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ON SEQUENCES OVER A FINITE ABELIAN GROUP WITH ZERO-SUM SUBSEQUENCES OF FORBIDDEN LENGTHS

BY

WEIDONG GAO (Tianjin), YUANLIN LI (St. Catharines, Ont.), PINGPING ZHAO (Tianjin) and JUJUAN ZHUANG (Dalian)

Abstract. Let G be an additive finite abelian group. For every positive integer ℓ, let $\mathrm{disc}_\ell(G)$ be the smallest positive integer t such that each sequence S over G of length $|S| \geq t$ has a nonempty zero-sum subsequence of length not equal to ℓ. In this paper, we determine $\mathrm{disc}_\ell(G)$ for certain finite groups, including cyclic groups, the groups $G = C_2 \oplus C_{2m}$ and elementary abelian 2-groups. Following Girard, we define $\mathrm{disc}(G)$ as the smallest positive integer t such that every sequence S over G with $|S| \geq t$ has nonempty zero-sum subsequences of distinct lengths. We shall prove that $\mathrm{disc}(G) = \max\{\mathrm{disc}_\ell(G) \mid \ell \geq 1\}$ and determine $\mathrm{disc}(G)$ for finite abelian p-groups G, where $p \geq r(G)$ and $r(G)$ is the rank of G.

1. Introduction. Throughout this paper, let G be an additive finite abelian group, C_n denote a cyclic group of n elements, and C_n^k denote the direct sum of k copies of C_n. It is well known that either $|G| = 1$, or $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$, where $r = r(G)$ is the rank of G and $n_r = \exp(G)$ is the exponent of G. Set

$$\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1).$$

Let p be a sufficiently large prime. In 1976, Erdős and Szemerédi [ES] proved that if S is a sequence of length $|S| = p$ over C_p whose support contains at least three distinct terms, then S has nonempty zero-sum subsequences of distinct lengths, confirming a conjecture of Graham for sufficiently large primes. In 2010, Gao, Hamidoune and Wang [GHW] extended the above result to all positive integers n. A different proof was given by Grynkiewicz [Gry] in 2011. In 2012 Girard [Gir] posed a natural problem of determining the smallest positive integer t, denoted by $\mathrm{disc}(G)$, such that every sequence S over G of length $|S| \geq t$ has nonempty zero-sum subsequences of distinct lengths. Recently, Gao, Zhao and Zhuang [GZZ] determined $\mathrm{disc}(G)$ for all

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elementary abelian 2-groups, all groups of rank at most two, and some other groups with large exponents. Around 2000, a similar invariant $E_k(G)$ was introduced by the first author and studied successfully by Schmid [S]. The invariant $E_k(G)$ is the smallest positive integer t such that every sequence S over G of length $|S| \geq t$ has a zero-sum subsequence T with $k \nmid |T|$. In this paper we determine $\mathrm{disc}(G)$ for finite abelian p-groups G with $p \geq r(G)$, and conduct a further detailed investigation of this problem by introducing the following constant.

Definition 1.1. For every positive integer ℓ, define $\mathrm{disc}_\ell(G)$ to be the smallest positive integer t such that every sequence S over G of length $|S| \geq t$ has a nonempty zero-sum subsequence T with $|T| \neq \ell$.

It is easy to see that $\mathrm{disc}(G) = \max\{\mathrm{disc}_\ell(G) \mid \ell \geq 1\}$ (see Proposition 2.1). Let $\mathsf{D}(G)$ denote the *Davenport constant* of G, which is defined as the smallest positive integer d such that every sequence over G of length at least d has a nonempty zero-sum subsequence. Our main results are as follows.

Theorem 1.2. *Let G be a finite abelian group. Then*

(1) $\mathrm{disc}_\ell(G) = \mathsf{D}(G) + 1$ *if* $\ell = 1$.

(2) $\mathrm{disc}_\ell(G) = \begin{cases} \mathsf{D}(G)+1, & G \text{ is not cyclic} \\ 2\mathsf{D}(G), & G \text{ is cyclic} \end{cases}$ *if* $\ell = \mathsf{D}(G)$.

(3) $\mathrm{disc}_\ell(G) = \mathsf{D}(G)$ *if* $\ell \geq \mathsf{D}(G) + 1$.

According to the above theorem, it would be sufficient to consider the case $\ell \in [2, \mathsf{D}(G) - 1]$ when studying $\mathrm{disc}_\ell(G)$. We derive the precise values of $\mathrm{disc}_\ell(G)$ for certain groups.

Theorem 1.3. *Let $\ell \in [2, \mathsf{D}(G) - 1]$ and m, n be positive integers. Then*

(1) $\mathrm{disc}_\ell(G) = n + 1$ *if G is a cyclic group of order $n \geq 3$.*

(2) $\mathrm{disc}_\ell(G) = \begin{cases} 2m+3, & \ell \in [2, 2m-2],\ \ell \text{ is even} \\ 2m+2, & \ell \in [3, 2m-1],\ \ell \text{ is odd} \\ 4m+1, & \ell = 2m \end{cases}$ *if* $G = C_2 \oplus C_{2m}$.

Theorem 1.4. *Let $G = C_2^r$ with $r \geq 2$, and let $\ell \in [2, r]$. Then*

$$\mathrm{disc}_\ell(G) = r + u_1 + 1,$$

where $u_1 = \max\left\{u \mid 2^{u-1} \mid \ell,\ \ell \cdot \frac{2^u - 1}{2^{u-1}} - u \leq r\right\}$.

Theorem 1.5. *Let p be a prime and G be a finite abelian p-group with $r(G) \geq 3$. If $p \geq r(G)$, then* $\mathrm{disc}(G) = \mathsf{D}(G) + \exp(G)$.

The paper is organized in the following way. In Section 2 we recall some basic notions, provide several preliminary results and give a proof of Theorem 1.2. In Sections 3 and 4 we determine $\mathrm{disc}_\ell(G)$ on cyclic groups C_n, the groups $G = C_2 \oplus C_{2m}$ and elementary abelian 2-groups, and prove our two main results: Theorems 1.3 and 1.4. Finally, in Section 5 both constants $\mathrm{disc}(G)$ and $\mathrm{disc}_\ell(G)$ for finite abelian p-groups are investigated.

2. Preliminaries. Throughout the paper, $\mathbb{N}$ denotes the set of positive integers. For real numbers $a \le b$, we set $[a,b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. By a *sequence over* G we mean a finite sequence of terms from G where the order is disregarded and repetition is allowed. We consider sequences as elements of the free abelian monoid $\mathcal{F}(G)$ over G, and our notation and terminology coincides with that of [GG, GH, Gryn]. A sequence $S = g_1 \cdot \ldots \cdot g_l = \prod_{i=1}^{l} g_i$ over G is called a *zero-sum sequence* if $\sum_{i=1}^{l} g_i = 0 \in G$. S is called *zero-sum free* if it contains no nonempty zero-sum subsequence. If S is a zero-sum sequence and each proper subsequence is zero-sum free, then S is called a *minimal zero-sum sequence.*

Let S be a sequence over G. We denote by $\mathrm{supp}(S)$ the subset of G consisting of all elements which occur in S. The sum of the elements in S is denoted by $\sigma(S)$, and the maximal number of repetitions of a term in S is denoted by $h(S)$. If T is a subsequence of S, we denote by ST^{-1} the sequence obtained from S by deleting T. Let

$$\sum(S) = \{\sigma(T) \mid 1 \ne T \mid S\},$$

where $T \mid S$ means T is a subsequence of S, and 1 denotes the empty sequence.

Now we recall some well-known results on the Davenport constant, which assert that $\mathsf{D}(G) = \mathsf{D}^*(G)$ if G satisfies any one of the following conditions (see [GG], [O], [Ol]):

1. G has rank at most two,
2. G is an abelian p-group,
3. $G = C_2 \oplus C_m \oplus C_n$ with $2 \mid m \mid n$.

We first give several easy observations on $\mathrm{disc}_\ell(G)$ and $\mathrm{disc}(G)$.

PROPOSITION 2.1. $\mathrm{disc}(G) = \max\{\mathrm{disc}_\ell(G) \mid \ell \ge 1\}$.

Proof. By the definition of $\mathrm{disc}(G)$, every sequence S over G of length $\mathrm{disc}(G)$ has nonempty zero-sum subsequences of distinct lengths, and thus, for every $\ell \ge 1$, S has a nonempty zero-sum subsequence of length not equal to ℓ, whence $\mathrm{disc}(G) \ge \mathrm{disc}_\ell(G)$, so $\mathrm{disc}(G) \ge \max\{\mathrm{disc}_\ell(G) \mid \ell \ge 1\}$. On the other hand, let T be a sequence over G of length $\mathrm{disc}(G) - 1$ such that T has no two nonempty zero-sum subsequences of distinct lengths. Note the obvious fact that $\mathrm{disc}(G) - 1 \ge \mathsf{D}(G)$, so all nonempty zero-sum subsequences of T have the same length, say ℓ_1. Hence, $\max\{\mathrm{disc}_\ell(G) \mid \ell \ge 1\} \ge \mathrm{disc}_{\ell_1}(G) \ge \mathrm{disc}(G)$. Therefore, $\mathrm{disc}(G) = \max\{\mathrm{disc}_\ell(G) \mid \ell \ge 1\}$. ∎

LEMMA 2.2. *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ $(1 < n_1 \mid \cdots \mid n_r)$. Then*

$$\max\{\mathsf{D}(G), \mathsf{D}^*(G) + 1\} \le \mathrm{disc}_\ell(G) \le \min\{\mathsf{D}(G) + \ell, \mathrm{disc}(G)\},$$

where $\ell \in [2, \mathsf{D}(G) - 1]$.

Proof. To prove the right-hand inequality, we first show $\mathrm{disc}_\ell(G) \le \mathsf{D}(G) + \ell$. In fact, let S be any sequence over G of length $\mathsf{D}(G) + \ell$, and S_1 be a nonempty zero-sum subsequence of S. If $|S_1| \neq \ell$, we are done. If $|S_1| = \ell$, we can obtain a nonempty zero-sum subsequence S_2 of SS_1^{-1} since $|SS_1^{-1}| = \mathsf{D}(G)$. Thus S_1S_2 is a nonempty zero-sum subsequence of length $|S_1S_2| \neq \ell$, implying $\mathrm{disc}_\ell(G) \le \mathsf{D}(G) + \ell$. In addition, by Proposition 2.1, $\mathrm{disc}_\ell(G) \le \mathrm{disc}(G)$ for every $\ell \ge 1$. Therefore, $\mathrm{disc}_\ell(G) \le \min\{\mathsf{D}(G) + \ell, \mathrm{disc}(G)\}$.

We now handle the left-hand inequality. Let $e_1, \ldots, e_r$ be a basis of G with $\mathrm{ord}(e_i) = n_i$ for each $i \in [1, r]$, and $S = (-\sigma(\prod_{i=1}^r e_i^{\ell_i}))\prod_{i=1}^r e_i^{n_i-1}$ be a sequence over G of length $\mathsf{D}^*(G)$, where $0 \le \ell_i \le n_i - 1$ and $\sum_{i=1}^r \ell_i = \ell - 1$. Then each nonempty zero-sum subsequence of S is of the form $(-\sigma(\prod_{i=1}^r e_i^{\ell_i}))\prod_{i=1}^r e_i^{\ell_i}$ with length ℓ, implying $\mathrm{disc}_\ell(G) \ge \mathsf{D}^*(G) + 1$. Obviously, $\mathrm{disc}_\ell(G) \ge \mathsf{D}(G)$. Therefore, $\mathrm{disc}_\ell(G) \ge \max\{\mathsf{D}(G), \mathsf{D}^*(G) + 1\}$. ∎

LEMMA 2.3. *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. If $\mathsf{D}(G) = \mathsf{D}^*(G)$, then*

$$\mathrm{disc}_{n_i}(G) = \mathsf{D}(G) + n_i \quad \textit{for each } i \in [1, r].$$

Proof. Let $e_1, \ldots, e_r$ be a basis of G, $\mathrm{ord}(e_j) = n_j$ for each $j \in [1, r]$. Let $i \in [1, r]$ and $S = e_i^{n_i}\prod_{j=1}^r e_j^{n_j-1}$ be a sequence over G of length $\sum_{j=1}^r (n_j - 1) + n_i = \mathsf{D}(G) + n_i - 1$. Then all nonempty zero-sum subsequences of S have the same length n_i, implying $\mathrm{disc}_{n_i}(G) \ge \mathsf{D}(G) + n_i$. On the other hand, by Lemma 2.2, $\mathrm{disc}_{n_i}(G) \le \mathsf{D}(G) + n_i$. Therefore, $\mathrm{disc}_{n_i}(G) = \mathsf{D}(G) + n_i$. ∎

Proof of Theorem 1.2. 1) Let T be a zero-sum free sequence over G of length $\mathsf{D}(G) - 1$. Then $S = T0$ is a sequence of length $\mathsf{D}(G)$ and $\{0\}$ is the only nonempty zero-sum subsequence of S. Hence $\mathrm{disc}_1(G) \ge \mathsf{D}(G) + 1$. On the other hand, every sequence of length $\mathsf{D}(G)+1$ has at least one nonempty zero-sum subsequence of length $k \ge 2$, and thus $\mathrm{disc}_1(G) \le \mathsf{D}(G) + 1$. Therefore, $\mathrm{disc}_1(G) = \mathsf{D}(G) + 1$.

2) If G is cyclic, then the result follows from Lemma 2.3. Next assume that G is not cyclic. Then $\mathsf{D}(G) \ge \mathsf{D}^*(G) > \exp(G)$. Let S be a sequence over G of length $\mathsf{D}(G)+1$. If all nonempty zero-sum subsequences of S have length $\mathsf{D}(G)$, then $|\mathrm{supp}(S)| = 1$. So we have a nonempty zero-sum subsequence of S of length $\exp(G) < \mathsf{D}(G)$, a contradiction. Hence, $\mathrm{disc}_{\mathsf{D}(G)}(G) \le \mathsf{D}(G)+1$. On the other hand, by the definition of $\mathsf{D}(G)$, there is a minimal zero-sum sequence over G of length $\mathsf{D}(G)$, hence $\mathrm{disc}_{\mathsf{D}(G)}(G) \ge \mathsf{D}(G) + 1$. Therefore, $\mathrm{disc}_{\mathsf{D}(G)}(G) = \mathsf{D}(G) + 1$.

3) Let $\ell \geq \mathsf{D}(G)+1$. By the definition of $\mathsf{D}(G)$, every sequence S over G of length $\mathsf{D}(G)$ has a nonempty zero-sum subsequence S_1 with $|S_1| \leq \mathsf{D}(G) < \ell$. Therefore, $\mathrm{disc}_\ell(G) \leq \mathsf{D}(G)$. Clearly, $\mathrm{disc}_\ell(G) \geq \mathsf{D}(G)$. Hence, we have the desired result. ■

3. $\mathrm{disc}_\ell(G)$ **on abelian groups** G **with** $r(G) \leq 2$**.** In this section, we determine $\mathrm{disc}_\ell(G)$ for cyclic groups G and for groups $G \cong C_2 \oplus C_{2m}$ for all $\ell, m \in \mathbb{N}$. We first list a few useful lemmas.

Lemma 3.1 ([GHW, Theorem 1.1]). *Let G be a cyclic group of order $|G| = n \geq 2$ and S a sequence over G of length n. If $|\mathrm{supp}(S)| \geq 3$, then S has nonempty zero-sum subsequences of distinct lengths.*

Lemma 3.2. *Let G be a cyclic group of order $|G| = n \geq 3$ and S a zero-sum free sequence over G of length $|S| = \ell \geq (n+1)/2$. Then there is a $g \in G$ with $\mathrm{ord}(g) = n$ such that $S = (n_1 g) \cdot \ldots \cdot (n_\ell g)$ where $1 = n_1 \leq \cdots \leq n_\ell$, $n_1 + \cdots + n_\ell < n$ and $\sum(S) = \{g, 2g, 3g, \ldots, (n_1 + \cdots + n_\ell)g\}$.*

Proof. This was first proved by Savchev and Chen [SCh], and Yuan [Y] independently, and one can also find a proof in [GR, Theorem 5.1.8]. ■

Lemma 3.3. *Let G be a cyclic group of order $|G| = n \geq 2$ and S a sequence over G of length $|S| = n$. Then S has a nonempty zero-sum subsequence T such that $|T| \leq h(S)$.*

Proof. See [GH, Theorem 5.7.3]. ■

Lemma 3.4. *Let G be a cyclic group of order $|G| = n \geq 2$ and S a sequence over G of length $n+1$ and with $|\mathrm{supp}(S)| = 2$. Then there exist nonempty zero-sum subsequences of S of distinct lengths.*

Proof. Let $g \in G$ with $\mathrm{ord}(g) = n$ and $S = (ag)^s(cg)^t$, where $a, c \in [0, n-1]$ are distinct, and $s, t \in [1, n]$ with $t \leq s$ and $s + t = n+1$. If $t = 1$, then $S = (ag)^n(cg)$. If $\mathrm{ord}(ag) \neq n$, then $S_1 = (ag)^n$ and $S_2 = (ag)^{\mathrm{ord}(ag)}$ are nonempty zero-sum subsequences of distinct lengths. If $\mathrm{ord}(ag) = n$, then $\sum((ag)^n) = G$. We conclude that there is a subsequence S_3 of $(ag)^n$ such that $\sigma(S_3) = -(cg)$, so $S_3(cg)$ is a nonempty zero-sum subsequence of length not equal to n. Next, suppose that $t \geq 2$. Then $t \leq s \leq n-1$. Assume to the contrary that all the nonempty zero-sum subsequences of S have the same length r. We distinguish two cases.

Case 1: $r \leq (n+1)/2$. Choose a nonempty zero-sum subsequence T of S. Then $|T| = r$ and ST^{-1} is a zero-sum free sequence with $|ST^{-1}| \geq (n+1)/2$. By Lemma 3.2, there is an $h \in G$ with $\mathrm{ord}(h) = n$ such that $ST^{-1} = h^u(xh)^v$, where $x \geq 2$ and $u + xv \leq n-1$. We set $T = h^\tau(xh)^w$. Clearly, $u, w \geq 1$. We first note that

$$x \geq u + 1. \tag{3.1}$$

Otherwise, $h^{\tau+x}(xh)^{w-1}$ is a nonempty zero-sum subsequence of S of length greater than r, a contradiction. We claim that $v \geq 1$. Otherwise, $u = |ST^{-1}| \geq (n+1)/2$, and by (3.1) we have $x \geq u+1 \geq n-u+2 > n-u$, so $n-x < u$. Therefore, $h^{n-x}(xh)$ is a nonempty zero-sum subsequence of length $n-x+1 \leq n-u = \tau+w-1 = r-1$, a contradiction. Now we have $u+vx \geq u+v(u+1) = 2(u+v)+(u-1)(v-1)-1 \geq n$, a contradiction.

Case 2: $r > (n+1)/2$, i.e.

$$r \geq \lceil n/2 \rceil + 1. \tag{3.2}$$

By Lemma 3.3, we have

$$r \leq s. \tag{3.3}$$

We first assert that $(a,n) = 1$. If $(a,n) \geq 2$, then $(ag)^{n/(a,n)}$ is a nonempty zero-sum subsequence of S of length $n/(a,n) < (n+1)/2 < r$, a contradiction. Hence, $\mathrm{ord}(ag) = \mathrm{ord}(g)/(\mathrm{ord}(g), a) = n$ and there is a $b \in [2, n-1]$ such that $cg = b(ag)$.

Subcase 2.1: $n-b \leq s$. Clearly, $(ag)^{n-b}(cg)$ is a nonempty zero-sum subsequence of S of length $n-b+1 = r$. By (3.2) and (3.3), we have $\lceil n/2 \rceil + 1 \leq r = n-b+1 \leq s$, so $n-s+1 \leq b \leq \lfloor n/2 \rfloor$. Thus $0 \leq n-2b < n-b \leq s$. Now, $(ag)^{n-2b}(cg)^2$ is a nonempty zero-sum subsequence of S of length $n-2b+2 \neq n-b+1$, a contradiction.

Subcase 2.2: $n-b \geq s+1$. Note that $b \leq n-s-1 \leq \lfloor n/2 \rfloor - 2 < s$ (since $s \geq r \geq \lceil n/2 \rceil + 1$).

If $tb < n$, we have $0 < n-tb = t-1-tb+s < s$ and $0 < n-tb+b = t-1-b(t-1)+s < s$. Thus $(ag)^{n-tb}(cg)^t$ and $(ag)^{n-tb+b}(cg)^{t-1}$ are nonempty zero-sum subsequences of S of distinct lengths, a contradiction.

If $tb \geq n$, then there is $t_1 \in [1, t-1]$ such that $t_1 b < n$ and $(t_1+1)b \geq n$, so $0 < n-t_1 b \leq b < s$. Therefore, $(ag)^{n-t_1 b}(cg)^{t_1}$ is a nonempty zero-sum subsequence of S of length $r = n-t_1 b+t_1$. Since $2b < n$, we have $t_1 \geq 2$. By (3.2) we have $2b-2 \leq t_1 b - t_1 = n-r \leq n - \lceil n/2 \rceil - 1$, so $2b \leq \lfloor n/2 \rfloor + 1 \leq s$ and $0 < n-t_1 b+b \leq b+b \leq s$. Therefore, $(ag)^{n-t_1 b+b}(cg)^{t_1-1}$ is a nonempty zero-sum subsequence of S of length $n-t_1 b+b+t_1-1 \neq n-t_1 b+t_1$, yielding a contradiction. Therefore, S must have nonempty zero-sum subsequences of distinct lengths. ∎

Lemma 3.5. *Let $G = C_m \oplus C_n$ with $2 \mid m \mid n$. Then*

$$\mathrm{disc}_\ell(G) = \begin{cases} 2m+n-1, & \ell = m, \\ m+n, & \ell \in [2, \mathsf{D}(G)-1] \text{ and } \ell \text{ is odd}, \\ m+2n-1, & \ell = n. \end{cases}$$

Proof. Note that $\mathsf{D}(G) = m+n-1 = \mathsf{D}^*(G)$. If $\ell = m$ or n, by Lemma 2.3 we derive the desired results.

If $\ell \in [2, \mathsf{D}(G)-1]$ and ℓ is odd, by Lemma 2.2 we find that $\mathrm{disc}_\ell(G) \geq \mathsf{D}^*(G)+1 = m+n$. We now prove $\mathrm{disc}_\ell(G) \leq m+n$. Let $S = \prod_{i=1}^{m+n} g_i$ be a sequence over G and assume that all the nonempty zero-sum subsequences of S have length ℓ. We consider another finite abelian group $G' = C_2 \oplus G$ and a new sequence $S' = \prod_{i=1}^{m+n} (x, g_i)$ over G' with $\mathrm{ord}(x) = 2$. Since $\mathsf{D}(G') = m+n = |S'|$, we obtain a nonempty zero-sum subsequence $T' = \prod_{k=1}^{|T'|} (x, g_{i_k})$ of S' with $\mathrm{ord}(x) \mid |T'|$. Clearly, the corresponding sequence $T = \prod_{k=1}^{|T'|} g_{i_k}$ is a nonempty zero-sum subsequence of S with $|T| = |T'|$ and $2 \mid |T|$. Thus $|T| \neq \ell$ as ℓ is odd, a contradiction. Hence, $\mathrm{disc}_\ell(G) \leq m+n$ and we are done. ∎

The Erdős–Ginzburg–Ziv constant $\mathsf{s}(G)$ is defined as the smallest positive integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ has a nonempty zero-sum subsequence T of length $\exp(G)$. The precise value of $\mathsf{s}(G)$ is known for groups G of rank $r(G) \leq 2$ (see [GH, Theorem 5.8.3]); for progress on groups of higher rank we refer to [FGZ]. Here we need this constant for elementary 2-groups.

Lemma 3.6 ([GH, Corollary 5.7.6]). *For every $r \in \mathbb{N}$, we have $\mathsf{s}(C_2^r) = 2^r + 1$.*

Proof of Theorem 1.3. (1) Let G be a cyclic group of order n. By Lemma 2.2, we have $\mathrm{disc}_\ell(G) \geq n+1$.

We next show that $\mathrm{disc}_\ell(G) \leq n+1$. Let S be a sequence over G of length $n+1$. We show that S has a nonempty zero-sum subsequence of length not equal to ℓ. We may always assume that $0 \notin S$. If $|\mathrm{supp}(S)| = 1$, set $S = g^{n+1}$; then g^n is a nonempty zero-sum subsequence of length $n \neq \ell$. If $|\mathrm{supp}(S)| \geq 2$, by Lemmas 3.1 and 3.4, we can obtain two nonempty zero-sum subsequences of S of distinct lengths, hence there must be a nonempty zero-sum subsequence of length not equal to ℓ.

(2) The results follow from Lemma 3.5 except when $\ell \in [4, 2m-2]$ and ℓ is even, so we consider this case. Let e_1, e_2 be a basis of G with $\mathrm{ord}(e_1) = 2$ and $\mathrm{ord}(e_2) = 2m$. Consider the sequence $S_0 = e_2^{2m-1} \cdot (e_1 + (m+1-\ell/2)e_2)^3$ over G of length $2m+2$. Clearly, all nonempty zero-sum subsequences of S_0 have ℓ. Hence, $\mathrm{disc}_\ell(G) \geq 2m+3$. To show equality, it is sufficient to prove that any sequence S over G of length $2m+3$ has a nonempty zero-sum subsequence of length not equal to ℓ.

Let $\phi : G = C_2 \oplus C_{2m} \to C_2 \oplus C_2$ be the natural homomorphism with $\ker(\phi) = C_m$ (up to isomorphism). Let S be a sequence over G of length $2m+3$. Applying Lemma 3.6 to $\phi(S)$ repeatedly, we get a decomposition $S = S_1 \cdot \ldots \cdot S_m \cdot S'$ with $|S_i| = 2$, $\sigma(S_i) \in \ker(\phi)$ for $i \in [1, m]$, and $|S'| = 3$ $(= \mathsf{D}(C_2^2))$, so we can find a subsequence S_{m+1} of S' such that $\sigma(\phi(S_{m+1}))$ $= 0$, i.e. $\sigma(S_{m+1}) \in \ker(\phi)$ and $|S_{m+1}| \in [1, 3]$. Set $T = \prod_{i=1}^{m+1} (\sigma(S_i))$. Then

T is a sequence over $\ker(\phi) = C_m$. For $\ell/2 \in [2, m-1]$, by (1), there is a nonempty zero-sum subsequence $T_1 = \prod_{j=1}^{t}(\sigma(S_{i_j}))$ of T over C_m of length $t \neq \ell/2$. If $|S_{i_j}| = 2$ for all $j \in [1,t]$, then $\prod_{j=1}^{t} S_{i_j}$ is a nonempty zero-sum subsequence of S of length not equal to ℓ; otherwise, $\sigma(S_{m+1}) \mid T_1$ and $|S_{m+1}|$ is 1 or 3, so $\prod_{j=1}^{t} S_{i_j}$ is a nonempty zero-sum subsequence of S of odd length (not equal to ℓ).

Therefore, $\mathrm{disc}_\ell(G) = 2m+3$, where $\ell \in [4, 2m-2]$ and ℓ is even, and we are done. ∎

4. $\mathrm{disc}_\ell(G)$ on elementary abelian 2-groups. In this section we determine $\mathrm{disc}_\ell(G)$ for elementary abelian 2-groups. A method similar to that used in [GZZ] will be adopted to derive the main result.

Lemma 4.1 ([GZZ, Lemma 4.2]). *Let t and r be positive integers with $t \geq 2$, and let $S = e_1 \cdot \ldots \cdot e_r x_1 \cdot \ldots \cdot x_t$ be a sequence of nonzero terms over C_2^r of length $r+t$, where $e_1, \ldots, e_r$ form a basis of C_2^r. For each $i \in [1,t]$, let $A_i \subset [1,r]$ be a nonempty subset such that $x_i = \sum_{j \in A_i} e_j$. If every nonempty zero-sum subsequence of S has the same length ℓ, then $|A_i| = \ell - 1$ and*

$$\Big|\bigcap_{i \in I} A_i\Big| = \frac{\ell}{2^{|I|-1}}$$

for every $I \subset [1,t]$ of cardinality $|I| \in [2,t]$. In particular, $\ell \equiv 0 \pmod{2^{t-1}}$.

Lemma 4.2. *Let $G = C_2^r$ with $r \geq 2$ and let $\ell \in [2,r]$. For $u \in \mathbb{N}$, if $2^{u-1} \mid \ell$ and $\ell \cdot \frac{2^u-1}{2^{u-1}} - u \leq r$, then there is a sequence S over G of length $r+u$ such that all nonempty zero-sum subsequences of S have length ℓ.*

Proof. To construct S, take a basis $e_1, \ldots, e_r$ of C_2^r. The assumption $\ell \cdot \frac{2^u-1}{2^{u-1}} - u \leq r$ allows us to find $2^u - 1$ disjoint subsets E_I of $[1,r]$, labeled by nonempty subsets $I \subset [1,u]$, satisfying the following conditions:

1. $|E_I| = \ell/2^{u-1}$ if $|I| \in [2,u]$.
2. $|E_I| = \ell/2^{u-1} - 1$ if $|I| = 1$.

(We note that if $l = 2^{u-1}$ there are exactly u empty subsets among the above $2^u - 1$ disjoint subsets, which are denoted by $E_{\{1\}}, \ldots, E_{\{u\}}$.)

We now define u subsets $A_1, \ldots, A_u$ of $[1,r]$ in the following way: $j \in [1,r]$ belongs to A_k if and only if there is a subset $I \subset [1,u]$ containing k such that $j \in E_I$. It follows that $A_i = \bigcup_{i \in I \subset [1,u]} E_I$ for each $i \in [1,u]$. Therefore,

$$|A_i| = \sum_{i \in I \subset [1,u]} |E_I| = \frac{\ell}{2^{u-1}} - 1 + \sum_{i \in I \subset [1,u],\, |I| \geq 2} \frac{\ell}{2^{u-1}} = \ell - 1 \tag{4.1}$$

for every $i \in [1, u]$, and

$$\Big|\bigcap_{i\in J} A_i\Big| = \sum_{J\subset I\subset[1,u]} |E_I| = \frac{\ell}{2^{u-1}} \cdot 2^{u-|J|} = \frac{\ell}{2^{|J|-1}} \tag{4.2}$$

for every $J \subset [1, u]$ with $|J| \geq 2$. Let

$$x_i = \sum_{j\in A_i} e_j$$

for each $i \in [1, u]$, and let

$$S = e_1 \cdot \ldots \cdot e_r x_1 \cdot \ldots \cdot x_u.$$

If T is any nonempty zero-sum subsequence of S, then

$$T = \prod_{i\in I} x_i \prod_{j\in A} e_j$$

for some nonempty subset I of $[1, u]$ and some subset A of $[1, r]$.

We next show the following result.

Claim.

$$|A| = \sum_{k=1}^{|I|} (-2)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq u,\, i_1,\ldots,i_k\in I} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

In fact, let $j \in \bigcup_{i\in I} A_i$ and $\lambda(j) = |\{i \in I \mid j \in A_i\}|$. Since T is zero-sum, clearly, $j \in A$ if and only if $\lambda(j)$ is odd. Let r_j be the number of times that j is counted on the right side of the equality in the Claim. Then

$$\begin{aligned} r_j &= \binom{\lambda(j)}{1} - 2\binom{\lambda(j)}{2} + 2^2\binom{\lambda(j)}{3} - \cdots + (-2)^{\lambda(j)-1}\binom{\lambda(j)}{\lambda(j)} \\ &= \frac{1-(1-2)^{\lambda(j)}}{2}. \end{aligned}$$

Therefore, $r_j = 1$ if $\lambda(j)$ is odd and $r_j = 0$ if $\lambda(j)$ is even. This proves the Claim.

By the above claim, (4.1) and (4.2), we obtain

$$\begin{aligned} |A| &= \sum_{k=1}^{|I|} (-2)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq u,\, i_1,\ldots,i_k\in I} |A_{i_1} \cap \cdots \cap A_{i_k}| \\ &= \binom{|I|}{1}(\ell-1) - 2\binom{|I|}{2} \cdot \frac{\ell}{2} + \cdots + (-2)^{|I|-1} \cdot \binom{|I|}{|I|}\frac{\ell}{2^{|I|-1}} \\ &= \ell - |I|, \end{aligned}$$

so $|A| + |I| = \ell$. Thus all the nonempty zero-sum subsequences of S have the same length ℓ, and we are done. ∎

Proof of Theorem 1.4. By Lemma 4.2, we have $\mathrm{disc}_\ell(G) \ge r + u_1 + 1$. Next we show $\mathrm{disc}_\ell(G) \le r + u_1 + 1$. Let S be a sequence over G of length $r + u_1 + 1$. We show that S contains a nonempty zero-sum subsequence of length not equal to ℓ. Assume to the contrary that every nonempty zero-sum subsequence of S has length ℓ. We may assume that $0 \notin S$. Suppose $\langle \mathrm{supp}(S) \rangle = C_2^{r_1} \subset G$. Then $r_1 \le r$ and $|S| = r_1 + t_1 = r + u_1 + 1$, where $t_1 \ge u_1 + 1 \ge 2$. Let $S = e_1 \cdot \ldots \cdot e_{r_1} x_1 \cdot \ldots \cdot x_{t_1}$ with $e_1, \ldots, e_{r_1}$ being a basis of $C_2^{r_1}$. For each $i \in [1, t_1]$, let $A_i \subset [1, r_1]$ be a nonempty subset such that $x_i = \sum_{j \in A_i} e_j$. Applying Lemma 4.1 on S we obtain

$$
\begin{aligned}
r_1 &\ge \Big| \bigcup_{i=1}^{t_1} A_i \Big| \\
&= \sum_{1 \le i \le t_1} |A_i| - \sum_{1 \le i < j \le t_1} |A_i \cap A_j| + \cdots + (-1)^{t_1 - 1} \Big| \bigcap_{i=1}^{t_1} A_i \Big| \\
&= t_1(\ell - 1) - \binom{t_1}{2} \cdot \frac{\ell}{2} + \cdots + (-1)^{t_1 - 1} \cdot \frac{\ell}{2^{t_1 - 1}} = \ell \cdot \frac{2^{t_1} - 1}{2^{t_1 - 1}} - t_1
\end{aligned}
$$

and

$$\ell \equiv 0 \pmod{2^{t_1 - 1}}.$$

Since $2^{t_1 - 1} \mid \ell$ and $t_1 \ge u_1 + 1$, according to the definition of u_1, we get $r_1 > r$, a contradiction. Therefore, S contains a nonempty zero-sum subsequence of length not equal to ℓ. So $\mathrm{disc}_\ell(G) \le r + u_1 + 1$, and thus $\mathrm{disc}_\ell(G) = r + u_1 + 1$. ∎

5. $\mathrm{disc}_\ell(G)$ **and** $\mathrm{disc}(G)$ **on abelian** p**-groups.** Let p be a prime, and G be a finite abelian p-group. In this section, we investigate both $\mathrm{disc}_\ell(G)$ and $\mathrm{disc}(G)$.

Definition 5.1. Let $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ be a sequence of length $|S| = l \in \mathbb{N}_0$ and let $g \in G$.

(a) For every $k \in \mathbb{N}_0$ let

$$\mathbf{N}_g^k(S) = \Big| \Big\{ I \subset [1, l] \,\Big|\, \sum_{i \in I} g_i = g \text{ and } |I| = k \Big\} \Big|$$

denote the number of subsequences T of S having sum $\sigma(T) = g$ and length $|T| = k$ (counted with the multiplicity of their appearance in S).

(b) We define

$$\mathbf{N}_g(S) = \sum_{k \ge 0} \mathbf{N}_g^k(S), \qquad \mathbf{N}_g^+(S) = \sum_{k \ge 0} \mathbf{N}_g^{2k}(S), \qquad \mathbf{N}_g^-(S) = \sum_{k \ge 0} \mathbf{N}_g^{2k+1}(S).$$

Thus $\mathbf{N}_g(S)$ denotes the number of subsequences T of S with $\sigma(T) = g$, $\mathbf{N}_g^+(S)$ denotes the number of such subsequences of even length, and $\mathbf{N}_g^-(S)$

denotes the number of such subsequences of odd length (each counted with the multiplicity of its appearance in S).

LEMMA 5.2 ([O, Theorem 1]). *Let G be a finite abelian p-group for some prime p, and let S be a sequence over G with $|S| \geq \mathsf{D}(G)$. Then $\mathbf{N}_g^+(S) \equiv \mathbf{N}_g^-(S) \pmod p$ for all $g \in G$.*

Proof. See [GH, Proposition 5.5.8]. ■

The following congruence was first used by Lucas [L]; we give a proof for the convenience of the reader.

LEMMA 5.3. *Let p be a prime, and let a, b be positive integers with p-adic expansions $a = a_n p^n + \cdots + a_1 p + a_0$ and $b = b_n p^n + \cdots + b_1 p + b_0$. Define $\binom{k}{0} = 1$ for $k \geq 0$. Then*

$$\binom{a}{b} \equiv \binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}} \cdot \ldots \cdot \binom{a_0}{b_0} \pmod p.$$

Proof. We have

$$\begin{aligned}(1+x)^a &= (1+x)^{a_n p^n + \cdots + a_1 p + a_0} \\ &\equiv (1+x^{p^n})^{a_n} \cdot \ldots \cdot (1+x^p)^{a_1}(1+x)^{a_0} \pmod p.\end{aligned}$$

Since $0 \leq a_i \leq p-1$, by equating the coefficients of x^b on both sides of the above equation, we obtain the desired result. ■

Proof of Theorem 1.5. Let $G = C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_r}}$, where $1 \leq n_1 \leq \cdots \leq n_r$. Then $\mathsf{D}(G) = \sum_{i=1}^r (p^{n_i} - 1) + 1$. By Proposition 2.1 and Lemma 2.3 we have $\mathrm{disc}(G) \geq \mathsf{D}(G) + \exp(G)$. So, it suffices to show that $\mathrm{disc}(G) \leq \mathsf{D}(G) + \exp(G)$.

Let S be a sequence over G of length $\mathsf{D}(G) + \exp(G)$. We need to show that S contains nonempty zero-sum subsequences of distinct lengths. Assume to the contrary that every nonempty zero-sum subsequence has length ℓ. By Theorem 1.2 we have $\ell \leq \mathsf{D}(G) - 1$. Therefore, $|S| - \mathsf{D}(G) + 1 \leq \ell \leq \mathsf{D}(G) - 1$, i.e. $p^{n_r} + 1 \leq \ell \leq \sum_{i=1}^r (p^{n_i} - 1)$.

Let $\phi : G \to G \oplus \langle e \rangle \cong G \oplus C_{p^{n_r}}$, where $\mathrm{ord}(e) = p^{n_r}$, be defined by $\phi(g) = g + e$. Since $|\phi(S)| = |S| > \mathsf{D}^*(G \oplus C_{p^{n_r}}) = \mathsf{D}(G \oplus C_{p^{n_r}})$, there is a subsequence S_1 of S such that $0 = \sigma(\phi(S_1)) = \sigma(S_1) + |S_1|e$. Hence S_1 is a nonempty zero-sum subsequence of S with length divisible by $\mathrm{ord}(e) = p^{n_r}$, so $\ell = kp^{n_r}$ for some $k \in [2, t]$, where $t = \lfloor \sum_{i=1}^r (p^{n_i} - 1)/p^{n_r} \rfloor \leq r - 1$.

Let T be a subsequence of S with $|T| \geq \sum_{i=1}^r (p^{n_i} - 1) + 1 = \mathsf{D}(G)$. By Lemma 5.2, $1 + (-1)^k \mathbf{N}_0^{kp^{n_r}}(T) = \mathbf{N}_0^+(T) - \mathbf{N}_0^-(T) \equiv 0 \pmod p$. Therefore,

$$\mathbf{N}_0^{kp^{n_r}}(T) \equiv (-1)^{k+1} \pmod p$$

for every $T \mid S$ with $|T| \geq \sum_{i=1}^r (p^{n_i} - 1) + 1$, and in particular

$$\mathbf{N}_0^{kp^{n_r}}(S) \equiv (-1)^{k+1} \pmod p.$$

Hence, by Lemma 5.3,

$$\begin{aligned}\sum_{T\mid S,\,|T|=\sum_{i=1}^r(p^{n_i}-1)+1}\mathbf{N}_0^{kp^{n_r}}(T) &\equiv \sum_{T\mid S,\,|T|=\sum_{i=1}^r(p^{n_i}-1)+1}(-1)^{k+1}\\ &= \binom{\sum_{i=1}^r(p^{n_i}-1)+p^{n_r}+1}{\sum_{i=1}^r(p^{n_i}-1)+1}(-1)^{k+1}\\ &\equiv (t+1)(-1)^{k+1} \pmod p.\end{aligned}$$

Note that for every nonempty zero-sum subsequence W of S of length $|W|=kp^{n_r}$, there exist $\binom{\sum_{i=1}^r(p^{n_i}-1)+1+p^{n_r}-kp^{n_r}}{\sum_{i=1}^r(p^{n_i}-1)+1-kp^{n_r}}$ subsequences T of S such that $W\mid T\mid S$ and $|T|=\sum_{i=1}^r(p^{n_i}-1)+1$. Thus, by Lemma 5.3,

$$\begin{aligned}\sum_{T\mid S,\,|T|=\sum_{i=1}^r(p^{n_i}-1)+1}\mathbf{N}_0^{kp^{n_r}}(T) &= \binom{\sum_{i=1}^r(p^{n_i}-1)+1+p^{n_r}-kp^{n_r}}{\sum_{i=1}^r(p^{n_i}-1)+1-kp^{n_r}}\mathbf{N}_0^{kp^{n_r}}(S)\\ &\equiv (t+1-k)(-1)^{k+1} \pmod p.\end{aligned}$$

Therefore,

$$(t+1-k)(-1)^{k+1}\equiv (t+1)\cdot(-1)^{k+1} \pmod p.$$

Thus $k\equiv 0 \pmod p$, contradicting $k\in[2,t]$ and $p\ge r(G)=r>t$. ∎

We next present some results on $\mathrm{disc}_\ell(G)$ for finite abelian p-groups.

LEMMA 5.4 ([Gir, Corollary 2.4]). *Let G be a finite abelian p-group, and let S be a sequence over G with $|S|=\mathsf{D}(G)+i-1$, where $i\ge 1$. If $i\ge 2$ and S contains a zero-sum subsequence S' with $p\nmid |S'|$, then S has nonempty zero-sum subsequences of distinct lengths.*

THEOREM 5.5. *Let p be a prime, G be a finite abelian p-group and let $\ell\in[2,\mathsf{D}(G)-1]$ with $p\nmid \ell$. Then $\mathrm{disc}_\ell(G)=\mathsf{D}(G)+1$.*

Proof. The result follows from Lemma 2.2 and Lemma 5.4 with $i=2$. ∎

THEOREM 5.6. *Let $\alpha\ge 1$ and $r\ge 3$ be integers, and let p be a prime such that $p\ge r$. Then*

$$(r+1)(p^\alpha-1)+1\le \mathrm{disc}_{kp^\alpha}(C_{p^\alpha}^r)\le (r+1)(p^\alpha-1)+2,$$

where $k\in[2,\lceil r/2\rceil]$.

Proof. Let $G=C_{p^\alpha}^r$ and let $e_1,\dots,e_r$ be a basis of G with $\mathrm{ord}(e_i)=p^\alpha$ for every $i\in[1,r]$. Since $k\le\lceil r/2\rceil$, we have $2k-1\le r$. Let $S=\prod_{i=1}^r e_i^{p^\alpha-1}\cdot(\sum_{j=1}^k e_j+\sum_{t=k+1}^{2k-1}(p^\alpha-1)e_t)^{p^\alpha-1}$ be a sequence over G of length $(r+1)(p^\alpha-1)$. It is easy to show that all the nonempty zero-sum subsequences of S have length kp^α. Hence, $\mathrm{disc}_{kp^\alpha}(G)\ge(r+1)(p^\alpha-1)+1$. It follows from Theorem 1.5 and Lemma 2.2 that $\mathrm{disc}_{kp^\alpha}(G)\le(r+1)(p^\alpha-1)+2$ as desired. ∎

We close the paper by making the following conjecture together with a remark on $\mathrm{disc}(G)$ for finite abelian p-groups.

CONJECTURE 5.7. For any finite abelian group G, there is an integer $t = t(G)$ depending only on G such that, if S is a sequence over G of length $\mathrm{disc}(G) - 1$ and every nonempty zero-sum subsequence of S has the same length ℓ, then $\ell = t(G)$.

REMARK 5.8. In terms of the results of [GZZ] we know that the above conjecture holds true for finite elementary abelian 2-groups, finite abelian groups of rank at most two, and some finite abelian groups with large exponent. It seems that it might be very difficult to determine the precise value of $\mathrm{disc}(G)$ for a general finite abelian p-group with $p < r(G)$. Even if $p = 2$ and $G = C_{2^\alpha}^r$ with $r \geq 3$, the invariant $\mathrm{disc}(G)$ has only recently been determined for the special case of $\alpha = 1$ (with a somewhat complicated proof [GZZ]).

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Weidong Gao, Pingping Zhao
Center for Combinatorics
LPMC-TJKLC, Nankai University
Tianjin 300071, P.R. China
E-mail: wdgao@nankai.edu.cn
ppz1989@126.com

Yuanlin Li
Department of Mathematics
Brock University
St. Catharines, Ontario, Canada L2S 3A1
E-mail: yli@brocku.ca

Jujuan Zhuang
Department of Mathematics
Dalian Maritime University
Dalian 116024, P.R. China
E-mail: jjzhuang@dlmu.edu.cn