$.

Proof. We set

$$\Lambda(a, 0) := \frac{a}{2\delta} + \frac{1}{2}$$

for all a, and for $\gamma \neq 0$ and $a \in \mathbb{R}$ we define

$$\Lambda(a,\gamma) := \frac{(\gamma - 2\delta) + \sqrt{(\gamma - 2\delta)^2 + 4\gamma(a+\delta)}}{2\gamma}.$$

It is plain that $P_\gamma(\Lambda(a,\gamma)) = a$. Continuity on $[-\delta,\delta] \times (\mathbb{R} \setminus \{0\})$ is clear, and for the proof that Λ is continuous on $[-\delta,\delta] \times \{0\}$ one uses the fact that $\sqrt{1+x} \approx 1 + x/2$, where the error term is of order x^2. ■

The following proposition, which will considerably simplify the investigations to come, is an easy consequence of the above lemma:

Proposition 2.2. *For* $f, g \in CK$ *and positive* ε, δ *the following two conditions are equivalent:*

- (i) $B(fg, \delta) \subset B(f, \varepsilon)B(g, \varepsilon)$.
- (ii) $fg - \underline{\delta}$ *and* $fg + \underline{\delta}$ *lie in* $B(f,\varepsilon)B(g,\varepsilon)$. (*For any real number* α *we denote by* $\underline{\alpha}$ *the constant function* $k \mapsto \alpha$.)

Proof. We only have to show that (i) follows from (ii). So suppose that we may choose $d_1^-, d_2^-, d_1^+, d_2^+ \in CK$ with norm at most ε such that

$$(f + d_1^-)(g + d_2^-) = fg - \underline{\delta}, \qquad (f + d_1^+)(g + d_2^+) = fg + \underline{\delta}.$$

If $d \in CK$ with $-\underline{\delta} \le d \le \underline{\delta}$, we have to find $d_1, d_2 \in CK$ with $\|d_1\|, \|d_2\| \le \varepsilon$ such that $(f + d_1)(g + d_2) = fg + d$.

This will be achieved by taking convex combinations of $d_1^\pm, d_2^\pm$ that vary continuously with k. Fix $k \in K$ and consider for $\lambda \in [0, 1]$ the convex combinations

$$d_1^\lambda := \lambda d_1^+(k) + (1 - \lambda) d_1^-(k), \qquad d_2^\lambda := \lambda d_2^+(k) + (1 - \lambda) d_2^-(k).$$

What is a suitable λ such that $(f(k) + d_1^\lambda)(g(k) + d_2^\lambda) = f(k)g(k) + d(k)$, i.e., $f(k)d_2^\lambda + g(k)d_1^\lambda + d_1^\lambda d_2^\lambda = d(k)$?

We observe that $\lambda \mapsto f(k)d_2^\lambda + g(k)d_1^\lambda + d_1^\lambda d_2^\lambda$ is a polynomial of degree at most two that assumes the value $-\delta$ (resp. δ) at 0 (resp. 1). Thus it is of the form $-\delta + (2\delta - \gamma(k))\lambda + \gamma(k)\lambda^2$; note that $\gamma(k)$ is composed from $f(k), g(k), d_1^\pm(k), d_2^\pm(k), d(k)$ so that $\gamma(k)$ varies continuously with k.

We define $\tilde{\lambda} : K \to [0, 1]$ by $\tilde{\lambda}(k) := \Lambda(d(k), \gamma(k))$ (with Λ as in Lemma 2.1). Then $d_1 := \tilde{\lambda} d_1^+ + (1 - \tilde{\lambda}) d_1^-$ and $d_2 := \tilde{\lambda} d_2^+ + (1 - \tilde{\lambda}) d_2^-$ will have the desired properties: these functions are continuous, their norm is bounded by ε, they satisfy $fd_2 + gd_1 + d_1 d_2 = d$ and consequently also $(f + d_1)(g + d_2) = fg + d$. ∎

The strategy of our proof. Suppose that, for some given positive ε_0, δ, we want to show that $fg - \underline{\delta}$ lies in $B(f, \varepsilon_0)B(g, \varepsilon_0)$. Consider the following picture where some level curves of the function $(x, y) \mapsto H(x, y) := xy$ are depicted:

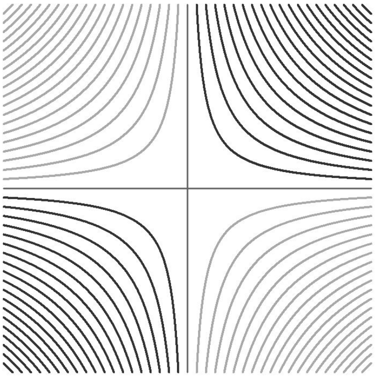

Fig. 5. Some level curves of $(x, y) \mapsto xy =: H(x, y)$ (positive: black; negative: gray)

We have to find continuous $d_1, d_2 : K \to \mathbb{R}$ such that for all k the point $(f(k) + d_1(k), g(k) + d_2(k))$ lies on a level surface where the associated value is precisely δ units smaller than the value of the level surface that contains $\phi_{f,g}(k)$. For every k there are infinitely many possibilities to choose $d_1(k), d_2(k)$, and the problem is to make these choices in such a way that they give rise to continuous functions d_1, d_2.

Definition 2.3. Let G be a subset of $\mathbb{R}^2$ and let $\varepsilon, \delta > 0$. A continuous function $\psi : G \to \mathbb{R}^2$ will be called an $(\varepsilon, -\delta)$ *vector field* if:

(i) $H(v + \psi(v)) = H(v) - \delta$ for all $v \in G$.
(ii) $\|\psi(v)\| \leq \varepsilon$ for all $v \in G$. (In this paper we will work with the maximum norm on $\mathbb{R}^2$.)

An (ε, δ) *vector field* is defined similarly, with $H(v+\psi(v)) = H(v)+\delta$ in (i).

Suppose that G is such that for every $\varepsilon > 0$ there is a positive δ for which one can find an $(\varepsilon, -\delta)$ vector field ψ. We also assume that the range of $\phi_{f,g}$ is contained in G. Then it is obvious that $fg - \underline{\delta}$ lies in $B(f, \varepsilon)B(g, \varepsilon)$: one only has to define d_1, d_2 by $(d_1, d_2) := \psi \circ \phi_{f,g}$; these functions will have the desired properties.

Therefore the problem arises to find G where such a ψ can be constructed. As an example consider the set $G := \{(x, y) \mid \|(x, y)\| \geq \alpha_0\}$, where $\alpha_0 > 0$. It is not hard to prove that for every $\varepsilon > 0$ there exists a positive δ such that G admits (ε, δ) and $(\varepsilon, -\delta)$ vector fields so that multiplication is locally open at (f, g) whenever the range of $\phi_{f,g}$ lies in G. Due to the compactness of K this observation can be applied whenever f, g have no common zeros. (This was already observed in Komisarski's paper [7].)

Everything would be very simple if $G = \mathbb{R}^2$ were admissible. That this cannot be true follows from the fact that there are (f, g) at which multiplication is *not* locally open, but a simple direct proof is also possible:

Proposition 2.4. *Let $\varepsilon > 0$. Then for no $\delta > 0$ does there exist an (ε, δ) vector field or an $(\varepsilon, -\delta)$ vector field.*

Proof. Suppose that for some positive δ an (ε, δ) vector field ψ exists. We have $\|\psi(-2\varepsilon, -2\varepsilon)\| \leq \varepsilon$ so that $(-2\varepsilon, -2\varepsilon) + \psi(-2\varepsilon, -2\varepsilon)$ must lie in the quadrant $\{(x, y) \mid x, y \leq 0\}$. Similarly $(2\varepsilon, 2\varepsilon) + \psi(2\varepsilon, 2\varepsilon)$ lies in $\{(x, y) \mid x, y \geq 0\}$ and consequently the curve $t \mapsto (t, t) + \psi(t, t)$ (from $[-\varepsilon, \varepsilon]$ to $\mathbb{R}^2$) meets $\{(x, y) \mid xy = 0\}$. Therefore $t \mapsto H((t, t) + \psi(t, t))$ vanishes at some $t \in [-2\varepsilon, 2\varepsilon]$, contrary to the assumption that it coincides with $t \mapsto H(t, t) + \delta = t^2 + \delta$, a strictly positive function.

Similarly one shows that the existence of an $(\varepsilon, -\delta)$ vector field implies a contradiction. (This time one works with $t \mapsto (t, -t) + \psi(t, -t)$.) ■

In view of this proposition we will have to argue more subtly: our solution will be to glue together several vector fields. The main ingredient will be the following elementary topological lemma, the proof of which is standard:

LEMMA 2.5. *Let L, M, N be topological Hausdorff spaces and $\phi : L \to M$ a continuous map. Suppose $L_1, \ldots, L_n$ are closed subsets of L such that $\bigcup_i L_i = L$ and $M_1, \ldots, M_n$ are closed subsets of M with $\bigcup_i M_i = M$ and $\phi(L_i) \subset M_i$ for $i = 1, \ldots, n$.*

Further suppose $\psi_i : M_i \to N$ are continuous maps such that, for all i, j, the maps ψ_i and ψ_j coincide on $M_i \cap M_j$.

Define $\Phi : L \to N$ as follows: for $l \in L_i$ (and any $i \in \{1, \ldots, n\}$), $\Phi(l) := \psi_i(\phi(l))$. Then Φ is well-defined and continuous.

We will use this lemma for $n = 3$ with $L := K$, $M := N := \mathbb{R}^2$ and $\phi := \phi_{f,g}$. The M_1, M_2, M_3 are the sets $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$ (resp. $M_{-\eta,1}$, $M_{-\eta,\varepsilon,1}$ and $M_{-\eta,\varepsilon,2}$), the sets L_1, L_2, L_3 are $\phi_{f,g}^{-1}(M_{\eta,1})$ (resp. $\phi_{f,g}^{-1}(M_{-\eta,1})$) and K_1, K_2 from Theorem 1; the ψ_1, ψ_2, ψ_3 will be suitable vector fields, to be defined next.

Some suitable vector fields

A. *Vector fields on $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$.* Fix $\varepsilon, \eta, \delta$ with $1 \ge \varepsilon > \eta > \delta > 0$; we will assume that $\eta \le \varepsilon^2$. The plane $\mathbb{R}^2$ is the union of $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$ (cf. Figure 1), and we will define $(\varepsilon, -\delta)$ vector fields on these subsets.

A.1. *The vector field on $M_{\eta,1}$.* This vector field is sketched in the following picture:

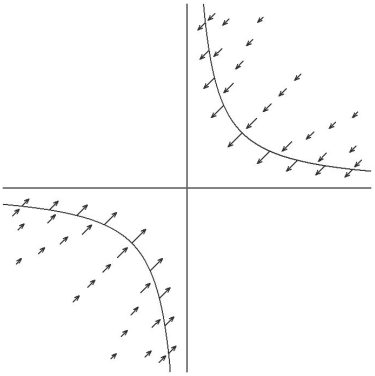

Fig. 6. The vector field on $M_{\eta,1}$

Rather than define it by a formula we prefer (here and in what follows) to describe first the rule how it is constructed; in this way the underlying ideas are more transparent. For $M_{\eta,1}$ the rule is simple:

> For $(x, y) \in M_{\eta,1}$ such that $x, y > 0$ (resp. $x, y < 0$) one goes in the direction $(-1, -1)$ (resp. $(1, 1)$) until one reaches

a point P where the H-value is precisely δ smaller than $H(x,y)$. We define $\psi_{\eta,\varepsilon,\delta}(x,y)$ to be the vector $P-(x,y)$.

LEMMA 2.6. *$\psi_{\eta,\varepsilon,\delta} : M_{\eta,1} \to \mathbb{R}^2$ is an $(\varepsilon,-\delta)$ vector field.*

Proof. We will only consider $(x,y) \in M_{\eta,1}$ with $x,y>0$; the case of $x,y<0$ can be treated in a similar way.

We start with the elementary observation that $xy \ge \eta$ implies $x+y \ge 2\sqrt{\eta}$, and we note that $\psi_{\eta,\varepsilon,\delta}(x,y)$ is the vector $(-t,-t)$, where t is the smallest positive solution of $(x-t)(y-t) = xy-\delta$. This t has the form

$$\frac{(x+y)-\sqrt{(x+y)^2-4\delta}}{2}$$

where $(x+y)^2-4\delta$ is positive by the preceding observation.

It follows that $\psi_{\eta,\varepsilon,\delta}$ is continuous and also that the length of $\psi_{\eta,\varepsilon,\delta}(x,y)$ only depends on $x+y$.

Finally we note that $\rho : [2\sqrt{\delta},\infty[\to \mathbb{R}$ defined by $c \mapsto (c-\sqrt{c^2-4\delta})/2$ has a negative derivative so that $\rho(c) \le \rho(2\sqrt{\delta}) = \sqrt{\delta} \le \varepsilon$ for all c, and it follows that $\|\psi_{\eta,\varepsilon,\delta}(x,y)\| \le \varepsilon$ for (x,y) under consideration. This proves the lemma. ■

A.2. *The vector field on $M_{\eta,\varepsilon,2}$.* We write $M_{\eta,\varepsilon,2}$ as the union of three subsets. The vector field will be constructed on each of these separately in such a way that on the intersection of two of them the definition is the same. Here are the subsets, they will be denoted by I, II, III:

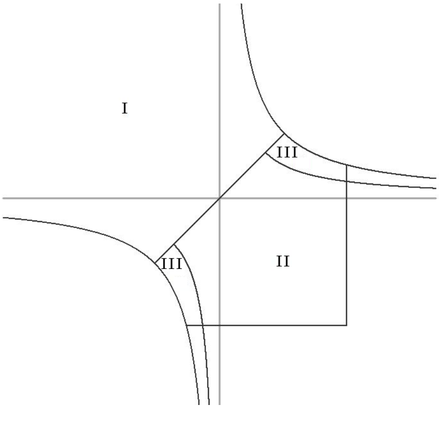

Fig. 7. The partition of $M_{\eta,\varepsilon,2}$

More precisely, we set

$$\begin{aligned} I &:= \{(x,y) \mid (x,y) \in M_{\eta,\varepsilon,2},\ y-x \ge 0\}, \\ II &:= \{(x,y) \mid x \le \varepsilon,\ y \ge -\varepsilon,\ xy \le \delta,\ y-x \le 0\}, \\ III &:= \{(x,y) \mid x \le \varepsilon,\ y \ge -\varepsilon,\ \delta \le xy \le \eta,\ y-x \le 0\}. \end{aligned}$$

On I our field looks as follows:

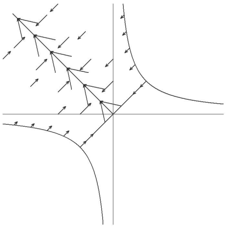

Fig. 8. The field on I

It is defined by the following rule:

1. If $x+y \geq 0$ go in the direction $(-1,-1)$. If before reaching $\{(x,y) \mid x+y=0\}$ you arrive at a point P with $H(P)=H(x,y)-\delta$, set $\psi_I(x,y):=P-(x,y)$.
   Otherwise, i.e., if you meet $\{(x,y) \mid x+y=0\}$ at a point Q where $H(Q)>H(x,y)-\delta$, then move from Q in the direction $(-1,1)$ to a point P where $H(P)=H(x,y)-\delta$, and set $\psi_I(x,y):=P-(x,y)$.
2. If $x+y \leq 0$ go in the direction $(1,1)$. Then proceed similarly to the above.

The sketch of the field on II is as follows:

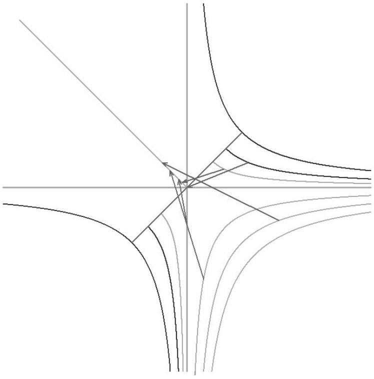

Fig. 9. The field on II

On this subset all vectors will point to some element in the diagonal

$$\{(x,y) \mid x+y=0,\ x \leq 0,\ y \geq 0\}.$$

Thus we have only one choice:

- For $(x,y) \in II$ calculate $t:=H(x,y)$: this number will lie in $[-\varepsilon^2,\delta]$. Set $s:=\delta-t \in [0,\delta+\varepsilon^2]$ and $P:=(-\sqrt{s},\sqrt{s})$ (so that $H(P)$

equals $t - \delta$). Define $\psi_{II}(x, y) := P - (x, y)$, which guarantees that $H(\psi_{II}(x, y)) = H(x, y) - \delta$ as desired.

It remains to define the field on III. This has to be done in such a way that the definition coincides

- with that of $\psi_{\eta,\varepsilon,\delta}$ on $M_{\eta,1} \cap III$;
- with that of ψ_I on $I \cap III$;
- with that of ψ_{II} on $II \cap III$.

Here is a sketch:

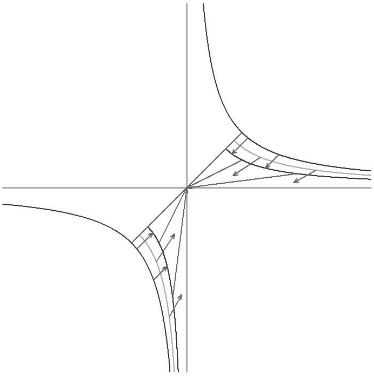

Fig. 10. The field on *III*

The procedure to find $\psi_{III}(x, y)$ is as follows:

- Let $(x, y) \in III$ be given and suppose first that $x, y > 0$. Then $\delta \leq H(x, y) \leq \eta$. We are looking for H-values that are smaller than $t := H(x, y)$ in the "south-west" direction. More precisely: for $t = \eta$ (i.e., if $x, y \in III \cap M_{1,\eta}$) we go in the direction $(-1, -1)$, for $t = \delta$ the direction points to $(0, 0)$, and for $t \in \,]\delta, \eta[$ the direction is interpolated continuously.
  Then we go in this direction to the first point P where $H(P) = H(x, y) - \delta$. $\psi_{III}(x, y)$ is the vector $P - (x, y)$. The "direction where one will find P" is chosen in such a way that it satisfies the above conditions and depends continuously on (x, y).
  For (x, y) with $x, y < 0$, $\psi_{III}(x, y)$ is defined similarly; this time P is found in the "north-east" direction.

We now put the pieces together:

LEMMA 2.7.

(i) ψ_I *is a* $(2\varepsilon, -\delta)$ *vector field on* I.
(ii) ψ_{II} *is a* $(2\varepsilon, -\delta)$ *vector field on* II.

(iii) *ψ_{III} is a $(2\varepsilon, -\delta)$ vector field on III.*

(iv) *Define $\psi_{\eta,\varepsilon,\delta,2} : M_{\eta,\varepsilon,2} \to \mathbb{R}^2$ by $\psi_{\eta,\varepsilon,\delta,2}(x,y) := \psi_I(x,y)$ (resp. $:= \psi_{II}(x,y)$, $\psi_{III}(x,y)$) if $(x,y) \in I$ (resp. $\in II, III$). Then $\psi_{\eta,\varepsilon,\delta,2}$ is a well-defined $(2\varepsilon, \delta)$ vector field on $M_{\eta,\varepsilon,2}$. It coincides with $\psi_{\eta,\varepsilon,\delta}$ on $M_{\eta,1} \cap M_{\eta,\varepsilon,2}$.*

Proof. (i) As in the proof of Lemma 2.6, we start by finding the associated formulas. Let $(x,y) \in I$ with $x+y \geq 0$. For $x+y \geq 2\sqrt{\delta}$ we get as above the expression $\psi_I(x,y) = (-t,-t)$, where $t = ((x+y) - \sqrt{(x+y)^2 - 4\delta})/2$.

If $x+y \in [0, 2\sqrt{\delta}[$ the definition is slightly more complicated. The vector Q can be written explicitly as $(-c, c)$, where $c = (y-x)/2 \geq 0$. The distance $\|Q - (x,y)\|$ can easily be estimated:

$$\|Q - (x,y)\| = \|(-(x+y)/2, -(x+y)/2)\| = |(x+y)/2| \leq \sqrt{\delta} \leq \varepsilon.$$

It remains to move from Q to P, i.e., a positive t has to be found such that $(-\alpha - t)(\alpha + t) = -\alpha^2 - \delta'$, where $\delta' := \delta + H(Q) \in [0, \delta]$. It follows that $t = -c + \sqrt{c^2 + \delta'}$, and since $c \mapsto \tilde{\rho}(c) := -c + \sqrt{c^2 + \delta'}$ has a negative derivative it follows that $t \leq \tilde{\rho}(0) = \sqrt{\delta'} \leq \sqrt{\delta} \leq \varepsilon$. We conclude that

$$\|\psi_I(x,y)\| = \|P - (x,y)\| \leq \|P - Q\| + \|Q - (x,y)\| \leq 2\varepsilon.$$

Since $\delta' \to 0$ as $x + y \to 2\sqrt{\delta}$, the function ψ_I is in fact continuous. For $(x,y) \in I$ with $x + y \leq 0$ we proceed similarly. On the diagonal $\{(x,y) \mid x + y = 0\}$ the ψ_I-value is the same as before ($\psi_I(-x,x)$ points in the direction $(-1,1)$), and thus we have shown that ψ_I is a $(2\varepsilon, -\delta)$ vector field.

(ii)&(iii) As in the case of ψ_I, we prove that both fields are continuous by giving explicit expressions. (Of course the "change of direction" in the definition of ψ_{III} must be prescribed by a continuous function.) For $(x,y) \in II$ one has $\|(x,y)\| \leq \varepsilon$, and the norm of P is at most $\sqrt{\delta} \leq \varepsilon$; it follows that ψ_{II} is a $(2\varepsilon, -\delta)$ vector field.

For ψ_{III}, (x,y) and P are both in $[0,\varepsilon] \times [0,\varepsilon]$ or both in $[-\varepsilon, 0] \times [-\varepsilon, 0]$ so that $\|\psi_{III}(x,y)\| \leq \varepsilon$.

(iv) It is an immediate consequence of the definition that $\psi_I, \psi_{II}, \psi_{III}$ coincide on the intersections of their domains (so that $\psi_{\eta,\varepsilon,\delta,2}$ is well defined and continuous) and that $\psi_{\eta,\varepsilon,\delta,2} = \psi_{\eta,\varepsilon,\delta}$ on $M_{\eta,1} \cap M_{\eta,\varepsilon,2}$. ∎

A.3. *The vector field on $M_{\eta,\varepsilon,1}$.* This is much simpler: we simply "rotate" the preceding definitions by 180 degrees. More precisely, $\psi_{\eta,\varepsilon,\delta,1} := M_{\eta,\varepsilon,1} \to \mathbb{R}^2$ is defined by

$$\psi_{\eta,\varepsilon,\delta,1} := R \circ \psi_{\eta,\varepsilon,\delta,2} \circ R,$$

where R stands for the rotation $(x,y) \mapsto (-x,-y)$. This is a $(2\varepsilon, -\delta)$ vector field.

B. *Vector fields on $M_{-\eta,1}$, $M_{-\eta,\varepsilon,1}$ and $M_{-\eta,\varepsilon,2}$.* This will simply be done by a suitable rotation of the fields on $M_{\eta,1}$, $M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$ (counter-clockwise by 90 degrees). We will denote the resulting fields, which are $(2\varepsilon,\delta)$ vector fields, by $\psi_{-\eta,\varepsilon,\delta}$, $\psi_{-\eta,\varepsilon,\delta,1}$ and $\psi_{-\eta,\varepsilon,\delta,2}$.

3. Proof of the main theorem. We are now ready for the proof of Theorem 1.1.

(i)⇒(ii). Suppose first that multiplication is locally open at (f,g) and that $\varepsilon_0>0$. We have to find an $\eta>0$ such that (ii)(C_1) and (ii)(C_2) are satisfied.

By assumption there exists a δ such that $fg-\underline{\delta}$ and $fg+\underline{\delta}$ are in $B(f,\varepsilon_0)B(g,\varepsilon_0)$. Let us first handle $fg-\underline{\delta}$. We choose $d_1,d_2\in CK$ with $\|d_1\|,\|d_2\|\le\varepsilon_0$ such that $(f+d_1)(g+d_2)=fg-\underline{\delta}$.

We will denote by Q^{++} (resp. Q^{+-}, Q^{-+}, Q^{--}) the quadrant $\{(x,y)\mid x\ge 0,\ y\ge 0\}$ (resp. $\{(x,y)\mid x\ge 0,\ y\le 0\}$, $\{(x,y)\mid x\le 0,\ y\ge 0\}$, $\{(x,y)\mid x\le 0,\ y\le 0\}$).

We choose any η such that $0<\eta<\delta$. Let $(x,y)\in M_{\eta,2}$ and let (x',y') be such that $H(x',y')=H(x,y)-\delta$. Then (x',y') must lie in $(Q^{+-}\cup Q^{-+})\setminus\{0\}$ since in the complement of this set there are no (x',y') with this property.

Another observation will be important: if (x,y) is in $M_{\eta,2}\setminus M_{\eta,\varepsilon_0,1}$ (resp. in $M_{\eta,2}\setminus M_{\eta,\varepsilon_0,2}$) and (x',y') is such that $H(x',y')=H(x,y)-\delta$ and $\|(x,y)-(x',y')\|\le\varepsilon_0$ then (x',y') necessarily lies in Q^{-+} (resp. in Q^{+-}).

Now we will show that (ii)(C_1) holds with this η. We set

$$K_1:=\{k\in K\mid \phi_{f,g}(k)\in M_{\eta,2},\ (f(k)+d_1(k),g(k)+d_2(k))\in Q^{+-}\},$$
$$K_2:=\{k\in K\mid \phi_{f,g}(k)\in M_{\eta,2},\ (f(k)+d_1(k),g(k)+d_2(k))\in Q^{-+}\}.$$

It is clear from the preceding remarks that $\phi^{-1}_{f,g}(M_{\eta,2})$ is the disjoint union of K_1 and K_2, and by the continuity of the maps under consideration these sets are closed.

Now let $k\in K_1$. If $\phi_{f,g}(k)\notin M_{\eta,\varepsilon_0,1}$, then by the preceding observation we would have $(f(k)+d_1(k),g(k)+d_2(k))\in Q^{-+}$, a contradiction. This shows that $\phi_{f,g}(K_1)\subset M_{\eta,\varepsilon_0,1}$, and similarly one proves that $\phi_{f,g}(K_2)\subset M_{\eta,\varepsilon_0,2}$.

In the same way one verifies that $fg+\underline{\delta}\in B(f,\varepsilon_0)B(g,\varepsilon_0)$ implies (ii)C_2.

(ii)⇒(i). Let $\varepsilon_0>0$. By Proposition 2.2 it will suffice to find a positive δ such that $fg\pm\underline{\delta}$ lie in $B(f,\varepsilon_0)B(g,\varepsilon_0)$.

By assumption there exists $\eta>0$ such that (ii)(C_1) and (ii)(C_2) hold. A moment's reflection shows that one may replace η by a smaller number: if $0<\eta'<\eta$, then $M_{\eta',2}\subset M_{\eta,2}$ so that $\phi^{-1}_{f,g}(M_{\eta',1})$ is the disjoint union of $K_1\cap\phi^{-1}_{f,g}(M_{\eta',1})$ and $K_2\cap\phi^{-1}_{f,g}(M_{\eta',1})$.

Therefore we may assume that $\eta\le\varepsilon_0^2/4$. We choose any δ such that $0<\delta<\eta$.

Let us prove by using (C_1) that $fg-\underline{\delta}$ lies in $B(f,\varepsilon_0)B(g,\varepsilon_0)$. Condition (C_1) implies that K can be written as $K=K'\cup K_1\cup K_2$ with closed subsets K',K_1,K_2 such that

- $K'=\phi_{f,g}^{-1}(M_{\eta,1})$;
- K_1 and K_2 are disjoint;
- $\phi_{f,g}(K_1)\subset M_{\eta,\varepsilon_0,1}$ and $\phi_{f,g}(K_2)\subset M_{\eta,\varepsilon_0,2}$.

We have to find continuous $d_1,d_2:K\to\mathbb{R}$ with $\|d_1\|,\|d_2\|\le\varepsilon_0$ such that $(f+d_1)(g+d_2)=fg-\underline{\delta}$. At this point the vector fields defined in Section 2 come into play (they are defined with $\varepsilon:=\varepsilon_0/2$). We set

$$(d_1(k),d_2(k)):=\begin{cases}\psi_{\eta,\varepsilon,\delta}\circ\phi_{f,g}(k) & \text{for } k\in K',\\ \psi_{\eta,\varepsilon,\delta,1}\circ\phi_{f,g}(k) & \text{for } k\in K_1,\\ \psi_{\eta,\varepsilon,\delta,2}\circ\phi_{f,g}(k) & \text{for } k\in K_2.\end{cases}$$

Our vector fields have been so constructed that this definition gives rise to well-defined maps. From Lemma 2.5 it follows that they are continuous; we emphasize that *here* one makes crucial use of the fact that $K_1\cap K_2=\emptyset$ (1).

Similarly it can be shown that (C_2) yields $fg+\underline{\delta}\in B(f,\varepsilon_0)B(g,\varepsilon_0)$. ∎

4. Consequences

The special case $K=[0,1]$. First we will prove that for $K=[0,1]$ our characterization coincides with that of [4]. Given continuous $f,g:[0,1]\to\mathbb{R}$ we define $\gamma:[0,1]\to\mathbb{R}^2$ by $t\mapsto(f(t),g(t))$, and we say that there is a *positive saddle point crossing* (resp. a *negative saddle point crossing*) if for some positive $\varepsilon>0$ one can find $t_1<t_2$ with $\|\gamma(t_1)\|,\|\gamma(t_2)\|\ge\varepsilon$, $\gamma(t_1)\in Q^{++}$ and $\gamma(t_2)\in Q^{--}$ or vice versa, and $\gamma(t)\in Q^{++}\cup Q^{--}$ for all $t\in[t_1,t_2]$ (resp. $\gamma(t_1)\in Q^{+-}$ and $\gamma(t_2)\in Q^{-+}$ or vice versa, and $\gamma(t)\in Q^{+-}\cup Q^{-+}$ for all $t\in[t_1,t_2]$).

PROPOSITION 4.1. (ii)(C_1) *and* (ii)(C_2) *hold for all positive* ε_0 *if and only if there are no positive and no negative saddle point crossings.*

Proof. (Under the assumption that our arguments were and are correct there is no need for the proof. It is nevertheless included in order to illustrate the difference between the two approaches.) Suppose first that there is a positive saddle point crossing. Choose ε,t_1,t_2 as in the definition and set $\varepsilon_0:=\varepsilon/2$. We claim that (ii)$(\mathrm{C}_1)$ cannot hold. Suppose that there were a positive η such that one could find K_1,K_2 as in this condition. Then $\gamma(t_1)$ must lie in K_2 and $\gamma(t_2)$ in K_1 (or vice versa). Also $\gamma(t)$ for all $t\in[t_1,t_2]$ would belong to $K_1\cup K_2$ so that $[t_1,t_2]$ could be written as the disjoint union of two closed nonvoid subsets, a contradiction.

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(1) A k in this intersection would destroy the hope to obtain a well-defined map. One would have $\phi_{f,g}(k)\in M_{\varepsilon,\eta,1}\cap M_{\varepsilon,\eta,2}$, but $\psi_{\eta,\delta,\varepsilon,1}$ and $\psi_{\eta,\delta,\varepsilon,2}$ do not coincide there.

Similarly the existence of a negative saddle point crossing would imply that (ii)(C_2) cannot be satisfied.

Suppose now that, for some $\varepsilon_0 > 0$, there is no positive η such that condition (ii)(C_2) holds. We will prove that there is a positive saddle point crossing. (Similarly the negation of (ii)(C_1) would give rise to a negative saddle point crossing.)

Claim. *For every $\eta > 0$ there are $t_{1,\eta}, t_{2,\eta} \in [0,1]$ such that for all t between $t_{1,\eta}$ and $t_{2,\eta}$ one has $\gamma(t) \in M_{-2\eta,2}$, but $\gamma(t_{1,\eta}) \notin M_{-2\eta,\varepsilon_0,1}$ and $\gamma(t_{2,\eta}) \notin M_{-2\eta,\varepsilon_0,2}$. (Thus, if $\eta = 0$ were admissible, there would be a positive saddle point crossing between t_1 and t_2.)*

Proof of claim. Consider the compact set $\Delta := \{t \mid t \in [0,1],\ \gamma(t) \in M_{-\eta,2}\}$. Then Δ is contained in the open set $O := \{t \mid t \in [0,1],\ H(\gamma(t)) > -2\eta\}$, and we may choose pairwise disjoint intervals $I_i = [s_i, t_i]$ for $i = 1, \ldots, n$ with $\Delta \subset \bigcup_i I_i \subset O$.

What happens for the t in some of the I_i? We claim that there must be at least one i where $G_i := \{\gamma(t) \mid t \in I_i\}$ is *neither* completely contained in $M_{-2\eta,\varepsilon_0,1}$ nor in $M_{-2\eta,\varepsilon_0,2}$. If always $G_i \subset M_{-2\eta,\varepsilon_0,1}$ or $G_i \subset M_{-2\eta,\varepsilon_0,2}$, we could set

$$K_1' := \bigcup\{I_i \mid G_i \subset M_{-2\eta,\varepsilon_0,1}\}, \qquad K_2' := \bigcup\{I_i \mid G_i \subset M_{-2\eta,\varepsilon_0,2}\},$$

and $K_1 := K_1' \cap \gamma^{-1}(M_{-\eta,2})$, $K_2 := K_2' \cap \gamma^{-1}(M_{-\eta,2})$ would be a disjoint partition of $\gamma^{-1}(M_{-\eta,2})$ as in (ii)(C_2), contrary to our hypothesis.

Choose any I_i where G_i is neither a subset of $M_{-2\eta,\varepsilon_0,1}$ nor of $M_{-2\eta,\varepsilon_0,2}$. This implies that we can select $t_{1,\eta}, t_{2,\eta}$ with the claimed properties: $t_{1,\eta}$ will be a t with $t \in M_{-2\eta,2} \setminus M_{-2\eta,\varepsilon_0,1}$, and $t_{2,\eta}$ will be a t with $t \in M_{-2\eta,2} \setminus M_{-2\eta,\varepsilon_0,2}$.

We are now ready to prove that there is a positive saddle point crossing. We choose $t_{1,\eta}, t_{2,\eta}$ for $\eta = 1, 1/2, 1/3, \ldots$ and select an accumulation point t_1 (resp. t_2) of these $t_{1,\eta}$ (resp. of these $t_{2,\eta}$). Then $\gamma(t)$ for t between t_1 and t_2 stays in $Q^{++} \cup Q^{--}$, the norm of $\gamma(t_1), \gamma(t_2)$ is at least ε_0 and one of these vectors lies in Q^{++} and the other in Q^{--}. ■

The case $f = \pm g$. It is easy to characterize those f where multiplication is locally open at (f, f):

Proposition 4.2. *For $f \in CK$ the following are equivalent:*

(i) *Multiplication is locally open at (f, f).*
(ii) *Multiplication is locally open at $(f, -f)$.*
(iii) *For every $\varepsilon_0 > 0$ there are disjoint closed subsets $K_1, K_2 \subset K$ such that $K_1 \cup K_2 = K$ and $f(k_1) \geq -\varepsilon_0$ and $f(k_2) \leq \varepsilon_0$ for $k_1 \in K_1$ and $k_2 \in K_2$.*

Proof. We will show that (i) and (iii) are equivalent; the proof of (ii)⇔(iii) is similar.

(i)⇒(iii). Supppose that (i) holds and let $\varepsilon_0 > 0$. By Theorem 1.1(ii) there exists $\eta > 0$ such that $\phi_{f,f}^{-1}(M_{-\eta,2})$ can be written as $K_1 \cup K_2$ with disjoint closed K_1, K_2 such that $\phi_{f,f}(K_1) \subset M_{-\eta,\varepsilon_0,2}$ and $\phi_{f,f}(K_2) \subset M_{-\eta,\varepsilon_0,1}$. We note that $\phi_{f,f}^{-1}(M_{-\eta,2}) = K$, i.e., K splits as $K_1 \cup K_2$, and $f(k) \ge -\varepsilon_0$ (resp. $f(k) \le \varepsilon_0$) for $k \in K_1$ (resp. $k \in K_2$).

(iii)⇒(i). The preceding part of the proof used the fact that (iii) is a reformulation of (ii)(C_2) of Theorem 1.1. But (ii)(C_1) is trivially true here: for ε_0 we set $\eta := \varepsilon^2$, and then $\phi_{f,f}^{-1}(M_{\eta,2})$ can be written as the union of $K_1 := \phi_{f,f}^{-1}(M_{\eta,2})$ and $K_2 := \emptyset$, with $\phi_{f,f}(K_1) \subset M_{\eta,\varepsilon_0,1}$ and $\phi_{f,f}(K_2) \subset M_{\eta,\varepsilon_0,2}$. ∎

The following corollary extends [3, Theorem 3]:

Corollary 4.3. *Let K be a connected compact Hausdorff space. Then the following assertions are equivalent:*

(i) *Multiplication is locally open at (f, f).*
(ii) *Multiplication is locally open at $(f, -f)$.*
(iii) *$f(k) \ge 0$ for all k or $f(k) \le 0$ for all k.*

Proof. We show that (i) and (iii) are equivalent; the equivalence of (ii) and (iii) is proved similarly.

(iii)⇒(i). Assume that, e.g., $f \ge 0$. Then in Proposition 4.1(iii) we can choose $K_2 := K$ and $K_1 := \emptyset$ for arbitrary ε_0. (Here it is not essential that K is connected.)

(i)⇒(iii). Suppose that (iii) does not hold, i.e., there are $k_1', k_2' \in K$ with $f(k_2') < 0 < f(k_1')$. Choose ε_0 with $f(k_2') < -\varepsilon_0 < 0 < \varepsilon_0 < f(k_1')$. We claim that (ii)($C_2$) is satisfied for no positive η. If there were K_1, K_2 as in (C_2), the space K would split as $K = K_1 \cup K_2$ with disjoint closed K_1, K_2. Since $k_1' \in K_1$ and $k_2' \in K_2$, this contradicts the fact that K is connected. ∎

The case $K = [-1,1]^2$. Next we show that for $K = Q := [-1,1]^2$ an interesting phenomenon can be observed. There—in contrast to the case of many other known bilinear mappings—the collection of "good" pairs (where multiplication is locally open) is not dense.

Fix $0 < \varepsilon \le 1/3$ and define $A_1, A_2 \subset Q$ as follows:

$$A_1 := \{(x,y) \mid x \ge -\varepsilon,\ y \le \varepsilon\} \qquad A_2 := \{(x,y) \mid x \le \varepsilon,\ y \ge -\varepsilon\}\ .$$

R, S, T, U denote the points $(-1,-1), (-\varepsilon,-\varepsilon), (\varepsilon,\varepsilon), (1,1)$, respectively.

Lemma 4.4. *Suppose that K_1, K_2 are compact subsets of Q with $K_1 \cap K_2 = \emptyset$ and $K_1 \subset (A_1)^0$, $K_2 \subset (A_2)^0$. (M^0 stands for the interior of a subset of a topological space.) Then there is a path ϕ from R to U in $Q \setminus (K_1 \cup K_2)$:*

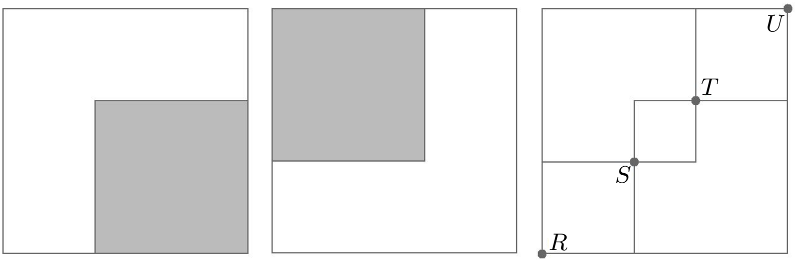

Fig. 11. The sets A_1, A_2 and the points R, S, T, U

the function $\phi : [0,1] \to Q$ *is continuous, it omits* $K_1 \cup K_2$*, and satisfies* $\phi(0) = R$ *and* $\phi(1) = U$*.*

Proof. (This assertion is "obvious", but it seems to be hard to give a short and elegant argument.) Since K_1 and K_2 are disjoint they have a positive distance 4τ, and since they lie in the interior of A_1 and A_2, respectively, we may assume that $K_1 + B(0, 2\tau) \subset A_1$ and $K_2 + B(0, 2\tau) \subset A_2$. (Recall that we work with the maximum norm in $\mathbb{R}^2$ so that our "balls" are in fact squares.) We will "blow up" K_1 and K_2 slightly to simplify the situation.

First step: Fix an $n \in \mathbb{N}$ so large that $1/n \leq \tau$. Let K_1' be the collection of all little squares $[i/n, (i+1)/n] \times [j/n, (j+1)/n]$ with $i, j \in \{-n, \ldots, n-1\}$ that intersect K_1. Define K_2' similarly.

Second step: We enlarge K_1' and K_2' further: $K_1'' := K_1' + B(0, \tau)$ and $K_2'' := K_2' + B(0, \tau)$. Then K_1'' and K_2'' are disjoint compact sets that are unions of tiny squares with sides parallel to the sides of Q and with $K_1 \subset K_1'' \subset A_1$ and $K_2 \subset K_2'' \subset A_2$.

We now construct a path that omits $K_1'' \cup K_2''$ (and therefore $K_1 \cup K_2$). To this end we consider the directed line L from R to U. Until S and after T it will not meet $K_1'' \cup K_2''$. If it omits this set also between S and T, we are done. Otherwise we modify this path as follows:

- Suppose that L meets K_1'' at a certain component C_1 of K_1''. Then surround C_1 in the clockwise direction without touching $K_1'' \cup K_2''$ until you arrive at L again. Continue the path on L towards R until R or until you arrive at the next obstacle.
  This will be possible since $K_1'' \subset A_1$.
- Proceed in a similar way if L meets a component C_2 of K_2''. This time, however, C_2 will have to be surrounded in the counter-clockwise direction.

Since there are at most finitely many components of $K_1'' \cup K_2''$, sooner or later one will arrive at U.

(A more rigorous proof could start with a result from function theory: if a compact subset L of $\mathbb{C}$ lies in an open set $O \subset \mathbb{C}$, then there is a cycle γ in $O \setminus L$ made up of line segments such that the winding number of x with respect to γ is one for every $x \in L$; see, e.g., [6, Proposition 1.1, Chapter VIII]. This should be applied with $L = K_i$ and $O = \{x \mid d(x, K_i) < \tau\}$ for $i = 1, 2$.) ∎

PROPOSITION 4.5. *Let $f_0, g_0 : Q \to \mathbb{R}$ be defined by $f_0(x,y) := x$ and $g_0(x,y) := y$. Whenever $f, g \in CQ$ are such that $\|f - f_0\|, \|g - g_0\| \le 1/9$, then multiplication is* not *locally open at (f,g).*

Proof. We fix any $f, g \in CQ$ with $\|f - f_0\|, \|g - g_0\| \le 1/9$, and we set $\Phi := \phi_{f,g}$. The map ϕ_{f_0,g_0} is the identity on Q so that $\|\Phi(q) - q\| \le 1/9$ for all q.

Now suppose that multiplication were locally open at (f,g). Choose $\eta > 0$ and K_1, K_2 for $\varepsilon_0 := 1/9$ according to (C_1) of Theorem 1.1: $\{q \mid \Phi(q) \in M_{\eta,2}\}$ is the disjoint union of K_1 and K_2, and $\Phi(q) \in M_{\eta,\varepsilon_0,1}$ (resp. $\Phi(q) \in M_{\eta,\varepsilon_0,2}$) for $q \in K_1$ (resp. $q \in K_2$). With $\varepsilon := 3\varepsilon_0$ we consider the sets A_1, A_2 introduced before Lemma 4.3.

We claim that K_1 lies in the interior of A_1. In fact, for $q \in K_1$ we have $f(q) \ge -\varepsilon_0$ and $g(q) \le \varepsilon_0$, and from $\|\Phi(q) - q\| \le \varepsilon_0$ we conclude that $f(q) \ge -2\varepsilon_0$ and $g(q) \le 2\varepsilon_0$. Similarly one proves that $K_2 \subset (A_2)^0$.

By the preceding lemma we find a continuous $\phi : [0,1] \to Q$ with $\phi(0) = R$, $\phi(1) = U$ and $\phi(t) \notin K_1 \cup K_2$ for all t. This means that no $\Phi \circ \phi(t)$ lies in $M_{\eta,2}$: recall that $\Phi^{-1}(M_{\eta,2}) = K_1 \cup K_2$. Therefore all $\Phi \circ \phi(t)$ belong to $M_{\eta,1}$.

Now $M_{\eta,1}$ is the disjoint union of $B_1 := \{(x,y) \mid (x,y) \in M_{\eta,1}, x, y \le 0\}$ and $B_2 := \{(x,y) \mid (x,y) \in M_{\eta,1}, x, y \ge 0\}$. Consequently, $[0,1]$ would be the disjoint union of two closed sets, $\Phi^{-1}(B_1)$ and $\Phi^{-1}(B_2)$, both nonempty: $R \in \Phi^{-1}(B_1)$ and $R \in \Phi^{-1}(B_2)$. This is not possible by the connectedness of $[0,1]$. ∎

We note that the existence of $f_0, g_0 \in C_{\mathbb{R}}Q$ for which multiplication is not locally open at any pair (f,g) in a suitable neighbourhood also follows from Komisarski's characterization (see [7]). Our proof, however, is more elementary (Komisarski's argument depends on the Hemmingsen lemma).

Quantitative results. It is also sometimes simple to derive quantitative results. Suppose that multiplication is locally open at (f,g) for some $f, g \in CK$; are there, for given $\varepsilon > 0$, estimates for $\delta > 0$ such that $B(fg,\delta) \subset B(f,\varepsilon)B(g,\varepsilon)$? As a typical result we show:

LEMMA 4.6. *Let $\varepsilon, \delta_0 > 0$ and suppose that there is a connected subset C of K with the following properties:*

(i) *$f(k)g(k) \ge -\delta_0$ for $k \in C$.*

(ii) *There are* $k_1, k_2 \in C$ *such that* $f(k_1), g(k_1) > \varepsilon$ *and* $f(k_2), g(k_2) < -\varepsilon$.

If $B(fg, \delta) \subset B(f, \varepsilon)B(g, \varepsilon)$ *for some positive* δ, *then* $\delta \leq \delta_0$.

Proof. Write $fg + \underline{\delta}$ as $(f + d_1)(g + d_2)$, where $d_1, d_2 \in CK$ are such that $\|d_1\|, \|d_2\| \leq \varepsilon$. Then $(f + d_1)(k_1) > 0$ and $(f + d_1)(k_2) < 0$, so that, since C is connected, there must exist a k_0 with $(f + d_1)(k_0) = 0$. This implies that $0 = (f + d_1)(k_0)(g + d_2)(k_0) = f(k_0)g(k_0) + \delta \geq -\delta_0 + \delta$, and so $\delta \leq \delta_0$. ∎

Complex-valued functions. It was sketched in [5] that multiplication is locally open at every pair (f, g) in $C_{\mathbb{C}}[0, 1]$, the space of *complex* continuous functions on $[0, 1]$. For arbitrary K the situation is more complicated; we will investigate it in a separate paper. Here we only note that there are examples where—in contrast to the case of $[0, 1]$—multiplication need not be locally open at every pair f, g:

PROPOSITION 4.7. *Let* $D = \{z \mid |z| \leq 1\}$ *be the complex unit disk. Define* $f, g : D \to \mathbb{C}$ *by* $f : z \mapsto z$ *and* $g : z \mapsto \overline{z}$. *Then pointwise multiplication on* $C_{\mathbb{C}}D$ *is not locally open at* (f, g).

Proof. Set $\varepsilon := 1/3$. We claim that $B(f, \varepsilon)B(g, \varepsilon)$ does not contain any $fg + \underline{\delta}$ with $\delta > 0$ such that fg is not in the interior of $B(f, \varepsilon)B(g, \varepsilon)$.

Let $d_1, d_2 : D \to \mathbb{C}$ be contiuous functions with $\|d_1\|, \|d_2\| \leq \varepsilon$, and fix $\delta > 0$. We have to show that $(f + d_1)(g + d_2)$ is different from $fg + \underline{\delta}$. The function $-d_1$ is a continuous map from D to D, so by the Brouwer fixed point theorem there exists a $z_0 \in D$ with $-d_1(z_0) = z_0$. This implies that $(f + d_1)(z_0) = 0$, and consequently $(f + d_1)(g + d_2)$ must differ from the strictly positive function $fg + \underline{\delta}$.

With a similar argument one can show that multiplication is not locally open at (f, g) where $f : z \mapsto z^n$ and $g : z \mapsto \overline{z}^n$ for an arbitrary $n \in \mathbb{N}$. ∎

5. Invitations for further research.

We have given, for real CK spaces, a characterization of those tuples at which multiplication is locally open. The following two problems should be studied next:

- Find a characterization for complex CK spaces.
- In [5] a characterization of those n-tuples $(f_1, \ldots, f_n)$ with $f_1, \ldots, f_n \in C[0, 1]$ was given at which multiplication is locally open. It would be interesting to generalize this theorem to the case of arbitrary real or complex CK-spaces.

Such results would surely be important steps towards understanding the obstructions that are responsible for the failure of local openness of multiplication at certain tuples in arbitrary Banach algebras.

References

[1] M. Balcerzak, E. Behrends and F. Strobin, *On certain uniformly open multilinear mappings*, Banach J. Math. 10 (2016), 482–494.

[2] M. Balcerzak, A. Majchrzycki and F. Strobin, *Multiplication in Banach spaces and multilinear continuous functionals*, preprint.

[3] M. Balcerzak, W. Wachowicz and W. Wilczyński, *Multiplying balls in the space of continuous functions on* $[0,1]$, Studia Math. 170 (2005), 203–209.

[4] E. Behrends, *Walk the dog, or: products of open balls in the space of continuous functions*, Funct. Approx. Comment. Math. 44 (2011), 153–164.

[5] E. Behrends, *Products of* n *open subsets in the space of continuous functions on* $[0,1]$, Studia Math. 204 (2011), 73–95.

[6] J. B. Conway, *Functions of One Complex Variable*, Grad. Texts in Math. 11, Springer, 1973.

[7] A. Komisarski, *A connection between multiplication in* $C(X)$ *and the dimension of* X, Fund. Math. 189 (2006), 149–154.

[8] A. Wachowicz, *Multiplying balls in* $C^{(N)}[0,1]$, Real Anal. Exchange 34 (2008/2009), 445–450.

Ehrhard Behrends
Mathematisches Institut
Freie Universität Berlin
Arnimallee 6
D-14195 Berlin, Germany
E-mail: behrends@math.fu-berlin.de