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HYPOTHESIS TESTING IN UNBALANCED TWO-FOLD NESTED RANDOM MODELS

Abstract. In many applications of linear random models to multilevel data, it is of interest to test whether the random effects variance components are zero. In this paper we propose approximate tests for testing significance of variance components in the unbalanced two-fold nested random model in the presence of non-normality. In the derivations of the asymptotic distributions of the test statistics, as an intermediate result, the explicit form of the asymptotic covariance matrix of the vector of mean squares in this model is obtained. We also study the influence of some special type of designs on the asymptotic covariance matrix and on the distribution of the proposed test statistics.

1. Introduction. In many medical, agricultural and biological studies multilevel data are collected. Multilevel data arise when there is hierarchical or clustered structure of the data. In [16], an experiment was described in which the relative effectiveness of two drugs with respect to some specified criterion was studied. The experiment was designed in such a way that drug 1 was to be administered to n patients from each of hospitals 1, 2, 3, while drug 2 was to be administered to n patients from hospitals 4, 5, 6. In this example, the hospital effects were nested under the drug factor. One can observe that, in the above example, each drug was tested in the same number of hospitals and in each hospital the drug was administered to the same number of patients, thus the experiment can be described by a balanced two-fold nested model. However, if in this example one assumes that drug 1 was administered to n_{11} patients from hospital 1 and n_{12} patients from

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hospital 2, while drug 2 was administered to n_{24}, n_{25} and n_{26} patients from hospitals 4, 5 and 6, respectively, then to describe the experiment one should use an unbalanced two-fold nested model. In general, if we denote by y_{ijk} the kth observation ($k = 1, \ldots, n_{ij}$) for the ith level of the nesting factor ($i = 1, \ldots, v$) measured in the jth level of the nested factor ($j = 1, \ldots, b_i$), then an unbalanced two-fold nested model can be written as

$$(1.1) \qquad\qquad y_{ijk} = \mu + \alpha_i + \beta_{ij} + e_{ijk},$$

where μ denotes the general mean, α_i is the nesting effect, β_{ij} is the nested effect, and e_{ijk} is a random error. In this paper, both α_i and β_{ij} are assumed to be random. Usually, it is assumed that $\alpha_i \sim N(0, \sigma_a^2)$, $\beta_{ij} \sim N(0, \sigma_b^2)$ and $e_{ijk} \sim N(0, \sigma_e^2)$. However, in practice the normality assumption is often violated. For this reason, in this paper we will assume that α_i, β_{ij} and e_{ijk} are independent random variables with zero mean and variances σ_a^2, σ_b^2 and σ_e^2, respectively.

For inference on variance components, many testing procedures have been proposed in the literature (see e.g. [6, 10, 13]). In recent years, many tests concerning the significance of variance components in balanced and unbalanced linear random models in the presence of non-normality were constructed by using joint distribution of the mean squares. In [1], for balanced data, Arkitras and Arnold showed that the vector of mean squares in a one-way model and a two-way model converges to a multivariate normal distribution with mean vector zero and an appropriate covariance matrix. Using this result they obtained the asymptotic distribution for the F-test in those two models. A similar result was obtained by Güven [8] for an unbalanced two-way random model without interaction in the presence of non-normality. Using the same approach, Güven [7] constructed an asymptotic F-test in a one-fold nested model. For an unbalanced two-fold nested model, in the normal case, some test procedures for inference on variance components have been presented in [10]; however, for non-normal unbalanced two-fold nested models no such test procedures have been proposed.

For this reason, in this paper we propose approximate tests for the hypotheses $H_0 : \sigma_b^2 = 0$ and $H_0 : \sigma_a^2 = 0$ in model (1.1). Using the Main Theorem in [14] and the delta method we obtain the asymptotic distributions of the test statistics. Additionally, as an intermediate result, we obtain the explicit form of the asymptotic covariance matrix of the vector of mean squares in model (1.1). Furthermore, we will study the influence of some special type of designs on the asymptotic covariance matrix and the distribution of the test statistics.

This paper is organized as follows. In Section 2 we introduce some notation which will be used throughout the paper. Section 3 presents the model, assumptions and some auxilary results. In Section 4 the main results are

established. In Section 5, we study the influence of some special type of designs on the asymptotic covariance matrix and asymptotic distribution of the F-statistics. Simulation studies are given in Section 6. In Section 7 we provide a discussion. The proofs are postponed until Section 8.

2. Notation. In this section we fix the notation which will be used in the subsequent sections. Throughout the paper, we will denote matrices and vectors by bold letters $\mathbf{A}$ and $\boldsymbol{a}$. The notations $\mathbf{J}_n$ and $\mathbf{1}_n$ stand respectively for the $n \times n$ matrix and $n \times 1$ vector of ones. $\bigoplus$ is the symbol of direct sum of matrices, i.e. for given square matrices $\mathbf{A}_i$ $(i = 1, \ldots, m)$, $\bigoplus_{i=1}^{m} \mathbf{A}_i = \mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$.

$E(\cdot)$, $\mathrm{Cov}(\cdot, \cdot)$ mean expected value and covariance, respectively. Depending on the context, $\mathrm{Var}(\cdot)$ will denote either the variance of a random variable or the covariance matrix of a random vector.

Let X be a zero-mean random variable. Then k_x will denote the kurtosis parameter for the distribution of X, defined by $k_x = E(X^4)/\sigma^4 - 3$.

Let $\{X_n\}$ be a sequence of random variables. Then

$$X_n = \mathrm{o}_p(1) \iff X_n \xrightarrow{p} 0, \quad \text{as } n \to \infty.$$

3. Model and assumptions

3.1. Statistical model. Let us define N, $n_{i.}$ and $b_.$ as

$$N = \sum_{i=1}^{v} \sum_{j=1}^{b_i} n_{ij}, \quad n_{i.} = \sum_{j=1}^{b_i} n_{ij}, \quad b_. = \sum_{i=1}^{v} b_i.$$

Then in the matrix notation of [10], model (1.1) can be expressed as

(3.1) $$\boldsymbol{y} = \mu \mathbf{1}_N + \mathbf{Z}_1 \boldsymbol{\alpha} + \mathbf{Z}_2 \boldsymbol{\beta} + \boldsymbol{e},$$

where $\boldsymbol{y}$ is a vector of observations, μ denotes the general mean, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors of random effects, and $\boldsymbol{e}$ denotes a vector of random errors. We assume that the random vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{e}$ are independent. The matrices $\mathbf{Z}_1$ $(N \times v)$ and $\mathbf{Z}_2$ $(N \times b_.)$ are design matrices for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively, and are equal to

(3.2)
$$\mathbf{Z}_1 = \mathrm{diag}(\mathbf{1}_{n_{1.}}, \ldots, \mathbf{1}_{n_{v.}}),$$
$$\mathbf{Z}_2 = \mathrm{diag}(\mathbf{1}_{n_{11}}, \ldots, \mathbf{1}_{n_{1b_1}}, \mathbf{1}_{n_{21}}, \ldots, \mathbf{1}_{n_{2b_2}}, \ldots, \mathbf{1}_{n_{v1}}, \ldots \mathbf{1}_{n_{vb_v}}).$$

Without loss of generality one may assume that $n_{ij} \geq 1$ for all i and j, since if $n_{ij} = 0$ then the corresponding level of nested effect does not occur in the experiment and hence does not have to be taken into account.

Often in model (3.1) the following hypotheses are of interest: $H_{01} : \sigma_b^2 = 0$ and $H_{02} : \sigma_a^2 = 0$. Alternatively, they can be written in the following way:

$$H_{01}' : \rho_2 = 0 \quad \text{and} \quad H_{02}' : \rho_1 = 0,$$

where $\rho_1 = \sigma_a^2/\sigma_e^2$ and $\rho_2 = \sigma_b^2/\sigma_e^2$.

Usually to test H_{01} and H_{02} the test statistics $F_1 = MSB/MSE$ and $F_2 = MSA/MSB$ respectively are used, where $MSA = SSA/(v-1)$, $MSB = SSB/(b. - v)$ and $MSE = SSE/(N - b.)$. The sums of squares SSA, SSB and SSE are

$$(3.3) \qquad SSA = \boldsymbol{y}'\mathbf{P}_1\boldsymbol{y}, \quad SSB = \boldsymbol{y}'\mathbf{P}_2\boldsymbol{y}, \quad SSE = \boldsymbol{y}'\mathbf{P}_3\boldsymbol{y},$$

where $\mathbf{P}_i$ $(i = 1, 2, 3)$ are orthogonal projection matrices and can be written as (see [10])

$$(3.4) \qquad \mathbf{P}_1 = \bigoplus_{i=1}^{v} \frac{1}{n_{i.}}\mathbf{J}_{n_{i.}} - \frac{1}{N}\mathbf{J}_N, \quad \mathbf{P}_2 = \bigoplus_{i=1}^{v}\left[\bigoplus_{j=1}^{b_i} \frac{1}{n_{ij}}\mathbf{J}_{n_{ij}} - \frac{1}{n_{i.}}\mathbf{J}_{n_{i.}}\right],$$

$$\mathbf{P}_3 = \mathbf{I}_N - \bigoplus_{i=1}^{v}\bigoplus_{j=1}^{b_i} \frac{1}{n_{ij}}\mathbf{J}_{n_{ij}}.$$

3.2. Assumptions and some useful results. In this subsection we give some assumptions and some general results needed in the subsequent sections.

The following assumptions will be needed throughout the paper.

(A1) For every $\varepsilon > 0$, we have $E|\alpha_i|^{4+\varepsilon} < \infty$, $E|\beta_{ij}|^{4+\varepsilon} < \infty$ and $E|e_{ijk}|^{4+\varepsilon} < \infty$ $(i = 1, \ldots, v; j = 1, \ldots, b_i; k = 1, \ldots, n_{ij})$.

(A2) The number v of treatments tends to infinity in such a way that the n_{ij}'s take values from a finite set of distinct positive integers.

(A3) Define $l_v(t)$ and $m_v(t)$ to be the number of occurrences of m_t in $(n_1, \ldots, n_{v.})$ and of r_l in $(\sum_j^{b_1} n_{1j}^{-1}, \ldots, \sum_j^{b_v} n_{vj}^{-1})$, respectively. Then $l_v(t)/v \to p_t$ and $m_v(t)/v \to q_t$.

(A4) There exists $M > 1$ such that $1 \le b_i \le M$ for each $1 \le i \le v$.

REMARK 3.1. As pointed out in [8], assumption (A2) implies that both $n_{i.}$ and $\sum_j^{b_i} n_{ij}^{-1}$ $(i = 1, \ldots, v)$ take values from finite discrete sets of positive numbers. Hence, one may assume that $n_{i.}$, $i = 1, \ldots, v$, take values from a set

$$(3.5) \qquad S_1 = \{m_1, \ldots, m_l\},$$

and $\sum_{j=1}^{b_i} n_{ij}^{-1}$, $i = 1, \ldots, v$, take values from

$$(3.6) \qquad S_2 = \{r_1, \ldots, r_l\}.$$

In the proofs of the main results the following lemmas will be needed. The first two are modifications of Lemmas 3.1 an 3.2 in [8] to the setting of this paper, and can be proved using similar arguments.

LEMMA 3.2. *Let*

$$\psi_k = \lim_{v \to \infty} \frac{1}{v} \sum_{i=1}^{v} n_{i\cdot}^k, \quad k = \mp 1, \mp 2, \quad \omega = \lim_{v \to \infty} \frac{1}{v} \sum_{i=1}^{v} \sum_{j=1}^{b_i} \frac{1}{n_{ij}}.$$

Then under assumptions (A2)–(A4),

$$\psi_k = \sum_{t=1}^{l} m_t^k p_t \quad and \quad \omega = \sum_{t=1}^{l} r_t^k q_t,$$

where l is the number of different elements of the sets S_1 and S_2 defined in (3.5) *and* (3.6).

LEMMA 3.3. *The number v of levels of the nesting factor and the number N of observations are of the same order, and the limits*

$$\zeta = \lim_{N \to \infty} \frac{v}{N}, \quad \gamma_k = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{v} n_{i\cdot}^k \quad for \ k = \mp 1, \mp 2,$$

$$\eta = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{v} \sum_{j=1}^{b_i} \frac{1}{n_{ij}}$$

are positive and finite.

LEMMA 3.4. *Under assumptions* (A2)–(A4) *the limits*

$$\varphi = \lim_{N \to \infty} \frac{b_{\cdot}}{N}, \quad \delta_{k2} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{v} \sum_{j=1}^{b_i} \frac{n_{ij}^2}{n_{i\cdot}^k} \quad for \ k = 0, 1, 2,$$

$$\delta_{l,l+2} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{v} \sum_{j=1}^{b_i} \frac{n_{ij}^{l+2}}{n_{i\cdot}^l} \quad for \ l = 1, 2,$$

$$\vartheta = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{v} \frac{1}{n_{i\cdot}^2} \Big(\sum_{j=1}^{b_i} n_{ij}^2\Big)^2, \quad \delta = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{v} \frac{b_{i\cdot}}{n_{i\cdot}}$$

are finite and positive.

REMARK 3.5. In practice the limits in Lemmas 3.2–3.4 are unknown. Thus, they should be replaced by their finite sample counterparts.

4. Main results. To derive the asymptotic distributions of the test statistics F_1 and F_2 (defined in Section 3.1) we will use the delta method. We first find the asymptotic covariance and the joint distribution of $\sqrt{N}\,[MSA, MSB, MSE]'$.

Let $\boldsymbol{\theta} = [MSA, MSB, MSE]'$.

THEOREM 4.1. *Under assumptions* (A1)–(A4), *the asymptotic covariance matrix of* $\sqrt{N}\,\boldsymbol{\theta}$ *is equal to*

$$\boldsymbol{\Sigma} = \sigma_e^4(\boldsymbol{\Psi}_1 + k_a\rho_1^2\boldsymbol{\Psi}_2 + k_b\rho_2^2\boldsymbol{\Psi}_3 + k_e\boldsymbol{\Psi}_4),$$

where

$$\boldsymbol{\Psi}_1 = \begin{bmatrix} \frac{2(\rho_1^2\gamma_2 + 2\rho_1\rho_2\delta_{02} + 2\rho_1 + 2\rho_2\delta_{12} + \rho_2^2\vartheta + \zeta)}{\zeta^2} & \frac{2\rho_2^2(\delta_{13} - \vartheta)}{\zeta(\varphi - \zeta)} & 0 \\ \frac{2\rho_2^2(\delta_{13} - \vartheta)}{\zeta(\varphi - \zeta)} & \frac{2(\rho_2^2(\delta_{02} - 2\delta_{13} + \vartheta) + 2\rho_2(1 - \delta_{12}) + \varphi - \zeta)}{(\varphi - \zeta)^2} & 0 \\ 0 & 0 & \frac{2}{1-\varphi} \end{bmatrix},$$

$$\boldsymbol{\Psi}_2 = \begin{bmatrix} \frac{\gamma_2}{\zeta^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\Psi}_3 = \begin{bmatrix} \frac{\delta_{24}}{\zeta^2} & \frac{\delta_{13} - \delta_{24}}{\zeta(\varphi - \zeta)} & 0 \\ \frac{\delta_{13} - \delta_{24}}{\zeta(\varphi - \zeta)} & \frac{\delta_{02} - 2\delta_{13} + \delta_{24}}{(\varphi - \zeta)^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{\Psi}_4 = \begin{bmatrix} \frac{\gamma_{-1}}{\zeta^2} & \frac{\delta - \gamma_{-1}}{\zeta(\varphi - \zeta)} & \frac{\varphi - \delta}{\zeta(1 - \varphi)} \\ \frac{\delta - \gamma_{-1}}{\zeta(\varphi - \zeta)} & \frac{\eta + \gamma_{-1} - 2\delta}{(\varphi - \zeta)^2} & \frac{\varphi - \eta - \zeta + \delta}{(1 - \varphi)(\varphi - \zeta)} \\ \frac{\varphi - \delta}{\zeta(1 - \varphi)} & \frac{\varphi - \eta - \zeta + \delta}{(1 - \varphi)(\varphi - \zeta)} & \frac{1 - 2\varphi + \eta}{(1 - \varphi)^2} \end{bmatrix}.$$

PROPOSITION 4.2. *Under assumptions* (A1)–(A4),

$$\sqrt{N}(\boldsymbol{\theta} - \boldsymbol{\Gamma}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\Gamma} = \left[\sigma_e^2(\rho_1\zeta^{-1} + \rho_2\delta_{12}\zeta^{-1} + 1), \sigma_e^2\left(1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}\right), \sigma_e^2\right]'.$

Combining Proposition 4.2 with the delta method (see e.g. [12]) we get the following:

THEOREM 4.3. *Suppose that assumptions* (A1)–(A4) *hold.*

(i) *We have*

$$\sqrt{N}\left(F_1 - \left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)\right) \xrightarrow{d} N(0, \sigma_{H_1}^2),$$

as $N \to \infty$, *where*

$$\sigma_{H_1}^2 = \frac{2\left(\rho_2^2(\delta_{02} - 2\delta_{13} + \vartheta) + 2\rho_2(1 - \delta_{12}) + \varphi - \zeta\right)}{(\varphi - \zeta)^2}$$

$$+ \frac{2}{1 - \varphi}\left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)^2 + k_b\rho_2^2\frac{\delta_{02} - 2\delta_{13} + \delta_{24}}{(\varphi - \zeta)^2}$$

$$+ k_e\left[\frac{\eta + \gamma_{-1} - 2\delta}{(\varphi - \zeta)^2} - \frac{2\left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)(\varphi - \eta - \zeta + \delta)}{(1 - \varphi)(\varphi - \zeta)}\right.$$

$$\left. + \frac{\left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)^2(1 - 2\varphi + \eta)}{(1 - \varphi)^2}\right].$$

(ii) *We have*

$$\sqrt{N}\left(F_2 - \left(1 + \frac{\rho_1\zeta^{-1} + \rho_2\delta_{12}\zeta^{-1} - \rho_2(1-\delta_{12})(\varphi-\zeta)^{-1}}{1 + \rho_2(1-\delta_{12})(\varphi-\zeta)^{-1}}\right)\right)$$
$$\xrightarrow{d} N(0, \sigma^2_{H_2}),$$

as $N \to \infty$, where

$$\sigma^2_{H_2} = a\left[\frac{2(\rho_1^2\gamma_2 + 2\rho_1\rho_2\delta_{02} + \rho_2^2\vartheta + \zeta)}{\zeta^2} - \frac{4d\rho_2^2(\delta_{13} - \vartheta)}{c\zeta}\right.$$
$$+ \frac{2d^2\left(\rho_2^2(\delta_{02} - 2\delta_{13} + \vartheta) + 2(1-\delta_{12}) + (\varphi-\zeta)\right)}{c^2}$$
$$+ k_a\rho_1^2\frac{\gamma_2}{\zeta^2} + k_b\rho_2^2\left(\frac{\delta_{24}}{\zeta^2} - \frac{2d(\delta_{13} - \delta_{24})}{c\zeta} + \frac{d^2(\delta_{02} - 2\delta_{13} + \delta_{24})}{c^2}\right)$$
$$\left.+ k_e\left(\frac{\gamma_{-1}}{\zeta^2} - \frac{2d(\delta - \gamma_{-1})}{c\zeta} + \frac{d^2(\eta + \gamma_{-1} - 2\delta)}{c^2}\right)\right]$$

and $c = \varphi - \zeta + \rho_2(1-\delta_{12})$, $a = (\varphi-\zeta)^2 c^{-2}$, $d = 1 + \rho_1\zeta^{-1} + \delta_{12}\rho_2\zeta^{-1}$.

COROLLARY 4.4. *Assume that the assumptions of Theorem 4.3 are satisfied.*

(i) *Under $H_{01} : \rho_2 = 0$,*

$$\sqrt{N}(F_1 - 1) \xrightarrow{d} N(0, \sigma^2_{H_{01}}),$$

where

$$\sigma^2_{H_{01}} = \frac{2}{\varphi - \zeta} + \frac{2}{1 - \varphi}$$
$$+ k_e\left(\frac{\eta + \gamma_{-1} - 2\delta}{(\varphi-\zeta)^2} - \frac{2(\varphi - \eta - \zeta + \delta)}{(1-\varphi)(\varphi-\zeta)} + \frac{1 - 2\varphi + \eta}{(1-\varphi)^2}\right).$$

(ii) *Under $H_{02} : \rho_1 = 0$,*

$$\sqrt{N}\left(F_2 - \left(1 + \frac{\rho_2\delta_{12}\zeta^{-1} - \rho_2(1-\delta_{12})(\varphi-\zeta)^{-1}}{1 + \rho_2(1-\delta_{12})(\varphi-\zeta)^{-1}}\right)\right) \xrightarrow{d} N(0, \sigma^2_{H_{02}}),$$

where

$$\sigma^2_{H_{02}} = a\left[\frac{2(\rho_2^2\vartheta + \zeta)}{\zeta^2} - \frac{4d\rho_2^2(\delta_{13} - \vartheta)}{c\zeta}\right.$$
$$+ \frac{2d^2\left(\rho_2^2(\delta_{02} - 2\delta_{13} + \vartheta) + 2(1-\delta_{12}) + (\varphi-\zeta)\right)}{c^2}$$
$$+ k_b\rho_2^2\left(\frac{\delta_{24}}{\zeta^2} - \frac{2d(\delta_{13} - \delta_{24})}{c\zeta} + \frac{d^2(\delta_{02} - 2\delta_{13} + \delta_{24})}{c^2}\right)$$
$$\left.+ k_e\left(\frac{\gamma_{-1}}{\zeta^2} - \frac{2d(\delta - \gamma_{-1})}{c\zeta} + \frac{d^2(\eta + \gamma_{-1} - 2\delta)}{c^2}\right)\right],$$

a and c are defined in Theorem 4.3, and $d = 1 + \rho_2\delta_{12}\zeta^{-1}$.

5. Special case. As we have seen in Section 4, the correlations between the mean squares in the unbalanced two-fold nested random model depend mainly on kurtoses, and most of the off-diagonal elements of $\mathbf{\Psi}_1$ are zero. However, when one looks at the correlation between MSA and MSB one can observe that $\Psi_{1,12} \neq 0$. A similar observation has been made by Cummings and Gaylor [4] (see also [10, Example 3.5.1]) for unbalanced linear mixed models under normality.

However in the normal case, for some designs the mean squares MSA and MSB are uncorrelated. Cummings and Gaylor [4] described several designs and studied their impact on the correlation between MSA and MSB as well as on the distribution of the sum of squares. In one of the designs they assumed that for a given i ($i = 1, \ldots, v$) and $j \neq j'$ one has $n_{ij} = n_{ij'}$. In [10], this type of design is called a *partially balanced design* (*PBD*).

In the same example, Cummings and Gaylor pointed out that for a partially balanced design, MSA and MSB are independent under normality. In our case, for a PBD, from Lemma 3.4 it follows that $\delta_{13} = \vartheta$ and the asymptotic covariance matrix $\mathbf{\Sigma}$ given Theorem 4.1 can be written as:

COROLLARY 5.1. *Suppose that the assumptions of Theorem* 4.1 *hold. Additionally, let $n_{ij} = n_{ij'}$ for $i = 1, \ldots, v$ and $j, j' = 1, \ldots, b_i$, $j \neq j'$. Then the asymptotic covariance matrix of $\sqrt{N}\,\boldsymbol{\theta}$ is equal to*

$$\mathbf{\Sigma} = \sigma_e^4(\widetilde{\mathbf{\Psi}}_1 + k_a \rho_1^2 \widetilde{\mathbf{\Psi}}_2 + k_b \rho_2^2 \widetilde{\mathbf{\Psi}}_3 + k_e \widetilde{\mathbf{\Psi}}_4),$$

where

$$\widetilde{\mathbf{\Psi}}_1 = \begin{bmatrix} \frac{2(\rho_1^2 \gamma_2 + 2\rho_1 \rho_2 \delta_{02} + 2\rho_1 + 2\rho_2 \delta_{12} + \rho_2^2 \vartheta + \zeta)}{\zeta^2} & 0 & 0 \\ 0 & \frac{2(\rho_2^2(\delta_{02} - \delta_{13}) + 2\rho_2(1 - \delta_{12}) + \varphi - \zeta)}{(\varphi - \zeta)^2} & 0 \\ 0 & 0 & \frac{2}{1 - \varphi} \end{bmatrix},$$

and $\widetilde{\mathbf{\Psi}}_2$, $\widetilde{\mathbf{\Psi}}_3$ and $\widetilde{\mathbf{\Psi}}_4$ are as in Theorem 4.1.

If, in Theorem 4.3, we assume that the experiment was organized as a PBD, then we get the following:

COROLLARY 5.2. *Suppose that the assumptions of Theorem* 4.1 *hold. Additionally, let $n_{ij} = n_{ij'}$ for $i = 1, \ldots, v$ and $j, j' = 1, \ldots, b_i$, $j \neq j'$.*

(i) *We have*

$$\sqrt{N}\left(F_1 - \left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)\right) \xrightarrow{d} N(0, \sigma_{H_1}^2),$$

as $N \to \infty$, where

$$\sigma^2_{H_1} = \frac{2\big(\rho_2^2(\delta_{02} - \delta_{13}) + 2\rho_2(1 - \delta_{12}) + \varphi - \zeta\big)}{(\varphi - \zeta)^2}$$

$$+ \frac{2}{1 - \varphi}\left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)^2 + k_b\rho_2^2\frac{\delta_{02} - 2\delta_{13} + \delta_{24}}{(\varphi - \zeta)^2}$$

$$+ k_e\left[\frac{\eta + \gamma_{-1} - 2\delta}{(\varphi - \zeta)^2} - \frac{2\big(1 + \frac{\rho_2(1-\delta_{12})}{\varphi - \zeta}\big)(\varphi - \eta - \zeta + \delta)}{(1 - \varphi)(\varphi - \zeta)}\right.$$

$$\left. + \frac{\big(1 + \frac{\rho_2(1-\delta_{12})}{\varphi - \zeta}\big)^2(1 - 2\varphi + \eta)}{(1 - \varphi)^2}\right].$$

(ii) *We have*

$$\sqrt{N}\left(F_2 - \left(1 + \frac{\rho_1\zeta^{-1} + \rho_2\delta_{12}\zeta^{-1} - \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}}{1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}}\right)\right)$$

$$\xrightarrow{d} N(0, \sigma^2_{H_2}),$$

as $N \to \infty$, where

$$\sigma^2_{H_2} = a\left[\frac{2(\rho_1^2\gamma_2 + 2\rho_1\rho_2\delta_{02} + \rho_2^2\vartheta + \zeta)}{\zeta^2}\right.$$

$$+ \frac{2d^2\big(\rho_2^2(\delta_{02} - \delta_{13}) + 2(1 - \delta_{12}) + (\varphi - \zeta)\big)}{c^2}$$

$$+ k_a\rho_1^2\frac{\gamma_2}{\zeta^2} + k_b\rho_2^2\left(\frac{\delta_{24}}{\zeta^2} - \frac{2d(\delta_{13} - \delta_{24})}{c\zeta} + \frac{d^2(\delta_{02} - 2\delta_{13} + \delta_{24})}{c^2}\right)$$

$$+ k_e\left(\frac{\gamma_{-1}}{\zeta^2} - \frac{2d(\delta - \gamma_{-1})}{c\zeta} + \frac{d^2(\eta + \gamma_{-1} - 2\delta)}{c^2}\right)\right]$$

and a, c, d are defined in Theorem 4.3.

6. Simulation studies. In this section we will study the performance of the approximate tests described in Section 5.

In the simulation study we will consider a design similar to the one in [10, Example 3.5.1, Design 3]. Suppose the number n_{ij} of observations corresponding to the jth level of the nested factor within the ith level of the nesting factor has the following values: $n_{11} = n_{21} = n_{31} = n_{41} = 1$, $n_{22} = 8$, $n_{32} = 10$, $n_{42} = 5$, $n_{43} = 6$, $n_{44} = 7$. Thus, $N = 40$, $v = 4$, $b_1 = 1$, $b_2 = b_3 = 2$ and $b_4 = 4$. In the simulations, we consider four combinations of distributions for the random effects and errors:

    (i) Normal-Normal-Normal (N-N-N),
    (ii) N-N-CE, where CE represents centralized exponential distribution, i.e. $X - 1$, where $X \sim \text{Exp}(1)$,

(iii) CE-CE-CE,

(iv) SN-SN-SN, where SN represents the skew-normal distribution (for the definition see [2]) with the following parameters:

I $[1]'_4$ for $\boldsymbol{\alpha}$, where $[1]'_4 = [1, 1, 1, 1]'$, $[1]'_9$ for $\boldsymbol{\beta}$ and $[1]'_{40}$ for $\boldsymbol{e}$;

II $[-1, 1/2, 1/3, 1]'$, $[-1, -1/2, 0, 1, 1, 1, 1, 1, 1]'$ and $[1]'_{40}$;

III $[1/2, -1/2, 1/2, -1/2]$, $[1]'_9$ and $[0]'_{40}$.

In all cases as a scale matrix $\mathbf{K}$ we took the identity matrix of the appropriate order, i.e. $\mathbf{K} = \mathbf{I}_l$, $l = 4, 9, 40$.

The kurtoses for the distributions used in the simulations are given in Table 1, where the kurtoses for skew-normal distributions were calculated from the formulas in [2, Section 4].

Table 1. Kurtoses of the distributions

Kurtosis	N-N-N	N-N-CE	CE-CE-CE	SN-SN-SN		
				I	II	III
k_a	0	0	6	0.3050	0.1853	0.0617
k_b	0	0	6	0.5098	0.4290	0.5098
k_e	0	6	6	0.7609	0.7609	0

First, we test the hypothesis $H_0 : \sigma_b^2 = 0$, or equivalently $H_0 : \rho_2 = 0$. In the simulations we assumed that the true values in model (3.1) are $\mu = \sigma_a^2 = \sigma_e^2 = 1$.

Based on 10000 simulations, the estimated size of test $P(F_1 > u_\alpha \,|\, \rho_2 = 0)$ was calculated as the number of times F_1 exceeds the cut-off point u_α divided by 10000, where u_α was calculated using Corollary 4.4(i) and it is given by

$$u_\alpha = \frac{\sigma_{01}}{\sqrt{N}} z_\alpha + 1,$$

and z_α is the upper $1 - \alpha$ quantile of the standard normal distribution.

The estimated sizes of the test for the nominal levels $\alpha = 0.01, 0.05, 0.1$ are reported in Table 2.

Table 2. Estimated size

Nominal level	N-N-N	N-N-CE	CE-CE-CE	SN-SN-SN		
				I	II	III
$\alpha = 0.01$	0.0466	0.0402	0.0427	0.0373	0.0342	0.0451
$\alpha = 0.05$	0.0837	0.0614	0.0649	0.0764	0.0736	0.0896
$\alpha = 0.1$	0.1267	0.0913	0.0927	0.1095	0.1131	0.1256

Next, we look at the approximate powers for the following alternatives $\rho_2 = 0.1, 0.25, 0.5, 0.75, 1, 2, 4$ for two nominal levels $\alpha = 0.01, 0.05$. The

approximate powers were calculated using Theorem 4.3(i) as

$$P(F_1 > u_\alpha \,|\, \rho_2 > 0) \cong 1 - \Phi\left(\frac{u_\alpha - \left(1 + \frac{\rho_2(1-\delta_{12})}{\varphi - \zeta}\right)}{\sigma_1}\right)$$

and are reported in Tables 3 and 4.

Table 3. Approximate power (nominal level 0.01)

	N-N-N	N-N-CE	CE-CE-CE	SN-SN-SN		
				I	II	III
$\rho_2 = 0.1$	0.0470	0.0441	0.0468	0.0730	0.0762	0.0885
$\rho_2 = 0.25$	0.2821	0.1362	0.1676	0.2489	0.2491	0.2811
$\rho_2 = 0.5$	0.5189	0.3291	0.3568	0.4785	0.4771	0.5182
$\rho_2 = 0.75$	0.6390	0.4675	0.4758	0.6036	0.6045	0.6321
$\rho_2 = 1$	0.7105	0.5580	0.5427	0.6734	0.6759	0.6995
$\rho_2 = 2$	0.8080	0.7186	0.6531	0.7794	0.7813	0.7929
$\rho_2 = 4$	0.8536	0.7967	0.7094	0.8300	0.8321	0.8378

Table 4. Approximate power (nominal level 0.05)

	N-N-N	N-N-CE	CE-CE-CE	SN-SN-SN		
				I	II	III
$\rho_2 = 0.1$	0.1962	0.1240	0.1302	0.1776	0.1788	0.1972
$\rho_2 = 0.25$	0.4149	0.2639	0.2870	0.3810	0.3818	0.4201
$\rho_2 = 0.5$	0.6113	0.4506	0.4609	0.5791	0.5785	0.6084
$\rho_2 = 0.75$	0.7038	0.5660	0.5522	0.6730	0.6747	0.6954
$\rho_2 = 1$	0.7541	0.6478	0.6036	0.7241	0.7266	0.7412
$\rho_2 = 2$	0.8276	0.7557	0.6833	0.8026	0.8044	0.8130
$\rho_2 = 4$	0.8628	0.8143	0.7237	0.8405	0.8427	0.8473

Now, for the same design and combinations of distributions we will study the performance of the approximation test for $H_0 : \rho_1 = 0$. We assumed that the true values of parameters in model (3.1) were $\mu = \sigma_e^2 = 1$ and $\sigma_b^2 = 1, 4$. Once again, based on 10000 simulations we calculated the estimated size of the test, but now the cut-off point u_α calculated using Corollary 4.4(ii) and was equal to

$$u_\alpha = \frac{\sigma_{02}}{\sqrt{N}} z_\alpha + \left(1 + \frac{\rho_2 \delta_{12} \zeta^{-1} - \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}}{1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}}\right),$$

where z_α is the upper $1 - \alpha$ quantile of the standard normal distribution.

For the same nominal levels the estimated sizes of the test are reported in Table 5.

Table 5. Estimated size

Nominal level	N-N-N	N-N-CE	CE-CE-CE	SN-SN-SN		
				I	II	III
$\rho_2 = 1$						
$\alpha = 0.01$	0.1327	0.1273	0.0948	0.1123	0.1115	0.1135
$\alpha = 0.05$	0.1790	0.1815	0.1371	0.1538	0.1541	0.1599
$\alpha = 0.1$	0.2227	0.2162	0.1629	0.1905	0.1927	0.1925
$\rho_2 = 4$						
$\alpha = 0.01$	0.1114	0.1046	0.0842	0.1008	0.1050	0.1015
$\alpha = 0.05$	0.1529	0.1534	0.1213	0.1439	0.1445	0.1442
$\alpha = 0.1$	0.1862	0.1884	0.1497	0.1788	0.1795	0.1815

Because the estimated sizes of this test were rather far from the nominal levels, we do not give the results on the power of the test.

7. Discussion. In this paper we derived the explicit form of the asymptotic covariance matrix of the vector of mean squares in a non-normal unbalanced two-fold nested random model, and we constructed approximate procedures for testing the hypotheses $H_0 : \sigma_b^2 = 0$ and $H_0 : \sigma_a^2 = 0$. Furthermore, we studied the influence of some special type of designs on the asymptotic covariance matrix and on the distributions of the test statistics.

All results were obtained under the assumption that the number of levels of nesting effects goes to infinity. In the proofs of the main results, besides the standard assumptions, we only required the convergence of some partial sums of sample sizes. We used similar assumptions to the ones used in [8], which are equivalent to assumptions (i)–(iii) in [14]. In the proofs of the main results we did not require assumptions (iv) and (v) of [14], because they follow from assumptions (A2)–(A4).

Using the covariance matrix of the mean squares, we constructed approximate procedures for testing the hypotheses $H_0 : \sigma_a^2 = 0$ and $H_0 : \sigma_b^2 = 0$ in an unbalanced two-fold nested model in the presence of non-normality. We showed that the test statistics are asymptotically normal. One can see that the asymptotic variances of the test statistics (see Theorem 4.3 and Corollary 4.4) as well the covariance matrix of the vector of mean squares (Theorem 4.1) depend on the kurtoses k_a, k_b and k_e, which in some cases could be known from some previous studies. However, in general, these parameters are unknown. To overcome this problem, one may use a similar approach to that proposed in [9]. We plan to explore this problem in a future work.

To check the performance of the proposed test procedures we carried out simulation studies. As mentioned earlier, all results were developed when the

sample size N goes to infinity. In practice, one applies the asymptotic results when $N \geq 30$ (see [5, p. 58]). In our simulations, all calculations were carried out for $N = 40$.

In the simulation studies the approximate test for testing $H_0 : \sigma_b^2 = 0$ was performing well. For all combinations the estimated sizes of the test were close to the nominal levels. The best results in terms of estimated size were obtained when all random effects and errors follow an exponential distribution. The results obtained for SN-SN-SN I and II also had the estimated size close to the nominal levels, but in comparison to CE-CE-CE had higher approximate power for both nominal levels studied. N-N-N had slightly bigger values of approximate power for both nominal levels than the skew-normal combinations. This can be partly explained by the fact that for SN-SN-SN both I and II had kurtoses close to one (see Table 1).

The results for testing $H_0 : \sigma_a^2 = 0$ were not as satisfactory as those for $H_0 : \sigma_b^2 = 0$. The best results, in terms of the estimated size, were obtained for CE-CE-CE. This can be partly explained by high values of kurtoses of random effects. Based on the simulation studies, one can observe that for bigger values of the nuisance parameter ρ_2 the estimated sizes of the test are closer to the nominal levels. This can be explained by the fact that the distribution of the test statistics depends on the nuisance parameter. To eliminate the influence of this parameter one could use the concept of generalized p-value (see e.g. [10], [13]). A different method of solving this problem would be to construct other test statistics. We plan to explore this problem.

Because most of the experiments are organized according to some design, we also studied the influence of PBD designs on the asymptotic covariance matrix $\boldsymbol{\Sigma}$ and on the distribution of the test statistics. It was pointed out that under the PBD assumption, the matrix $\boldsymbol{\Psi}_1$ is diagonal, which is in agreement with the results obtained in [10, Example 3.5.1].

Finally, it should be noted that testing significance of variance components in random two-fold nested models is only one of several hypothesis testing problems for which further statistical research is needed. For example, the problem of testing the null hypothesis $H_0 : \sigma_a^2 = \sigma_b^2$ against the alternative $H_1 : \sigma_a^2 > \sigma_b^2$ should be investigated. Such a test would be very interesting for practitioners, e.g. for researchers working in agronomy.

8. Proofs. In this section we give the proofs of the statements formulated in Sections 3 and 4.

Proof of Lemma 3.4. We have

$$v \leq b. = \sum_{i=1}^{v} b_i \leq M \sum_{i=1}^{v} 1 = Mv \quad \text{(by (A3))}.$$

Thus, by Lemma 3.3 it follows that

$$\lim_{N\to\infty}\frac{b.}{N}\le M\zeta<\infty.$$

Since $\zeta>0$, one obtains

$$\lim_{N\to\infty}\frac{b.}{N}\ge\zeta>0.$$

Now, from $1\le n_{ij}\le n_{i\cdot}$ and (A3), for $k=0,1,2$ one has

$$\frac{1}{N}\sum_{i=1}^{v}\frac{1}{n_{i\cdot}^{k}}\le\frac{1}{N}\sum_{i=1}^{v}\sum_{j=1}^{b_i}\frac{n_{ij}^{2}}{n_{i\cdot}^{k}}\le\frac{1}{N}\sum_{i=1}^{v}\sum_{j=1}^{b_i}\frac{n_{i\cdot}^{2}}{n_{i\cdot}^{k}}$$

$$=\begin{cases}\dfrac{1}{N}\sum_{i=1}^{v}b_{i\cdot}n_{i\cdot}^{2}\le\dfrac{M}{N}\sum_{i=1}^{v}n_{i\cdot}^{2}, & k=0,\\[2mm]\dfrac{1}{N}\sum_{i=1}^{v}b_{i\cdot}n_{i\cdot}\le\dfrac{M}{N}\sum_{i=1}^{v}n_{i\cdot}, & k=1,\\[2mm]\dfrac{b.}{N}, & k=2.\end{cases}$$

Thus, finiteness of δ_{02} follows from $\gamma_2,\delta_{12}<\infty$ since $\gamma_1<\infty$, whereas finiteness of δ_{22} follows from $\varphi<\infty$. The positivity of δ_{k2}, $k=1,2$, follows from $\gamma_k>0$, $k=1,2$ (see Lemma 3.3), while positivity of δ_{02} follows from $\zeta>0$.

Using similar arguments one can show that δ and $\delta_{l,l+2}$ are positive finite.

Finally, using the fact that $1\le\sum_{j=1}^{b_i}n_{ij}^{2}\le n_{i\cdot}^{2}$ and assumption (A3) one obtains

$$\frac{1}{N}\sum_{i=1}^{v}n_{i\cdot}^{2}\le\frac{1}{N}\sum_{i=1}^{v}\frac{(\sum_{j=1}^{b_i}n_{ij}^{2})^{2}}{n_{i\cdot}^{2}}\le\frac{M}{N}\sum_{i=1}^{v}n_{i\cdot}^{2}.$$

Hence, by Lemma 3.3 it follows that ϑ is positive and finite. ∎

Proof of Theorem 4.1. Denote

$$\boldsymbol{\theta}=[MSA,MSB,MSE]',\quad\boldsymbol{\Theta}=[SSA,SSB,SSE]'.$$

Then,

$$\sqrt{N}\,\boldsymbol{\theta}=\boldsymbol{\Omega}_n N^{-1/2}\boldsymbol{\Theta},$$

and by basic properties of the covariance matrix (see e.g. [3])

$$\mathrm{Var}(\sqrt{N}\,\boldsymbol{\theta})=\boldsymbol{\Omega}_N\frac{1}{N}\,\mathrm{Var}(\boldsymbol{\Theta})\boldsymbol{\Omega}_N,$$

where

$$(8.1)\qquad\boldsymbol{\Omega}_N=\begin{bmatrix}\dfrac{N}{v-1} & 0 & 0\\[2mm] 0 & \dfrac{N}{b.-v} & 0\\[2mm] 0 & 0 & \dfrac{N}{N-b.}\end{bmatrix}.$$

By Lemmas 3.3 and 3.4,

(8.2)
$$\lim_{N\to\infty} \mathbf{\Omega}_N = \begin{bmatrix} \frac{1}{\zeta} & 0 & 0 \\ 0 & \frac{1}{\varphi-\zeta} & 0 \\ 0 & 0 & \frac{1}{1-\varphi} \end{bmatrix} = \mathbf{\Omega}.$$

Hence, to prove the theorem one has to calculate the asymptotic covariance matrix $\mathbf{\Delta}$ of $N^{-1/2}\mathbf{\Theta}$.

Using (3.1), the sums of squares given in (3.3) can be rewritten in terms of the random vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{e}$, as

$$SSA = [\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{e}'] \begin{bmatrix} \mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1 & \mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_2 & \mathbf{Z}_1'\mathbf{P}_1 \\ \mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_1 & \mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2 & \mathbf{Z}_2'\mathbf{P}_1 \\ \mathbf{P}_1\mathbf{Z}_1 & \mathbf{P}_1\mathbf{Z}_2 & \mathbf{P}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{e} \end{bmatrix},$$

(8.3)
$$SSB = [\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{e}'] \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2 & \mathbf{Z}_2'\mathbf{P}_2 \\ \mathbf{0} & \mathbf{P}_2\mathbf{Z}_2 & \mathbf{P}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{e} \end{bmatrix},$$

$$SSE = [\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{e}'] \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{e} \end{bmatrix},$$

where the matrices $\mathbf{Z}_i$ and $\mathbf{P}_i$, $i = 1, 2, 3$, are defined in (3.2) and (3.4).

Because $[\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{e}]'$ is a zero mean random vector, by [15, Lemma 1] one finds that the variances of SSA, SSB and SSE are

$$\begin{aligned} \mathrm{Var}(SSA) = \sigma_e^2 \big(&2\rho_1^2 \,\mathrm{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1)^2 + 4\rho_1\rho_2 \,\mathrm{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_2\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_1) \\ &+ 4\rho_1 \,\mathrm{tr}(\mathbf{Z}_1'\mathbf{P}_1^2\mathbf{Z}_1) + 2\rho_2^2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2)^2 + 4\rho_2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_1^2\mathbf{Z}_2) \\ &+ 2\,\mathrm{tr}(\mathbf{P}_1^2) + k_a\rho_1^2 \,\mathrm{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1 \,\mathrm{diag}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1)) \\ &+ k_e \,\mathrm{tr}(\mathbf{P}_1 \,\mathrm{diag}(\mathbf{P}_1)) + k_b\rho_2^2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2 \,\mathrm{diag}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2)) \big), \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(SSB) = \sigma_e^2 \big(&2\rho_2^2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2)^2 + 4\rho_2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_2^2\mathbf{Z}_2) + 2\sigma_e^4 \,\mathrm{tr}(\mathbf{P}_2^2) \\ &+ k_b\rho_2^2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2 \,\mathrm{diag}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2)) + k_e \,\mathrm{tr}(\mathbf{P}_2 \,\mathrm{diag}(\mathbf{P}_2)) \big), \end{aligned}$$

$$\mathrm{Var}(SSE) = \sigma_e^4 (2\,\mathrm{tr}(\mathbf{P}_3^2) + k_e \,\mathrm{tr}(\mathbf{P}_3 \,\mathrm{diag}(\mathbf{P}_3))),$$

and the covariances (between appropriate sum of squares) are

$$\begin{aligned} \mathrm{Cov}(SSA, SSB) = \sigma_e^4 \big(&2\rho_1^2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2) + 2\sigma_e^4 \,\mathrm{tr}(\mathbf{P}_1\mathbf{P}_2) \\ &+ 4\rho_2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{P}_2\mathbf{Z}_2) + k_b\rho_2^2 \,\mathrm{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2 \,\mathrm{diag}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2)) \\ &+ k_e \,\mathrm{tr}(\mathbf{P}_1 \,\mathrm{diag}(\mathbf{P}_2)) \big), \end{aligned}$$

$$\mathrm{Cov}(SSA, SSE) = \sigma_e^4 (2\,\mathrm{tr}(\mathbf{P}_1\mathbf{P}_3) + k_e \,\mathrm{tr}(\mathbf{P}_1 \,\mathrm{diag}(\mathbf{P}_3))),$$

$$\mathrm{Cov}(SSB, SSE) = \sigma_e^4 (2\,\mathrm{tr}(\mathbf{P}_2\mathbf{P}_3) + k_e \,\mathrm{tr}(\mathbf{P}_2 \,\mathrm{diag}(\mathbf{P}_3))).$$

Now, by (3.2) and (3.4) one can easily show that

$$(8.4) \qquad \mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1 = \operatorname{diag}(n_{1\cdot}, \ldots, n_{v\cdot}) - \frac{1}{N}[n_{1\cdot}, \ldots, n_{v\cdot}]'[n_{1\cdot}, \ldots, n_{v\cdot}],$$

$$(8.5) \qquad \mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2 = \bigoplus_{i=1}^{v} \frac{\boldsymbol{n}_{i\cdot}\boldsymbol{n}_{i\cdot}'}{n_{i\cdot}} - \frac{1}{N}\boldsymbol{n}\boldsymbol{n}',$$

where $\boldsymbol{n}_{i\cdot} = [n_{i1}, \ldots, n_{ib_i}]'$ and $\boldsymbol{n} = [n_{11}, \ldots, n_{1b_1}, \ldots, n_{v1}, \ldots, n_{vb_v}]'$.

Hence, by (8.4),

$$\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1)^2 = \sum_{i=1}^{v} n_{i\cdot}^2 - 2\frac{1}{N}\sum_{i=1}^{v} n_{i\cdot}^3 + \frac{1}{N^2}\sum_{i=1}^{v}\sum_{l=1}^{v} n_{i\cdot}^2 n_{l\cdot}^2,$$

$$\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1 \operatorname{diag}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1)) = \sum_{i=1}^{v} n_{i\cdot}^2\left(1 - \frac{1}{N}\right)^2,$$

Since the matrix $\mathbf{P}_1$ is idempotent, one has

$$\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1^2\mathbf{Z}_1) = \operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1) = \sum_{i=1}^{v}\left(n_{i\cdot} - \frac{n_{i\cdot}}{N}\right).$$

Thus, by Lemma 3.3,

$$(8.6) \qquad \begin{aligned} &\lim_{N\to\infty} \frac{1}{N}\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1)^2 = \gamma_2 = \lim_{N\to\infty} \frac{1}{N}\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1 \operatorname{diag}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1)), \\ &\lim_{N\to\infty} \frac{1}{N}\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1^2\mathbf{Z}_1) = \lim_{N\to\infty} \frac{1}{N}\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1) = 1. \end{aligned}$$

Using similar arguments, one can show that

$$\begin{aligned} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2)^2 &= \sum_{i=1}^{v} \frac{(\sum_{j=1}^{b_i} n_{ij}^2)^2}{n_{i\cdot}^2} + \frac{1}{N^2}\left(\sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij}^2\right)^2 - \frac{2}{N}\boldsymbol{n}'\left(\bigoplus_{i=1}^{v} \frac{\boldsymbol{n}_i\boldsymbol{n}_i'}{n_{i\cdot}}\right)\boldsymbol{n} \\ &= \sum_{i=1}^{v} \frac{(\sum_{j=1}^{b_i} n_{ij}^2)^2}{n_{i\cdot}^2} + \frac{1}{N^2}\left(\sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij}^2\right)^2 - \frac{2}{N}\sum_{i=1}^{v} \frac{(\sum_{i=1}^{b_i} n_{ij}^2)^2}{n_{\cdot}}. \end{aligned}$$

Moreover,

$$\operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2 \operatorname{diag}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2)) = \sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij}^4\left(\frac{1}{n_{i\cdot}} - \frac{1}{N}\right)^2,$$

$$\operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1^2\mathbf{Z}_2) = \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2) = \sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij}^2\left(\frac{1}{n_{i\cdot}} - \frac{1}{N}\right).$$

Thus, by Lemmas 3.3 and 3.4,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2)^2 = \vartheta,$$

(8.7)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2 \operatorname{diag}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2)) = \delta_{24},$$

$$\lim_{N\to\infty} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1^2\mathbf{Z}_2) = \lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2) = \delta_{12}.$$

Because the matrix $\mathbf{P}_1$ is idempotent, one has

$$\operatorname{tr}\mathbf{P}_1^2 = \operatorname{tr}\mathbf{P}_1 = \sum_{i=1}^{v} n_{i\cdot}\left(\frac{1}{n_{i\cdot}} - \frac{1}{N}\right) = v - 1.$$

Also, one can easily show that

$$\operatorname{tr}(\mathbf{P}_1 \operatorname{diag}(\mathbf{P}_1)) = \sum_{i=1}^{v} n_{i\cdot}\left(\frac{1}{n_{i\cdot}} - \frac{1}{N}\right)^2.$$

Hence,

(8.8)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}\mathbf{P}_1^2 = \zeta, \qquad \lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{P}_1 \operatorname{diag}(\mathbf{P}_1)) = \gamma_{-1}.$$

After some algebra using the properties of matrix trace (see [11]) it follows that

$$\operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_2\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_1) = \operatorname{tr}\left(\left(\bigoplus_{i=1}^{v}\bigoplus_{j=1}^{b_i}\mathbf{J}_{n_{ij}}\right)\mathbf{P}_1\left(\bigoplus_{i=1}^{v}\mathbf{J}_{n_{i\cdot}}\right)\mathbf{P}_1\right)$$

$$= \sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij}^2 - \frac{1}{N}\sum_{i=1}^{v} n_{i\cdot}^2,$$

and by Lemma 3.4,

(8.9)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_2\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_1) = \delta_{02}.$$

Thus by (8.6)–(8.9),

(8.10)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Var}(SSA) = \sigma_e^2(2\rho_1^2\gamma_2 + 4\rho_1\rho_2\delta_{02} + 4\rho_1 + 4\rho_2\delta_{12} + 2\sigma_b^4\vartheta$$

$$+ 2\sigma_e^4 + k_a\sigma_a^4\gamma_2 + k_b\sigma_b^4\delta_{24} + k_e\sigma_e^4\gamma_{-1}).$$

Using (3.2) and (3.4), one writes

(8.11)
$$\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2 = \operatorname{diag}(n_{11}, \ldots, n_{1b1}, \ldots, n_{v1}, \ldots, n_{vb_v}) - \bigoplus_{i=1}^{v} \frac{\boldsymbol{n}_{i\cdot}\boldsymbol{n}_{i\cdot}'}{n_{i\cdot}}.$$

Hence,

$$\operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2 \operatorname{diag}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2)) = \sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij}^2 \left(1 - \frac{n_{ij}}{n_{i\cdot}}\right)^2,$$

and after some algebra

$$\operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2)^2 = \sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij}^2 - 2\sum_{i=1}^{v}\sum_{j=1}^{b_i} \frac{n_{ij}^3}{n_{i\cdot}} + \sum_{i=1}^{v} \frac{(\sum_{j=1}^{b_i} n_{ij}^2)^2}{n_{i\cdot}^2}.$$

Hence, by Lemma 3.4,

(8.12)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2 \operatorname{diag}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2)) = \delta_{02} - 2\delta_{13} + \delta_{24},$$
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2)^2 = \delta_{02} - 2\delta_{13} + \vartheta.$$

The matrix $\mathbf{P}_2$ is idempotent, so $\operatorname{tr}\mathbf{P}_2^2 = \operatorname{tr}\mathbf{P}_2$, where $\operatorname{tr}\mathbf{P}_2 = b. - v$, and

$$\operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2^2\mathbf{Z}_2) = \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2) = \sum_{i=1}^{v}\sum_{j=1}^{b_i} \left(n_{ij} - \frac{n_{ij}^2}{n_{i\cdot}}\right).$$

Moreover,

$$\operatorname{tr}(\mathbf{P}_2 \operatorname{diag}(\mathbf{P}_2)) = \sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij} \left(\frac{1}{n_{ij}} - \frac{1}{n_{i\cdot}}\right).$$

This implies that

(8.13)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{P}_2^2) = \varphi - \zeta,$$
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{P}_2 \operatorname{diag}(\mathbf{P}_2)) = \eta - 2\delta + \gamma_{-1},$$
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2^2\mathbf{Z}_2) = \lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2) = 1 - \delta_{12}.$$

Combining (8.12) and (8.13) one gets

(8.14)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Var}(SSB) = \sigma_e^4 \big(2\rho_2^2(\delta_{02} - 2\delta_{13} + \vartheta) + 4\rho_2(1 - \delta_{12})$$
$$+ 2(\varphi - \zeta) + k_b\rho_2^2(\delta_{02} - 2\delta_{13} + \delta_{24}) + k_e(\eta - 2\delta + \gamma_{-1})\big).$$

The matrix $\mathbf{P}_3$ is idempotent, so

$$\operatorname{tr}(\mathbf{P}_3^2) = \operatorname{tr}(\mathbf{P}_3) = \sum_{i=1}^{v}\sum_{j=1}^{b_i} n_{ij} \left(1 - \frac{1}{n_{i\cdot}}\right) = N - b.,$$

and furthermore

$$\mathrm{tr}(\mathbf{P}_3 \, \mathrm{diag}(\mathbf{P}_3)) = \sum_{i=1}^{v} \sum_{j=1}^{b_i} n_{ij} \left(1 - \frac{1}{n_{ij}}\right)^2 = N - 2b. + \sum_{i=1}^{v} \sum_{j=1}^{b_i} \frac{1}{n_{ij}}.$$

This implies that

$$(8.15) \quad \lim_{N \to \infty} \frac{1}{N} \mathrm{tr}(\mathbf{P}_3^2) = 1 - \varphi, \qquad \lim_{N \to \infty} \frac{1}{N} \mathrm{tr}(\mathbf{P}_3 \, \mathrm{diag}(\mathbf{P}_3)) = 1 - 2\varphi + \eta.$$

Hence,

$$(8.16) \qquad \lim_{N \to \infty} \mathrm{Var}(SSE) = 2\sigma_e^4 (1 - \varphi) + k_e \sigma_e^4 (1 - 2\varphi + \eta).$$

Now, by the definition of $\mathbf{P}_i$, $i = 1, 2, 3$,

$$\mathrm{tr}(\mathbf{P}_1 \, \mathrm{diag}(\mathbf{P}_2)) = \sum_{i=1}^{v} \sum_{j=1}^{b_i} n_{ij} \left(\frac{1}{n_{i.}} - \frac{1}{N}\right)\left(\frac{1}{n_{ij}} - \frac{1}{n_{i.}}\right),$$

$$\mathrm{tr}(\mathbf{P}_1 \, \mathrm{diag}(\mathbf{P}_3)) = \sum_{i=1}^{v} \sum_{j=1}^{b_i} n_{ij} \left(\frac{1}{n_{i.}} - \frac{1}{N}\right)\left(1 - \frac{1}{n_{ij}}\right),$$

$$\mathrm{tr}(\mathbf{P}_2 \, \mathrm{diag}(\mathbf{P}_3)) = \sum_{i=1}^{v} \sum_{j=1}^{b_i} n_{ij} \left(\frac{1}{n_{ij}} - \frac{1}{n_{i.}}\right)\left(1 - \frac{1}{n_{ij}}\right).$$

This implies

$$(8.17) \quad \begin{aligned} \lim_{N \to \infty} \frac{1}{N} \mathrm{tr}(\mathbf{P}_1 \, \mathrm{diag}(\mathbf{P}_2)) &= \delta - \gamma_{-1}, \\ \lim_{N \to \infty} \frac{1}{N} \mathrm{tr}(\mathbf{P}_1 \, \mathrm{diag}(\mathbf{P}_3)) &= \zeta - \delta, \\ \lim_{N \to \infty} \frac{1}{N} \mathrm{tr}(\mathbf{P}_2 \, \mathrm{diag}(\mathbf{P}_3)) &= \varphi - \eta - \zeta + \delta. \end{aligned}$$

By (8.5) and (8.11) after some algebra one obtains

$$(8.18) \quad \begin{aligned} \mathrm{tr}(\mathbf{Z}_2' \mathbf{P}_1 \mathbf{Z}_2 \mathbf{Z}_2' \mathbf{P}_2 \mathbf{Z}_2) &= \sum_{i=1}^{v} \sum_{j=1}^{b_i} \frac{n_{ij}^3}{n_{i.}} - \sum_{i=1}^{v} \frac{(\sum_{j=1}^{b_i} n_{ij}^2)^2}{n_{i.}^2} \\ &\quad - \frac{1}{N} \sum_{i=1}^{v} \sum_{j=1}^{b_i} n_{ij}^3 + \frac{1}{N} \sum_{i=1}^{v} \frac{(\sum_{j=1}^{b_i} n_{ij}^2)^2}{n_{i.}}, \\ \mathrm{tr}(\mathbf{Z}_2' \mathbf{P}_1 \mathbf{Z}_2 \, \mathrm{diag}(\mathbf{Z}_2' \mathbf{P}_2 \mathbf{Z}_2)) &= \sum_{i=1}^{v} \sum_{j=1}^{b_i} n_{ij}^3 \left(1 - \frac{n_{ij}}{n_{i.}}\right)\left(\frac{1}{n_{i.}} - \frac{1}{N}\right). \end{aligned}$$

Because the matrices $\mathbf{P}_i$, $i = 1, 2, 3$, are pairwise orthogonal, i.e. $\mathbf{P}_i \mathbf{P}_j = \mathbf{0}$

for $i \neq j$, by (8.17) and (8.18) one obtains

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Cov}(SSA, SSB)$$
$$= \sigma_e^4\big(2\rho_2^2(\delta_{13} - \vartheta) + k_b\rho_2^2(\delta_{13} - \delta_{24}) + k_e(\delta - \gamma_1)\big),$$

(8.19)
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Cov}(SSA, SSE) = k_e\sigma_e^4(\zeta - \delta),$$

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Cov}(SSB, SSE) = k_e\sigma_e^4(\varphi - \eta - \zeta + \delta).$$

Thus, by (8.10), (8.14), (8.16) and (8.19) the asymptotic covariance matrix of $N^{-1/2}\boldsymbol{\Theta}$ is given by

(8.20)
$$\boldsymbol{\Delta} = \sigma_e^4(\boldsymbol{\Xi}_1 + \boldsymbol{\Xi}_2 + \boldsymbol{\Xi}_3 + \boldsymbol{\Xi}_4),$$

where

$$\boldsymbol{\Xi}_1 = \begin{bmatrix} 2(\rho_1^2\gamma_2+2\rho_1\rho_2\delta_{02}+2\rho_1+2\rho_2\delta_{12}+\rho_2^2\vartheta+\zeta) & 2\rho_2^2(\delta_{13}-\vartheta) & 0 \\ 2\rho_2^2(\delta_{13}-\vartheta) & 2(\rho_2^2(\delta_{02}-2\delta_{13}+\vartheta)+2\rho_2(1-\delta_{12})+\varphi-\zeta) & 0 \\ 0 & 0 & 2(1-\varphi) \end{bmatrix},$$

$$\boldsymbol{\Xi}_2 = \begin{bmatrix} \gamma_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Xi}_3 = \begin{bmatrix} \delta_{24} & \delta_{13}-\delta_{24} & 0 \\ \delta_{13}-\delta_{24} & \delta_{02}-2\delta_{13}+\delta_{24} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{\Xi}_4 = \begin{bmatrix} \gamma_{-1} & \delta-\gamma_{-1} & \varphi-\delta \\ \delta-\gamma_{-1} & \eta+\gamma_{-1}-2\delta & \varphi-\eta-\zeta+\delta \\ \varphi-\delta & \varphi-\eta-\zeta+\delta & 1-2\varphi+\eta \end{bmatrix}.$$

This completes the proof of Theorem 4.1. ∎

Proof of Proposition 4.2. First, we will show that

(8.21)
$$\sqrt{N}(\boldsymbol{\theta} - E(\boldsymbol{\theta})) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma}) \quad \text{as } N \to \infty.$$

In view of the Main Theorem in [14], in order to prove (8.21) it suffices to show that the assumptions of that theorem are satisfied. Note that assumptions (A2)–(A4) are equivalent to (i)–(iii) in [14]. Thus, one only has to show that assumptions (iv) and (v) are satisfied.

By the proof of Theorem 4.1 (see (8.20)), under (A1)–(A4),

$$\frac{1}{N} \operatorname{Var}(\boldsymbol{\Theta}) \to \boldsymbol{\Delta} \quad \text{as } N \to \infty.$$

Hence, condition (v) of the Main Theorem is satisfied.

Furthermore, by (A2)–(A4), $\boldsymbol{\Omega}_N \to \boldsymbol{\Omega}$ as $N \to \infty$ (see the proof o Theorem 4.1). Thus, taking $F_N = \boldsymbol{\Omega}_N^{-1}$ and $F = \boldsymbol{\Omega}^{-1}$ in assumption (iv) in [14],

one obtains

$$\sqrt{N}(\boldsymbol{\theta} - E(\boldsymbol{\theta})) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Omega}\boldsymbol{\Delta}\boldsymbol{\Omega}') \quad \text{as } N \to \infty.$$

By Theorem 4.1 one gets the claim.

Now, after some algebra using the identity $E(\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}) = \text{tr}(\boldsymbol{A}\boldsymbol{\Lambda}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$, we have

$$E(\boldsymbol{\theta}_N) = \sigma_e^2 \begin{bmatrix} \frac{1}{v-1}(\rho_1 \text{tr}(\mathbf{Z}_1'\mathbf{P}_1\mathbf{Z}_1) + \rho_2 \text{tr}(\mathbf{Z}_2'\mathbf{P}_1\mathbf{Z}_2) + \text{tr}(\mathbf{P}_1)) \\ \frac{1}{b.-v}(\rho_2 \text{tr}(\mathbf{Z}_2'\mathbf{P}_2\mathbf{Z}_2) + \text{tr}(\mathbf{P}_2)) \\ \frac{1}{N-b.} \text{tr}(\mathbf{P}_3) \end{bmatrix}.$$

Then, by (8.6)–(8.9), (8.13) and (8.15) it follows that

$$(8.22) \qquad E(\boldsymbol{\theta}_N) \to \boldsymbol{\Gamma} \quad \text{as } N \to \infty.$$

Combining (8.21) and (8.22) we get the assertion. ∎

Proof of Theorem 4.3. (i) Note that $F_1 = MSB/MSE$ is a function of MSA, MSB and MSE. Thus, differentiating F_1 with respect to MSA, MSB and MSE at

$$(8.23) \qquad \left(\sigma_e^2(\rho_1\zeta^{-1} + \rho_2\delta_{12}\zeta^{-1} + 1), \sigma_e^2(1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}), \sigma_e^2\right),$$

one finds that the vector ∇F_1 of first derivatives is equal to

$$\nabla F_1 = \left[0, \frac{1}{\sigma_e^2}, -\frac{1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}}{\sigma_e^2}\right]'.$$

Hence, by using the delta method (see e.g. [12]) one has

$$\sqrt{N}\left(F_1 - \left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)\right) = \sqrt{N}(\nabla F_1)'(\boldsymbol{\theta} - \boldsymbol{\Gamma}) + o_P(1).$$

Thus, by [3, Theorem 4.2 and Proposition 5.2],

$$\sqrt{N}\left(F_1 - \left(1 + \frac{\rho_2(1 - \delta_{12})}{\varphi - \zeta}\right)\right) \xrightarrow{d} N(0, \sigma_{H_1}^2),$$

where $\sigma_{H_1}^2 = (\nabla F_1)'\boldsymbol{\Sigma}(\nabla F_1)$. After some algebra one gets (i).

(ii) Using similar arguments to those in (i) one finds that

$$\sqrt{N}(F_2 - \widetilde{F}) = \sqrt{N}(\nabla F_2)'(\boldsymbol{\theta} - \boldsymbol{\Gamma}) + o_P(1),$$

where $\widetilde{F}$ and ∇F_2 are the value of F_2 and the vector of first derivatives of F_2 calculated at (8.23), and are equal to

$$\widetilde{F} = 1 + \frac{\rho_1\zeta^{-1} + \rho_2\delta_{12}\zeta^{-1} - \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}}{1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1}},$$

$$\nabla F_2 = \left[\frac{1}{\sigma_e^2(1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1})}, \frac{-(\rho_1\zeta^{-1} + \rho_2\delta_{12}\zeta^{-1} + 1)}{\sigma_e^2(1 + \rho_2(1 - \delta_{12})(\varphi - \zeta)^{-1})^2}, 0\right]'.$$

Hence, by Theorem 4.2, $\sqrt{N}(F_2 - \widetilde{F})$ is asymptotically normal with variance $\sigma_{H_2}^2 = (\nabla F_2)' \boldsymbol{\Sigma} (\nabla F_2)$. After straightforward computations one gets (ii). ∎

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