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Some remarks on generalised lush spaces

by

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Abstract. D. Tan, X. Huang and R. Liu [Studia Math. 219 (2013)] recently introduced the notion of generalised lush (GL) spaces, which, at least for separable spaces, is a generalisation of the concept of lushness introduced by Boyko et al. [Math. Proc. Cambridge Philos. Soc. 142 (2007)]. The main result of D. Tan et al. is that every GL-space has the so called Mazur–Ulam property (MUP).

In this note, we prove some further properties of GL-spaces, for example, every M-ideal in a GL-space is again a GL-space, ultraproducts of GL-spaces are again GL-spaces, and if the bidual X^{**} of a Banach space X is GL, then X itself has the MUP.

1. Introduction. Our notation is as follows: if not otherwise stated, X denotes a real Banach space, X^* its dual, B_X its closed unit ball and S_X its unit sphere. For a subset A of X, we denote by $\overline{A}$ its norm-closure and by $\operatorname{co} A$ resp. $\operatorname{aco} A$ its convex resp. absolutely convex hull. By $\mathrm{d}(x, A)$ we denote the distance from a point $x \in X$ to the set A. For any functional $x^* \in S_{X^*}$ and $\varepsilon > 0$ let $S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1 - \varepsilon\}$ be the slice of B_X induced by x^* and ε.

We begin by recalling the classical Mazur–Ulam theorem (see [21]), which states that every bijective isometry T between two real normed spaces X and Y must be affine, i.e. $T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y)$ for all $x, y \in X$ and $\lambda \in [0, 1]$ (equivalently, $T - T(0)$ is linear). A simplified proof of this theorem was given in [33]. See also the recent paper [25] for an even more simplified argument.

In 1972, Mankiewicz [19] proved the following generalisation of the Mazur–Ulam theorem: if $A \subseteq X$ and $B \subseteq Y$ are convex with non-empty interior or open and connected, then every bijective isometry $T : A \to B$ can be extended to a bijective affine isometry $\tilde{T} : X \to Y$. This implies in particular that every bijective isometry from B_X onto B_Y is the restriction of a linear

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isometry from X onto Y. Tingley [32] asked whether the same is true if one replaces the unit balls of X and Y by their respective unit spheres. As a first step towards solving this problem, he proved that for X and Y finite-dimensional, every bijective isometry $T : S_X \to S_Y$ satisfies $T(-x) = -T(x)$ for all $x \in S_X$.

Though Tingley's problem remains open even in two dimensions, affirmative answers have been obtained for many special classes of spaces. In particular, the answer is "yes" if Y is an (a priori) arbitrary Banach space and X is any of the classical Banach spaces $\ell^p(I)$, $c_0(I)$, for $1 \le p \le \infty$ and I any index set, or $L^p(\mu)$, for $1 \le p \le \infty$ and μ a σ-finite measure (see [7], [10], [34], [28], [29] and references therein). The answer is also known to be positive for Y arbitrary and $X = C(K)$ if K is a compact metric space (see [9]).

The Mazur–Ulam property was introduced in [5]: a (real) Banach space X is said to have the *Mazur–Ulam property* (MUP) if for every Banach space Y every bijective isometry between S_X and S_Y can be extended to a linear isometry between X and Y.

Recall that a Banach space X is called a *CL-space* resp. *an almost CL-space* if for every maximally convex subset F of S_X one has $B_X = \operatorname{aco} F$ resp. $B_X = \overline{\operatorname{aco}} F$. CL-spaces were introduced by Fullerton [11], and almost CL-spaces by Lima [17], [18]. Lima also proved that real $C(K)$ and $L^1(\mu)$ spaces (where K is any compact Hausdorff space, and μ any finite measure) are CL-spaces. The complex spaces $C(K)$ are also CL while $L^1(\mu)$ in the complex case is in general only almost CL (see [20]).

Cheng and Dong [5] stated that every CL-space whose unit sphere has a smooth point and every polyhedral space ([1]) have the MUP. Unfortunately their proof is not completely correct, as is mentioned in the introduction of [14], where Kadets and Martín proved that every *finite-dimensional* polyhedral space has the MUP. Tan and Liu [31] showed that every almost CL-space whose unit sphere admits a smooth point has the MUP.

Next we recall the definition of lushness, introduced in [4] (in connection with a problem concerning the numerical index of a Banach space). The space X is said to be *lush* provided that for any $x, y \in S_X$ and every $\varepsilon > 0$ there exists $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\mathrm{d}(y, \operatorname{aco} S(x^*, \varepsilon)) < \varepsilon.$$

For example, every almost CL-space is lush but the converse is not true (see [4, Example 3.4]).

Tan, Huang and Liu [30] proposed the following definition of generalised lush spaces: X is called a *generalised lush* (GL) *space* if for every $x \in S_X$

([1]) A Banach space is called *polyhedral* if the unit ball of each of its finite-dimensional subspaces is a polyhedron, i.e. the convex hull of finitely many points.

and every $\varepsilon > 0$ there is some $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\mathrm{d}(y, S(x^*, \varepsilon)) + \mathrm{d}(y, -S(x^*, \varepsilon)) < 2 + \varepsilon \quad \forall y \in S_X.$$

It is proved in [30] that every almost CL-space and every separable lush space is a GL-space (see [30, Examples 2.4 and 2.5]). Also, by [30, Example 2.7] the space $\mathbb{R}^2$ equipped with the hexagonal norm $\|(x, y)\| = \max\{|y|, |x| + (1/2)|y|\}$ is a GL-space which is not lush.

Very recently, M. Cúth [6] used a separable reduction technique to prove that every lush Asplund space is a GL-space, but it is still not known whether every lush space is actually GL.

Concerning the connection of GL-spaces with the MUP, the following two propositions are proved in [30].

Proposition 1.1 ([30, Proposition 3.2]). *If X is a GL-space, Y any Banach space and $T : S_X \to S_Y$ is a (not necessarily onto) isometry, then*

$$\|T(x) - \lambda T(y)\| \geq \|x - \lambda y\| \quad \forall x, y \in S_X, \forall \lambda \geq 0. \tag{1.1}$$

Proposition 1.2 ([30, Proposition 3.4]). *If X and Y are Banach spaces and $T : S_X \to S_Y$ is an onto isometry which satisfies* (1.1) *then T can be extended to a linear isometry from X onto Y.*

It follows that every GL-space (in particular, every almost CL-space and every separable lush space) has the MUP [30, Theorem 3.3].

The authors of [30] further call a Banach space X a *local GL-space* if for every separable subspace Y of X there is a subspace Z of X which is GL and contains Y. Since lushness is separably determined (see [3, Theorem 4.2]) every lush space is a local GL-space [30, Example 3.7]. From their Propositions 3.2 and 3.4 the authors of [30] conclude that even every local GL-space has the MUP [30, Theorem 3.8], thus *every* lush space (separable or not) has the MUP [30, Corollary 3.9].

Many stability properties for GL-spaces have already been established in [30]: for example, if X is GL then so is the space $C(K, X)$ of all continuous functions from K into X, where K is any compact Hausdorff space (see [30, Theorem 2.10]). Also, the property GL is preserved under c_0-, ℓ^1- and ℓ^∞-sums (see [30, Theorem 2.11]). In the next section, we will prove some further stability results.

2. Stability results

2.1. Ultraproducts. We begin with an easy observation on ultraproducts of GL-spaces. First we recall the definition of ultraproducts of Banach spaces (see for example [13]). Given a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, for every bounded sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers there exists (by a compactness argument) $a \in \mathbb{R}$ such that for every $\varepsilon > 0$ one has $\{n \in \mathbb{N} : |a_n - a| < \varepsilon\} \in \mathcal{U}$.

Of course a is uniquely determined. It is called the *limit* of $(a_n)_{n\in\mathbb{N}}$ *along* $\mathcal{U}$ and denoted by $\lim_{n,\mathcal{U}} a_n$.

Now for a given sequence $(X_n)_{n\in\mathbb{N}}$ of Banach spaces denote by $\ell^\infty((X_n)_{n\in\mathbb{N}})$ the space of all sequences $(x_n)_{n\in\mathbb{N}}$ in the product $\prod_{n\in\mathbb{N}} X_n$ such that $\sup_{n\in\mathbb{N}}\|x_n\| < \infty$. We set

$$\mathcal{N}_{\mathcal{U}} := \Big\{(x_n)_{n\in\mathbb{N}} \in \ell^\infty((X_n)_{n\in\mathbb{N}}) : \lim_{n,\mathcal{U}} \|x_n\| = 0\Big\},$$

$$\prod_{n,\mathcal{U}} X_n := \ell^\infty((X_n)_{n\in\mathbb{N}})/\mathcal{N}_{\mathcal{U}}.$$

Equipped with the (well-defined) norm $\|[(x_n)_{n\in\mathbb{N}}]\|_{\mathcal{U}} := \lim_{n,\mathcal{U}}\|x_n\|$, this quotient becomes a Banach space. It is called the *ultraproduct* of $(X_n)_{n\in\mathbb{N}}$ (with respect to $\mathcal{U}$). By the way, it is easy to see that the subspace $\mathcal{N}_{\mathcal{U}}$ is closed in $\ell^\infty((X_n)_{n\in\mathbb{N}})$ with respect to the usual sup-norm and that $\|\cdot\|_{\mathcal{U}}$ coincides with the usual quotient norm. For more information on ultraproducts the reader is referred to [13].

In [3, Corollary 4.4] it is shown that the ultraproduct of a sequence of lush spaces is again lush, in fact it even has a stronger property, called ultra-lushness in [3]. We can easily prove an analogous result for GL-spaces. First we need a small remark.

Remark 2.1. If X is a GL-space, $x \in S_X$ and $\varepsilon > 0$ then there is some $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\mathrm{d}(y, S(x^*, \varepsilon)) + \mathrm{d}(y, -S(x^*, \varepsilon)) \le (2 + \varepsilon)\|y\| + 2\big|1 - \|y\|\big| \quad \forall y \in X.$$

Proof. Analogous to the proof of [30, Lemma 2.9]. ∎

Proposition 2.2. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $(X_n)_{n\in\mathbb{N}}$ a sequence of GL-spaces. Let $Z = \prod_{n,\mathcal{U}} X_n$. Then the following holds: for every $z \in S_Z$ there is a functional $z^* \in S_{Z^*}$ with $z^*(z) = 1$ such that for every $y \in S_Z$ there are $z_1, z_2 \in S_Z$ with $z^*(z_1) = 1 = -z^*(z_2)$ and $\|y - z_1\| + \|y - z_2\| = 2$. In particular, Z is also a GL-space.*

Proof. Let $z = [(x_n)_{n\in\mathbb{N}}] \in S_Z$. Without loss of generality we may assume $x_n \ne 0$ for all $n \in \mathbb{N}$. By the previous remark we can find, for every $n \in \mathbb{N}$, a functional $x_n^* \in S_{X_n^*}$ such that $x_n/\|x_n\| \in S(x_n^*, 2^{-n})$ and for every $v \in X_n$,

$$\mathrm{d}(v, S(x_n^*, 2^{-n})) + \mathrm{d}(v, -S(x_n^*, 2^{-n})) \le (2 + 2^{-n})\|v\| + 2\big|1 - \|v\|\big|. \tag{2.1}$$

Define $z^* : Z \to \mathbb{R}$ by $z^*([(v_n)]) := \lim_{n,\mathcal{U}} x_n^*(v_n)$. Then z^* is a well-defined element of S_{Z^*} with $z^*(z) = 1$ (because $x_n^*(x_n) > (1 - 2^{-n})\|x_n\|$ for all n).

Now given any $y = [(y_n)] \in S_Z$ we can find, by (2.1), sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\ell^\infty((X_n)_{n\in\mathbb{N}})$ such that $-v_n, u_n \in S(x_n^*, 2^{-n})$ and

$$\|u_n - y_n\| + \|v_n - y_n\| < (2 + 2^{-n})\|y_n\| + 2\big|1 - \|y_n\|\big| + 2^{-n}.$$

Also, because $-v_n, u_n \in S(x_n^*, 2^{-n})$, the sum on the left-hand side of the above equation is at least $2(1-2^{-n})$. Altogether it follows that $z_1 := [(u_n)]$ and $z_2 := [(v_n)]$ satisfy our requirements. ∎

Recall that for two isomorphic Banach spaces X and Y, their *Banach–Mazur distance* is defined by

$$\mathrm{d}(X,Y) := \inf\big\{\|T\|\,\|T^{-1}\| : T \text{ is an isomorphism between } X \text{ and } Y\big\}.$$

The next result shows that the class of GL-spaces is closed with respect to the Banach–Mazur distance.

PROPOSITION 2.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of GL-spaces and X a Banach space which is isomorphic to each X_n such that $\mathrm{d}(X_n, X) \to 1$. Then X is also a GL-space.*

Proof. By passing to a subsequence we may assume that $\mathrm{d}(X_n, X) < 1 + 1/n$ for each $n \in \mathbb{N}$. Hence there are isomorphisms $T_n : X_n \to X$ with $\|T_n\| = 1$ and $\|T_n^{-1}\| \le 1 + 1/n$.

Now let $\varepsilon > 0$ and $x \in S_X$. Set $x_n := T_n^{-1}x$ for each n. Then $1 \le \|x_n\| \le 1 + 1/n$. Let $y_n := x_n/\|x_n\|$. Since X_n is a GL-space, we can find, for each $n \in \mathbb{N}$, a functional $x_n^* \in S_{X_n^*}$ such that $y_n \in S(x_n^*, \varepsilon/2)$ and

$$\mathrm{d}(z, S(x_n^*, \varepsilon/2)) + \mathrm{d}(z, -S(x_n^*, \varepsilon/2)) < 2 + \varepsilon/2 \quad \forall z \in S_{X_n}. \tag{2.2}$$

We define $y_n^* := (T_n^*)^{-1}x_n^* \in X^*$ for each n. Then $\|y_n^*\| \le 1 + 1/n$ and

$$y_n^*(x) = x_n^*(x_n) = \|x_n\| x_n^*(y_n) > 1 - \varepsilon/2,$$

since $\|x_n\| \ge 1$ and $y_n \in S(x_n^*, \varepsilon/2)$.

Take $N \in \mathbb{N}$ such that $(1-\varepsilon/2)(1+1/N)^{-1} \ge 1 - \varepsilon$ and $1/N \le \varepsilon/4$. Let $y^* := y_N^*/\|y_N^*\|$. It follows that $y^*(x) > (1-\varepsilon/2)(1+1/N)^{-1} \ge 1-\varepsilon$, i.e. $x \in S(y^*, \varepsilon)$.

Now if $y \in S_X$, we define $z := T_N^{-1}y$. Then $1 \le \|z\| \le 1 + 1/N$. Let $z_0 := z/\|z\| \in S_{X_N}$. By (2.2) we can find $u_1 \in S(x_N^*, \varepsilon/2)$ and $u_2 \in -S(x_N^*, \varepsilon/2)$ such that $\|z_0 - u_1\| + \|z_0 - u_2\| < 2 + \varepsilon/2$. Let $v_i := T_N u_i \in B_X$ for $i = 1, 2$. It is easily checked that $v_1 \in S(y^*, \varepsilon)$ and $v_2 \in -S(y^*, \varepsilon)$. We further have

$$\begin{aligned}\|v_1 - y\| + \|v_2 - y\| &= \big\|T_N u_1 - \|z\| T_N z_0\big\| + \big\|T_N u_2 - \|z\| T_N z_0\big\| \\ &\le \|T_N(u_1 - z_0)\| + \|T_N(u_2 - z_0)\| + 2(\|z\| - 1)\|T_N z_0\| \\ &\le \|u_1 - z_0\| + \|u_2 - z_0\| + 2(\|z\| - 1) < 2 + \varepsilon/2 + 2/N \le 2 + \varepsilon.\end{aligned}$$

This shows that X is indeed a GL-space. ∎

2.2. F-ideals. First we recall the following notions (see [12, Chapter I, Definition 1.1]): a linear projection $P : X \to X$ is called an *M-projection* if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \quad \forall x \in X.$$

P is called an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\| \quad \forall x \in X.$$

A closed subspace Y of X is said to be an *M-summand* [*L-summand*] in X if it is the range of some M-projection [L-projection] on X. Equivalently, Y is an M-summand [L-summand] in X if and only if there is some closed subspace Z in X such that $X = Y \oplus_\infty Z$ [$X = Y \oplus_1 Z$]. Also, Y is called an *M-ideal* in X if $Y^\perp$ is an L-summand in X^* (where $Y^\perp := \{x^* \in X^* : x^*|_Y = 0\}$ is the *annihilator* of Y).

Every M-summand is also an M-ideal, but not conversely (see for instance [12, Chapter I, Example 1.4(a)]).

As is pointed out in [12], the notion of an "L-ideal" (i.e. a subspace whose annihilator is an M-summand in the dual) is not introduced because every "L-ideal" is already an L-summand (see [12, Chapter I, Theorem 1.9]). For more information on M-ideals and L-summands the reader is referred to [12].

Of course it is also possible to consider more general types of summands and ideals (see the overview in [12, p. 45f] and the papers [22], [23], [24] and [26]; we will recall just the basic definitions here). Firstly, a norm F on $\mathbb{R}^2$ is called *absolute* if $F(a, b) = F(|a|, |b|)$ for all $(a, b) \in \mathbb{R}^2$, and *normalised* if $F(1, 0) = 1 = F(0, 1)$. In the following, F will always denote an absolute, normalised norm on $\mathbb{R}^2$. If X and Y are two Banach spaces, their *F-sum* $X \oplus_F Y$ is defined as the direct product $X \times Y$ equipped with the norm $\|(x, y)\| = F(\|x\|, \|y\|)$, which is again a Banach space. For every $1 \leq p \leq \infty$, the p-norm F_p on $\mathbb{R}^2$ is of course an absolute, normalised norm and the corresponding sum is just the usual p-sum of Banach spaces.

An important property of absolute, normalised norms is their monotonicity, i.e. for all $a, b, c, d \in \mathbb{R}$,

$$|a| \leq |c| \text{ and } |b| \leq |d| \;\Rightarrow\; F(a, b) \leq F(c, d).$$

A proof of this fact can be found for instance in [2, Lemma 2]. It follows in particular that $|a|, |b| \leq F(a, b)$ for all $a, b \in \mathbb{R}$. We will use this later without further mention.

A linear projection $P : X \to X$ is called an *F-projection* if

$$\|x\| = F(\|Px\|, \|x - Px\|) \quad \forall x \in X,$$

and of course a closed subspace Y of X is said to be an *F-summand* in X if it is the range of an F-projection (equivalently, $X = Y \oplus_F Z$ for some closed subspace Z). Finally, Y is called an *F-ideal* if $Y^\perp$ is an F^*-summand in X^*, where F^* is the reversed dual norm of F, i.e.

$$F^*(a, b) = \sup\{|av + bu| : (u, v) \in \mathbb{R}^2 \text{ with } F(u, v) \leq 1\} \quad \forall (a, b) \in \mathbb{R}^2.$$

Then the L- resp. M-summands [M-ideals] are just the F_1- resp. F_∞-summands [F_∞-ideals]. Every F-summand is also an F-ideal (see [26, Lemma 8]). It is known that F-summands and F-ideals coincide (in every Banach space) if and only if the point $(0,1)$ is an extreme point of the unit ball of $(\mathbb{R}^2, F)$ (see [26, Corollary 10 and Remark 12] and [22, Section 2]).

It was proved in [27] that every L-summand and every M-ideal in a lush space are again lush. In [30, Theorem 2.11] it is shown that the c_0-sum of a family of Banach spaces is GL if *and only if* each summand is GL. So M-summands in GL-spaces are again GL. We will extend this result to a class of F-ideals which includes in particular all M-ideals. The main tool in the proof is, as in [27], the principle of local reflexivity (see [1, Theorem 11.2.4]).

Theorem 2.4. *If F is an absolute, normalised norm on $\mathbb{R}^2$ such that $(0,1)$ is an extreme point of the unit ball of $(\mathbb{R}^2, F^*)$, X is a GL-space and Y is an F-ideal in X, then Y is also a GL-space.*

Proof. Let $X^* = Y^\perp \oplus_{F^*} U$ for a suitable closed subspace $U \subseteq X^*$. It easily follows that U can be canonically identified with $X^*/Y^\perp$, which in turn can be canonically identified with Y^*, thus $X^* = Y^\perp \oplus_{F^*} Y^*$.

Now let $y \in S_Y$ and $0 < \varepsilon < 1$. Since $(0,1)$ is an extreme point of $B_{(\mathbb{R}^2, F^*)}$, by an easy compactness argument there is a $0 < \delta < \varepsilon$ such that

$$F^*(a,b) = 1 \text{ and } b \geq 1-\delta \ \Rightarrow\ |a| \leq \varepsilon. \tag{2.3}$$

Because X is GL we can find $x^* \in S_{X^*}$ such that $y \in S(x^*,\delta)$ and

$$\mathrm{d}(v, S(x^*,\delta)) + \mathrm{d}(v, -S(x^*,\delta)) < 2+\delta \quad \forall v \in S_X. \tag{2.4}$$

Write $x^* = (y^\perp, y^*)$ with $y^\perp \in Y^\perp$, $y^* \in Y^*$ and $1 = \|x^*\| = F^*(\|y^\perp\|, \|y^*\|)$. Then $y^*(y) = x^*(y) > 1-\delta > 1-\varepsilon$. Since $\|y^*\| \leq 1$ we see that $y \in S(y^*/\|y^*\|, \varepsilon)$. It also follows that $\|y^*\| > 1-\delta$ and hence by (2.3) we must have $\|y^\perp\| \leq \varepsilon$.

Next we fix $z \in S_Y$. By (2.4) we can find $x_1 \in S(x^*,\delta)$ and $x_2 \in -S(x^*,\delta)$ such that

$$\|x_1 - z\| + \|x_2 - z\| < 2+\delta. \tag{2.5}$$

We have $X^{**} = Y^{**} \oplus_F (Y^\perp)^*$, so if we consider X canonically embedded in its bidual we can write $x_i = (y_i^{**}, f_i) \in Y^{**} \oplus_F (Y^\perp)^*$ for $i = 1,2$. It follows that

$$1-\delta < x^*(x_1) = f_1(y^\perp) + y_1^{**}(y^*).$$

Taking into account that $\|y^\perp\| \leq \varepsilon$ and $\|f_1\| \leq 1$ we obtain

$$y_1^{**}(y^*) > 1-\delta-\varepsilon > 1-2\varepsilon. \tag{2.6}$$

Analogously one can see that

$$-\,y_2^{**}(y^*) > 1-2\varepsilon. \tag{2.7}$$

It also follows from (2.5) that

$$F(\|y_1^{**}-z\|,\|f_1\|)+F(\|y_2^{**}-z\|,\|f_2\|)<2+\delta$$

and hence

$$\|y_1^{**}-z\|+\|y_2^{**}-z\|<2+\delta<2+\varepsilon. \tag{2.8}$$

We set $E=\operatorname{span}\{y_1^{**},y_2^{**},z\}$ and choose $0<\eta<2\varepsilon$ such that

$$\frac{1-2\varepsilon}{1+\eta}>1-3\varepsilon \quad \text{and} \quad (1+\eta)(2+\varepsilon)<2+2\varepsilon.$$

Now the principle of local reflexivity comes into play. It yields a finite-dimensional subspace $V\subseteq Y$ and an isomorphism $T:E\to V$ such that $\|T\|,\|T^{-1}\|\le 1+\eta$, $T|_{E\cap Y}=\mathrm{id}$ and $y^*(Ty^{**})=y^{**}(y^*)$ for all $y^{**}\in E$. Let $y_i=Ty_i^{**}$ for $i=1,2$. Then $y^*(y_i)=y_i^{**}(y_i)$ and $\|y_i\|\le 1+\eta$. By (2.6), (2.7) and the choice of η we obtain $\|y_i\|>1-2\varepsilon$ as well as

$$\frac{y_1}{\|y_1\|}\in S\left(\frac{y^*}{\|y^*\|},3\varepsilon\right) \quad \text{and} \quad \frac{y_2}{\|y_2\|}\in -S\left(\frac{y^*}{\|y^*\|},3\varepsilon\right). \tag{2.9}$$

From (2.8) and the choice of η we get

$$\begin{aligned}\|y_1-z\|+\|y_2-z\|&=\|Ty_1^{**}-Tz\|+\|Ty_2^{**}-Tz\|\\&<(1+\eta)(2+\varepsilon)<2+2\varepsilon.\end{aligned}$$

Since $1-2\varepsilon<\|y_i\|\le 1+\eta<1+2\varepsilon$ we have $\|y_i-y_i/\|y_i\|\|<2\varepsilon$ and thus

$$\left\|\frac{y_1}{\|y_1\|}-z\right\|+\left\|\frac{y_2}{\|y_2\|}-z\right\|<2+6\varepsilon,$$

which in view of (2.9) finishes the proof. ■

As mentioned before, Theorem 2.4 shows in particular that M-ideals in GL-spaces are again GL, which was proved for lushness in [27]. The proof of [27] readily extends to the case of more general ideals that we consider above (we skip the details).

Theorem 2.5. *If F is an absolute, normalised norm on $\mathbb{R}^2$ such that $(0,1)$ is an extreme point of the unit ball of $(\mathbb{R}^2,F^*)$, X is a lush space and Y is an F-ideal in X, then Y is also lush.*

Theorem 2.11 in [30] also states that the ℓ^1-sum of any family of Banach spaces is GL if *and only if* every summand is GL. The "only if" part of this statement just means that L-summands in GL-spaces are again GL-spaces. However, the proof of this part given in [30] contains a slight mistake: the statement "$\|u_\lambda\|>1/2-\varepsilon/2$ and $\|v_\lambda\|>1/2-\varepsilon/2$" cannot be deduced from the two preceding lines (2.3) and (2.4) as claimed in [30]. For a counterexample just consider the sum $X:=\mathbb{R}\oplus_1\mathbb{R}$ and take $x:=(1,0)\in S_X$. Then the norm-one functional $x^*:X\to\mathbb{R}$ defined by $x^*(a,b):=a+b$ satisfies $x^*(x)=1$ and $\mathrm{d}(y,S)+\mathrm{d}(y,-S)=2$ for all

$y \in S_X$, where $S := \{z \in S_X : x^*(z) = 1\}$ (we even have $\operatorname{aco} S = B_X$). Now for $y := (-1, 0), u := (u_1, u_2) := (0, 1)$ and $v := y$ we have $-v, u \in S$ and $\|y - u\|_1 + \|y - v\|_1 = 2$. So if the claim in the proof of [30] were true, we would obtain the contradiction $|u_1| \geq 1/2$.

We will therefore include a slightly different proof that generalised lushness is inherited by L-summands.

PROPOSITION 2.6. *If X is a GL-space and Y is an L-summand in X, then Y is also a GL-space.*

Proof. Write $X = Y \oplus_1 Z$ for a suitable closed subspace $Z \subseteq X$. Let $y \in S_Y$ and $0 < \varepsilon < 1$. Take $0 < \delta < \varepsilon^2$. Since X is GL, there is a functional $x^* = (y^*, z^*)$ in the unit sphere of $X^* = Y^* \oplus_\infty Z^*$ such that $y \in S(x^*, \delta)$ and

$$\mathrm{d}(v, S(x^*, \delta)) + \mathrm{d}(v, -S(x^*, \delta)) < 2 + \delta \quad \forall v \in S_X. \tag{2.10}$$

Since $x^*(y) = y^*(y)$ it follows that $y \in S(y^*/\|y^*\|, \delta) \subseteq S(y^*/\|y^*\|, \varepsilon)$.

Now fix $u \in S_Y$. Because of (2.10) we can find $x_1 \in S(x^*, \delta)$ and $x_2 \in -S(x^*, \delta)$ such that $\|u - x_1\| + \|u - x_2\| < 2 + \delta$. Write $x_i = y_i + z_i$ with $y_i \in Y$, $z_i \in Z$ for $i = 1, 2$. It then follows that

$$\|u - y_1\| + \|z_1\| + \|u - y_2\| + \|z_2\| < 2 + \delta. \tag{2.11}$$

We distinguish two cases. First we assume that $\|y_1\|, \|y_2\| \geq \varepsilon$. Since $x_1 \in S(x^*, \delta)$ we have

$$y^*(y_1) = x^*(x_1) - z^*(z_1) > 1 - \delta - \|z_1\| \geq \|y_1\| - \delta$$

and hence

$$y^*\left(\frac{y_1}{\|y_1\|}\right) > 1 - \frac{\delta}{\|y_1\|} \geq 1 - \frac{\delta}{\varepsilon} > 1 - \varepsilon,$$

thus $y_1/\|y_1\| \in S(y^*/\|y^*\|, \varepsilon)$. Analogously one can see that $y_2/\|y_2\| \in -S(y^*/\|y^*\|, \varepsilon)$. Furthermore, because of (2.11) and since $\|y_i\| + \|z_i\| = \|x_i\| > 1 - \delta$, we have

$$\begin{aligned}\left\|u - \frac{y_1}{\|y_1\|}\right\| + \left\|u - \frac{y_2}{\|y_2\|}\right\| &\leq \|u - y_1\| + \|u - y_2\| + \big|1 - \|y_1\|\big| + \big|1 - \|y_2\|\big| \\ &\leq \|u - y_1\| + \|u - y_2\| + \|z_1\| + \|z_2\| + 2\delta \\ &< 2 + 3\delta < 2 + 3\varepsilon.\end{aligned}$$

In the second case we have $\|y_1\| < \varepsilon$ or $\|y_2\| < \varepsilon$. If $\|y_1\| < \varepsilon$ it follows that $\|z_1\| = \|x_1\| - \|y_1\| > 1 - \delta - \varepsilon > 1 - 2\varepsilon$ and hence, because of (2.11),

$$\begin{aligned}\|u - y_2\| + \|z_2\| &< 2 + \delta - (1 - 2\varepsilon) - \|u - y_1\| \\ &< 1 + 3\varepsilon - (1 - \|y_1\|) < 4\varepsilon.\end{aligned}$$

Then in particular $|y^*(u) - y^*(y_2)| < 4\varepsilon$ and thus (since $-x_2 \in S(x^*\delta))$ we have

$$\begin{aligned} y^*(u) &< 4\varepsilon + y^*(y_2) = 4\varepsilon + x^*(x_2) - z^*(z_2) < 4\varepsilon - (1-\delta) - z^*(z_2) \\ &< 5\varepsilon - 1 + \|z_2\| \leq 5\varepsilon - \|y_2\| \leq 5\varepsilon + \|u - y_2\| - 1 < 9\varepsilon - 1. \end{aligned}$$

Hence $-u \in S(y^*/\|y^*\|, 9\varepsilon)$. But then

$$\mathrm{d}(u, S(y^*/\|y^*\|, 9\varepsilon)) + \mathrm{d}(u, -S(y^*/\|y^*\|, 9\varepsilon)) = \mathrm{d}(u, S(y^*/\|y^*\|, 9\varepsilon)) \leq 2.$$

If $\|y_2\| < \varepsilon$ an analogous argument shows that $u \in S(y^*/\|y^*\|, 9\varepsilon)$ and thus the proof is complete. ∎

2.3. Inheriting from the bidual. Next we would like to prove that every Banach space X whose bidual is GL has itself the MUP. First consider the following (at least formal) weakening of the definition of GL-spaces.

DEFINITION 2.7. A real Banach space X is said to have *property* $(*)$ provided that for every $\varepsilon > 0$ and all $x, y_1, y_2 \in S_X$ there exists a functional $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\mathrm{d}(y_i, S(x^*, \varepsilon)) + \mathrm{d}(y_i, -S(x^*, \varepsilon)) < 2 + \varepsilon \quad \text{for } i = 1, 2.$$

The exact same proof as in [30] shows that [30, Proposition 3.2] (Proposition 1.1 in our introduction) holds true not only for GL-spaces but for all spaces with property $(*)$ and consequently every space with property $(*)$ has the MUP.

In a preprint version of this paper ([2]), the author proved that property $(*)$ passes from X^{**} to X and thus in particular X has the MUP if X^{**} is a GL-space.

The referee proposed to consider instead the following definition.

DEFINITION 2.8. A real Banach space X is said to have *property* $(**)$ provided that for every $\varepsilon > 0$ and all $x, y \in S_X$ there exists a functional $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\mathrm{d}(y, S(x^*, \varepsilon)) + \mathrm{d}(y, -S(x^*, \varepsilon)) < 2 + \varepsilon.$$

Obviously, we have GL $\Rightarrow (*) \Rightarrow (**)$. Now one can make the following observations (again these are due to the referee).

(i) A similar argument to one in [30, Example 2.5] shows that *all* lush spaces (not only the separable ones) have property $(**)$.

(ii) If we denote by (K, w^*) the weak*-closure of the set $\mathrm{ex}(B_{X^*})$ of extreme points of B_{X^*} endowed with the weak*-topology, then one can show as in the proof of [30, Proposition 2.2] that for every

([2]) See arXiv:1309.4358 and the author's PhD thesis, available at http://www.diss.fu-berlin.de/diss/receive/FUDISS_thesis_000000099968.

$y \in S_X$ and all $\varepsilon, \delta > 0$ the set

$$\{x^* \in K : \mathrm{d}(y, S(x^*, \delta)) + \mathrm{d}(y, -S(x^*, \delta)) < 2 + \varepsilon\}$$

is open and dense in (K, w^*).

(iii) Using (ii), the fact that $\mathrm{ex}(B_{X^*})$ is norming for X, and the fact that the intersection of two open, dense subsets is again dense, one can show that properties $(*)$ and $(**)$ are actually equivalent.

(iv) Using (ii), the fact that $\mathrm{ex}(B_{X^*})$ is norming for X, and the Baire category theorem, one can proceed as in the proof of [30, Proposition 2.2] to show that for *separable* spaces, properties $(**)$ and GL are equivalent.

Now we will prove that property $(**)$ passes from X^{**} to X (this is virtually the same proof that was already presented in the above mentioned preprint version, except that now we work just with one point y instead of two points y_1, y_2).

THEOREM 2.9. *If X^{**} has property $(**)$, then so does X. In particular, if X^{**} is GL, then X has the MUP.*

Proof. The principle of local reflexivity is the key to the proof. If we fix $x, y \in S_X$ and $\varepsilon > 0$ and consider X canonically embedded into its bidual, then since the latter has property $(**)$ we can find $x^{***} \in S_{X^{***}}$ such that $x \in S(x^{***}, \varepsilon)$ and $u^{**} \in S(x^{***}, \varepsilon), v^{**} \in -S(x^{***}, \varepsilon)$ with

$$\|y - u^{**}\| + \|y - v^{**}\| < 2 + \varepsilon. \tag{2.12}$$

If we also consider X^* canonically embedded into X^{***}, then by Goldstine's theorem B_{X^*} is weak*-dense in $B_{X^{***}}$, so we can find $\tilde{x}^* \in B_{X^*}$ such that

$$|u^{**}(\tilde{x}^*) - x^{***}(u^{**})| \le \varepsilon,$$
$$|v^{**}(\tilde{x}^*) - x^{***}(v^{**})| \le \varepsilon,$$
$$|\tilde{x}^*(x) - x^{***}(x)| \le \varepsilon.$$

We set $x^* = \tilde{x}^*/\|\tilde{x}^*\|$. It follows that $x \in S(x^*, 2\varepsilon)$ as well as $x^* \in S(u^{**}, 2\varepsilon)$ and $-x^* \in S(v^{**}, 2\varepsilon)$.

Now let $E := \mathrm{span}\{x, y, u^{**}, v^{**}\} \subseteq X^{**}$ and choose $0 < \delta < \varepsilon$ such that

$$\frac{1 - 2\varepsilon}{1 + \delta} > 1 - 3\varepsilon \quad \text{and} \quad (2 + \varepsilon)(1 + \delta) < 2 + 2\varepsilon.$$

By the principle of local reflexivity there is a finite-dimensional subspace F of X and an isomorphism $T : E \to F$ such that $\|T\|, \|T^{-1}\| \le 1 + \delta$, $T|_{X \cap E} = \mathrm{id}$ and $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$.

Set $\tilde{u} := Tu^{**}$ and $\tilde{v} := Tv^{**}$ as well as $u := \tilde{u}/\|\tilde{u}\|$ and $v := \tilde{v}/\|\tilde{v}\|$. We then have

$$\frac{1 - \varepsilon}{1 + \delta} \le \|\tilde{u}\|, \|\tilde{v}\| \le 1 + \delta. \tag{2.13}$$

It follows that

$$x^*(u) = \frac{x^*(Tu^{**})}{\|\tilde{u}\|} \geq \frac{u^{**}(x^*)}{1+\delta} > \frac{1-2\varepsilon}{1+\delta} > 1 - 3\varepsilon,$$

so $u \in S(x^*, 3\varepsilon)$ and similarly also $-v \in S(x^*, 3\varepsilon)$. Furthermore, because of (2.12), we have

$$\begin{aligned}\|y - \tilde{u}\| + \|y - \tilde{v}\| &= \|Ty - Tu^{**}\| + \|Ty - Tv^{**}\| \\ &< (1+\delta)(2+\varepsilon) < 2 + 2\varepsilon.\end{aligned}$$

From (2.13) we get $\|u - \tilde{u}\|, \|v - \tilde{v}\| \leq \varepsilon + \delta < 2\varepsilon$. Hence

$$\|y - u\| + \|y - v\| < 2 + 6\varepsilon,$$

and we are done. ∎

By a similar argument one could also prove that lushness passes from X^{**} to X. This fact has already been established in [16, Proposition 4.3], albeit with a different proof (the proof in [16] is based on an equivalent formulation of lushness [16, Proposition 2.1] and does not use the principle of local reflexivity).

3. GL-spaces and rotundity. We start with the following observation on Hilbert spaces (the proof is easy and will therefore be omitted).

Remark 3.1. Let H be a Hilbert space and set

$$A := \{(x, x^*) \in S_H \times S_{H^*} : x \in \ker x^*\}.$$

Then

$$\mathrm{d}(x, S(x^*, \varepsilon)) = \sqrt{2\big(1 - \sqrt{2\varepsilon - \varepsilon^2}\big)} \tag{3.1}$$

for all $0 < \varepsilon < 1$ and all $(x, x^*) \in A$. Consequently,

$$\lim_{\varepsilon \to 0} \mathrm{d}(x, S(x^*, \varepsilon)) = \sqrt{2} \quad \text{uniformly in } (x, x^*) \in A.$$

It immediately follows from Remark 3.1 that a Hilbert space (of dimension at least two) is not a GL-space. It is possible to generalise this statement in a certain sense. To do so, first recall some basic rotundity notions (see for example [8, Chapters 8–9]).

A Banach space X is said to be *strictly convex* if $x, y \in S_X$ and $\|x + y\| = 2$ already implies $x = y$. X is called *uniformly rotund* (or *uniformly convex*) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in the unit sphere of X the condition $\|x_n + y_n\| \to 2$ implies $\|x_n - y_n\| \to 0$. Finally, X is *locally uniformly rotund* (or *locally uniformly convex*) if for every $x \in S_X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X the condition $\|x_n + x\| \to 2$ implies $\|x_n - x\| \to 0$. Such points x are called *LUR points* of S_X (an easy normalisation argument shows that in the definition of LUR points we can replace the condition $\|x_n\| = 1$ for all n by $\|x_n\| \to 1$).

Of course uniform rotundity implies local uniform rotundity, which in turn implies strict convexity, but the converses are false in general (though by an easy compactness argument it can be seen that the three notions coincide for finite-dimensional Banach spaces). The most prominent examples of uniformly rotund spaces are the Hilbert spaces and, more generally, the spaces $L^p(\mu)$ for any measure μ and any $1 < p < \infty$.

In a preprint version of this paper ([3]), the author proved that the unit sphere of any infinite-dimensional GL-space with (not necessarily countable) 1-unconditional basis does not have any LUR points. The referee provided a more general argument showing that the unit sphere of any GL-space of dimension at least two has no LUR points. In the following, we present the referee's proof.

LEMMA 3.2. *Let* $\dim(X) \geq 2$ *and* $x \in S_X$. *Then there exists* $y \in S_X$ *with* $\|x + y\| + \|x - y\| > 2$.

Proof. First assume that S_X is not covered by the sets $A := x + B_X$ and $B := -x + B_X$. Then there exists $y \in S_X$ with $\|x - y\| > 1$ and $\|x + y\| > 1$, hence $\|x + y\| + \|x - y\| > 2$.

Now suppose that S_X is contained in $A \cup B$. Since $\dim(X) \geq 2$, the unit sphere is connected and hence the closed sets $A \cap S_X$ and $B \cap S_X$ have nonempty intersection. Let $z \in A \cap B \cap S_X$.

Then $\|x - z\| \leq 1$, $\|x + z\| \leq 1$ and

$$2 = 2\|x\| \leq \|x - z\| + \|x + z\| \leq 2.$$

It follows that $\|x - z\| = \|x + z\| = 1 = \|x\|$. Hence the whole line segment $\{x + tz : t \in [-1, 1]\}$ is contained in S_X.

Set $y := x + z \in S_X$. Then

$$\|x + y\| + \|x - y\| = \|2x + z\| + \|z\| = 2\|x + z/2\| + 1 = 3 > 2. \ \blacksquare$$

LEMMA 3.3. *Let* $x \in S_X$ *be an LUR point and* $\varepsilon > 0$. *Then there is some* $\delta > 0$ *such for every* $x^* \in S_{X^*}$ *with* $x \in S(x^*, \delta)$ *one has* $\operatorname{diam}(S(x^*, \delta)) \leq \varepsilon$, *where* diam *denotes diameter.*

Proof. Assume to the contrary that for every $\delta > 0$ there exists some $x^* \in S_{X^*}$ with $x \in S(x^*, \delta)$ but $\operatorname{diam}(S(x^*, \delta)) > \varepsilon$. Then one can find a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that $x, x_n \in S(x_n^*, 1/n)$ and $\|x - x_n\| \geq \varepsilon/2$ for every $n \in \mathbb{N}$. It follows that

$$2 \geq \|x + x_n\| \geq x^*(x + x_n) \geq 2\left(1 - \frac{1}{n}\right) \quad \forall n \in \mathbb{N}.$$

Hence $\|x + x_n\| \to 2$ and since x is an LUR point, we would obtain $\|x_n - x\| \to 0$, contradicting the fact that $\|x - x_n\| \geq \varepsilon/2$ for every $n \in \mathbb{N}$. $\blacksquare$

([3]) See footnote 2 above.

THEOREM 3.4. *Let X be a GL-space with* $\dim(X) \geq 2$. *Then S_X does not have any LUR points.*

Proof. Assume that there is an LUR point $x \in S_X$. By Lemma 3.2 there exists $y \in S_X$ such that $\|x+y\| + \|x-y\| > 2$. Choose $\varepsilon > 0$ with $\|x+y\| + \|x-y\| > 2 + 3\varepsilon$.

By Lemma 3.3 there is a $\delta > 0$ such that every slice $S(x^*, \delta)$ containing x has diameter at most ε. Without loss of generality one can assume $\delta < \varepsilon$.

Since X is GL, there exists $x^* \in S_{X^*}$ with $x \in S(x^*, \delta)$ and $u, v \in S(x^*, \delta)$ such that

$$\|y-u\| + \|y+v\| < 2 + \delta.$$

But on the other hand, since $\operatorname{diam}(S(x^*, \delta)) \leq \varepsilon$,

$$\begin{aligned}\|y-u\| + \|y+v\| &\geq \|x-y\| + \|x+y\| - \|x-u\| - \|x-v\| \\ &> 2 + 3\varepsilon - 2\varepsilon = 2 + \varepsilon > 2 + \delta,\end{aligned}$$

and this contradiction finishes the proof. ■

4. Open problems. Following again the referee's suggestion, we list some open problems below.

PROBLEM 4.1. *Is every lush space a GL-space?*

As mentioned in the introduction, the answer is "yes" for separable spaces [30] and for Asplund spaces [6], but the question remains open in general.

PROBLEM 4.2. *Is every space with property* (∗∗) *actually a GL-space?*

We know that the answer is "yes" for separable spaces (Subsection 2.3), but it is open for nonseparable ones.

PROBLEM 4.3. *If F is an absolute, normalised norm on $\mathbb{R}^2$ which is GL, and X, Y are GL-spaces, does this imply that $X \oplus_F Y$ is GL?*

PROBLEM 4.4. *If X is GL, does this imply that X^* is GL?*

Example 3.1 in [4] shows that the answer to the analogous question for lush spaces is "no".

PROBLEM 4.5. *If X^* is GL, does this imply that X is GL?*

According to [16], the answer to the analogous question for lush spaces is "no".

PROBLEM 4.6. *Does there exist a strictly convex GL-space with dimension at least two?*

By [15, Corollary 4.6], a real lush space cannot be strictly convex, unless it is one-dimensional.

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