

# SELECTED RESULTS OF THE THEORY OF VALUE DISTRIBUTION AND GROWTH OF MEROMORPHIC FUNCTIONS

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**Abstract.** The paper discusses development of the theory of value distribution and growth of meromorphic functions, focusing on two basic notions: exceptional values and asymptotic values. Some historical context is given and contemporary achievements are presented. In particular, recent results concerning exceptional functions and asymptotic functions are considered.

**1. Basic notions of value distribution theory.** We say that a complex function of a complex variable is meromorphic in a region  $D$  if it is holomorphic in  $D$ , possibly apart from isolated points where it has got poles. As Rolf Nevanlinna describes them, they are functions *of rational character*. Usually, value distribution theory considers functions meromorphic in the whole complex plane or in a disc  $|z| < R < \infty$ , typically the unit disc.

By the fundamental theorem of algebra, a non-constant polynomial in  $\mathbb{C}$  takes up every complex value the same number of times. It is no longer true for entire functions in general or for meromorphic functions in the complex plane. There are certain rules however, governing frequency with which the values are assumed or omitted. By the Picard theorem, a transcendental entire function assumes every finite value, possibly except one. In consequence, a transcendental meromorphic function in the plane assumes every value in  $\overline{\mathbb{C}}$  infinitely often apart from at most two exceptions ([52]).

Modern theory of value distribution of meromorphic functions was introduced in the 1920's in the papers of Rolf Nevanlinna. Each function  $f$  meromorphic in  $|z| < R \leq \infty$

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is characterized by a real-valued function

$$T(r, f) := m(r, \infty, f) + N(r, \infty, f),$$

called the *Nevanlinna's characteristic function of  $f$* . Here

$$m(r, \infty, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

denotes a mean proximity to infinity function ( $\log^+ x := \max(\log x, 0)$ ) and

$$N(r, \infty, f) := \int_0^r [n(t, \infty, f) - n(0, \infty, f)] \frac{dt}{t} + n(0, \infty, f) \log r,$$

a function counting poles ( $n(t, \infty, f)$  is the number of poles of  $f$  in  $\{z : |z| \leq t\}$ , counted with multiplicity). Thus two components contribute to the characteristic: the number of poles in a disc and the average level of modulus of a function on its boundary circle. While  $N(r, \infty, f)$  grows with  $r$ ,  $m(r, \infty, f)$  in general does not. Nevertheless,  $T(r, f)$  is a non-decreasing, convex function of  $\log r$ .

For a meromorphic function  $f$  and a number  $a \in \mathbb{C}$  we can also consider accompanying functions  $m(r, a, f) := m(r, \infty, \frac{1}{f-a})$  and  $N(r, a, f) := N(r, \infty, \frac{1}{f-a})$  measuring mean proximity of  $f$  to  $a$  on the circle and the density of distribution of  $a$ -points of  $f$  in the disc, respectively. The magnitude of  $m(r, a, f)$  can be viewed as measuring mean deviation of  $f$  from the value  $a$  on the circle  $|z| = r$ . Nevanlinna, by application of the notions presented above brought the Poisson–Jensen formula into a form suitable to the new applications ([48]).

**THEOREM 1.1.** *For any function  $f$  meromorphic in the disc  $|z| < R \leq \infty$  the equality*

$$m(r, a, f) + N(r, a, f) = T(r, f) + \phi(r, a) \quad (1)$$

*holds for each  $a \in \mathbb{C}$ , where  $|\phi(r, a)| \leq \log^+ |a| + |\log |c|| + \log 2$ ,  $c$  is the first non-vanishing coefficient of the Laurent expansion of  $f - a$  at zero.*

Theorem 1.1, known as the first main theorem of Nevanlinna, says that not the number of  $a$ -points (which is true for polynomials), but the sum  $m(r, a, f) + N(r, a, f)$  is to some degree independent from the choice of a value  $a \in \overline{\mathbb{C}}$ . This statement is universally true in the plane. In the disc, however, it is possible that  $T(r, f) = O(1)$  ( $r \rightarrow R$ ), in which situation the first main theorem is not very informative. In general, however, if a function  $f$  has got relatively few  $a$ -points, then most of the time it has to be relatively close to the value  $a$ .

The second fundamental theorem of Nevanlinna shows that for most values  $a$  the main role in the invariant sum (1) belongs to the counting function  $N(r, a, f)$  ([48]).

**THEOREM 1.2.** *Let  $f$  be a meromorphic function in  $|z| < R \leq \infty$  and  $\{a_k\}_{k=1}^q \subset \overline{\mathbb{C}}$  a finite set of distinct values. The following inequality is true*

$$\sum_{k=1}^q m(r, a_k, f) \leq 2T(r, f) + S(r, f), \quad (2)$$

where  $S(r, f)$  satisfies the conditions:

- (i) If  $R = \infty$ , then  $S(r, f) = O(\log(rT(r, f)))$  for  $r \rightarrow \infty$ , possibly outside a set  $E \subset [0, \infty)$  such that  $\text{mes } E < \infty$ .
- (ii) If  $R < \infty$ , then  $S(r, f) = O(\log(\frac{1}{R-r}T(r, f)))$  for  $r \rightarrow R$ , possibly outside a set  $E \subset [0, R)$ , such that  $\int_E \frac{dr}{R-r} < \infty$ .

Looking at the remainder term  $S(r, f)$  we can make the following observations. If  $f$  is a function meromorphic in the complex plane then  $S(r, f)$  is relatively small in comparison with  $T(r, f)$  unless  $f$  is a rational function. If  $f$  is meromorphic in a disc the remainder is small if  $f$  fulfils the condition

$$\alpha := \limsup_{r \rightarrow R^-} \frac{T(r, f)}{-\log(R - r)} = \infty. \tag{3}$$

We then call  $f$  an *admissible function*. In a way condition (3) in the disc and condition of transcendency in the plane draw boundaries for application of Nevanlinna theory.

The rate of growth of a polynomial is completely determined by its degree. One of possible ways to describe growth of a meromorphic function is by means of its characteristic. The values

$$\varrho := \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda := \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}$$

are called, respectively, *order* and *lower order* of a meromorphic in the plane function  $f$ . Similarly,

$$\varrho := \limsup_{r \rightarrow R^-} \frac{\log^+ T(r, f)}{-\log(R - r)} \quad \text{and} \quad \lambda := \liminf_{r \rightarrow R^-} \frac{\log^+ T(r, f)}{-\log(R - r)}$$

are *order* and *lower order* of a meromorphic in the disc  $|z| < R$  function  $f$ . In the planar case all rational functions are of zero order,  $\exp z$  is of order 1 and  $\exp(\exp z)$  is of infinite order. For entire functions the order of growth defined with respect to the logarithm of maximum modulus is always equal with the order defined with respect to  $T(r, f)$ .

On the other hand, all functions which are meromorphic in the plane have got bounded characteristic when they are considered in a disc. Moreover, growth order of a holomorphic function defined with respect to

$$\log M(r, f) := \log \max_{|z|=r} |f(z)|$$

does not always coincide with order in comparison with  $T(r, f)$ . For example,  $f(z) = \exp((1 - z)^{-1})$  is of zero order in the unit disc with respect to Nevanlinna characteristic ( $T(r, f) = O(1)$  ( $r \rightarrow 1^-$ )) and of order 1 with respect to  $\log M(r, f)$ .

**2. Exceptional values.** Value distribution theory refers to  $a \in \overline{\mathbb{C}}$  as an exceptional (defective, deficient) value of  $f$  if this value is assumed by  $f$  relatively less frequently than other values. This relative absence of a value can be understood in various ways. We say that  $a$  is a *Picard defective value* if there is only a finite number of  $a$ -points. We shall denote the set of all such values as  $P(f)$ <sup>1</sup>. Obviously, rational functions in the plane

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<sup>1</sup>Notation concerning various sets of exceptional values applied in this survey may differ from those encountered elsewhere. As far as I know there is no uniformly accepted notation in this respect.

take up each value at most a finite number of times, which makes the term ‘exceptional value’ inappropriate in this case.

A *Nevanlinna defective value*, on the other hand, is a value for which the conditions

$$\delta(a, f) := \liminf_{r \rightarrow \infty} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a, f)}{T(r, f)} > 0$$

in the planar case, or

$$\delta(a, f) := \liminf_{r \rightarrow R^-} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r \rightarrow R} \frac{N(r, a, f)}{T(r, f)} > 0$$

in the disc case hold. We put  $D(f)$  for the set of all defective values in this sense. The first theorem of Nevanlinna implies that for all  $a \in \overline{\mathbb{C}}$  there is

$$0 \leq \delta(a, f) \leq 1. \quad (4)$$

The second theorem, on the other hand, means that the set  $D(f)$  is at most countable and

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a, f) \leq 2, \quad (5)$$

apart from the case of non-admissible functions in a disc.

The second main theorem holds for all meromorphic functions in the plane. Nevertheless, there are some limits concerning the set of Nevanlinna defective values, connected with order of a function. By Wiman’s theorem if the lower order of an entire function  $f$  is less than  $1/2$ , then  $\delta(a, f) = 0$  for all  $a \neq \infty$  (see [25], p. 218). Moreover, a meromorphic function of lower order 0 cannot have more than one defective value. On the other hand, we have the following result of Arakelyan ([2]).

**THEOREM 2.1.** *Let  $\varrho$  be an arbitrary number  $1/2 < \varrho < \infty$  and let  $M$  be an arbitrary, at most countable subset of  $\overline{\mathbb{C}}$ . Then there exists an entire function  $f$  such that  $\varrho(f) = \varrho$  and  $D(f) \subset M$ .*

Later on Eremenko proved that for an entire function of finite order  $\varrho > 1/2$  it is possible to have any arbitrary, at most countable set of deficiencies ([16]). The inverse problem of Nevanlinna deficiencies was solved by Drasin by application of quasiconformal mappings in 1976 ([14]). For functions in the unit disc there are two main results concerning the inverse problem by Krutin ([36]) and by Girnyk ([21]).

**THEOREM 2.2.** *Let  $\varrho$  be a non-negative real number, and let  $\{\delta_k\}$  be a sequence of positive numbers less than or equal to one satisfying  $\sum_{k=1}^{\infty} \delta_k \leq 2$ . Let  $\{a_k\}_{k=1}^{\infty}$  be a sequence of distinct complex numbers. Then there is a meromorphic in the unit disc function  $f$  with  $\varrho(f) = \varrho$  such that  $\delta(a_k, f) > \delta_k/4$ ,  $k = 1, 2, \dots$ .*

**THEOREM 2.3.** *Let  $\{\delta_k\}$  be a sequence of positive numbers less than or equal to one satisfying  $\sum_{k=1}^{\infty} \delta_k \leq 1$ . Then there is a function  $f$  holomorphic in the unit disc such that  $\delta(a_k, f) = \delta_k$  and  $\delta(b, f) = 0$  for  $b \neq a_k$  ( $k = 1, 2, \dots$ ).*

Obviously, for transcendental meromorphic functions  $P(f) \subset D(f)$ . For example, for  $f(z) = \exp z$  we have  $P(f) = D(f) = \{0, \infty\}$ . Moreover, we learn from the second main theorem that a meromorphic function may omit at most two values and this way

we obtain Picard theorem as a corollary of the second main theorem. There is even a stronger theorem bounding Picard exceptional values with order (see [25], p. 114).

**THEOREM 2.4.** *A transcendental meromorphic function of zero or non-integer order cannot have more than one Picard exceptional value.*

The defect relation for meromorphic functions in the disc in general is

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a, f) \leq 2 + \frac{1}{\alpha},$$

where  $\alpha$  is the limit defined in (3) (see [48, 56]). Notice that if  $\alpha \neq 0$  the set of Nevanlinna defective values is countable even for non-admissible functions and the set of Picard exceptional values is finite. It is possible, however, to construct a meromorphic function with three and more omitted values, depending on the order of growth.

Let us now consider *Valiron defective values*. We say that  $a \in \overline{\mathbb{C}}$  is exceptional in this sense if the condition

$$\Delta(a, f) := \limsup_{r \rightarrow \infty} \frac{m(r, a, f)}{T(r, f)} > 0$$

in the planar case, or

$$\Delta(a, f) := \limsup_{r \rightarrow R^-} \frac{m(r, a, f)}{T(r, f)} > 0$$

in the disc case holds. We denote the set of defective values in the sense of Valiron by  $V(f)$ . It follows that for admissible functions

$$P(f) \subset D(f) \subset V(f).$$

Although  $V(f)$  need not be countable, Valiron himself proved in [61] that the two-dimensional Lebesgue measure of  $V(f)$  for meromorphic functions in the plane is zero. The set  $V(f)$  is also of zero capacity ([48]).

Proximity of a meromorphic function to a certain value  $a$  may be measured not only by quantity  $m(r, a, f)$ , but also by means of a different metric. For this purpose Petrenko in 1969 introduced a *function of deviation*

$$\mathcal{L}(r, a, f) := \begin{cases} \max_{|z|=r} \log^+ |f(z)| & \text{for } a = \infty, \\ \max_{|z|=r} \log^+ |f(z) - a|^{-1} & \text{for } a \neq \infty. \end{cases}$$

We say that  $a \in \overline{\mathbb{C}}$  is a *deficient value in the sense of Petrenko* if

$$\beta(a, f) := \liminf_{r \rightarrow \infty} \frac{\mathcal{L}(r, a, f)}{T(r, f)} > 0,$$

or similarly,

$$\beta(a, f) := \liminf_{r \rightarrow R^-} \frac{\mathcal{L}(r, a, f)}{T(r, f)} > 0,$$

in the disc case. The quantity  $\beta(a, f)$  is called *Petrenko's deviation*. We denote by  $\Omega(f) := \{a \in \overline{\mathbb{C}} : \beta(a, f) > 0\}$  the set of all values deficient in the sense of Petrenko with respect to  $f$  (see [51, 44]).

It is easy to notice that for all  $a \in \overline{\mathbb{C}}$  we have  $\delta(a, f) \leq \beta(a, f)$ . Therefore  $D(f) \subset \Omega(f)$ . For meromorphic in the plane functions of finite lower order  $\lambda$  we have upper estimates of  $\beta(a, f)$  parallel with Nevanlinna defect relations (4) and (5) ([50]).

**THEOREM 2.5.** *If  $f$  is a meromorphic function of finite lower order  $\lambda$ , then for all  $a \in \overline{\mathbb{C}}$*

$$\beta(a, f) \leq B(\lambda) := \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & \text{if } \lambda \leq 0.5, \\ \pi\lambda & \text{if } \lambda > 0.5. \end{cases} \tag{6}$$

$$\sum_{a \in \overline{\mathbb{C}}} \beta(a, f) \leq 816\pi(\lambda + 1)^2. \tag{7}$$

It should be added that the hypothesis that for entire functions of order  $\rho$  with  $0.5 \leq \rho < \infty$ ,

$$\beta(\infty, f) \leq \pi\rho,$$

was stated by Paley in 1932 and proved in 1969 by Govorov ([26]). In 1990 Marchenko and Shcherba proved a sharp estimate of the sum of Petrenko deviations for functions meromorphic in the plane ([46]), which was the solution of Petrenko’s problem posed in [51].

**THEOREM 2.6.** *If  $f$  is a meromorphic function of finite lower order  $\lambda$ , then*

$$\sum_{a \in \overline{\mathbb{C}}} \beta(a, f) \leq 2B(\lambda).$$

An interesting question was whether for finite lower order functions  $D(f) \neq \Omega(f)$ . The first example of a plane meromorphic function of finite lower order with

$$\beta(0, f) > \delta(0, f) = 0$$

was shown by Grishin in 1975 ([27]). In 1981 Sodin in [57] proved that it is possible to find a meromorphic function of any positive order without Nevanlinna defective values and with any chosen, at most countable set of positive deviations. The following result of Gol’dberg, Eremenko and Sodin from 1987 gives the complete answer to this question ([23, 24]).

**THEOREM 2.7.** *Let  $E_1 \subset E_2 \subset \overline{\mathbb{C}}$  be at most countable sets and let  $\rho > 0$  be any fixed number. There is a plane meromorphic function of order  $\rho$  such that*

$$D(f) = E_1, \quad \Omega(f) = E_2.$$

Valiron proved that, unlike to the set of Nevanlinna’s defects,  $V(f)$  may be of cardinality continuum (see [25], p. 153). However, the set  $V(f)$  is of zero capacity ([48]). The following result showing the relationship between  $V(f)$  and  $\Omega(f)$  in case of functions of finite lower order belongs to Shea and was presented by Fuchs ([20], see also [51]).

**THEOREM 2.8.** *Let  $f(z)$  be a meromorphic in the complex plane function of finite lower order  $\lambda$ . Then for each  $a \in \mathbb{C}$  we have*

$$\beta(a, f) \leq B(\lambda, \Delta) := \begin{cases} \pi\lambda\sqrt{\Delta(2-\Delta)} & \text{if } \lambda \notin \Lambda(\Delta), \\ \frac{\pi\lambda}{\sin \pi\lambda} (1 - (1-\Delta)\cos \pi\lambda) & \text{if } \lambda \in \Lambda(\Delta), \end{cases}$$

where  $\Lambda(\Delta) = \{ \lambda : 0 \leq \lambda \leq 0.5, \sin \frac{\pi\lambda}{2} < \sqrt{\frac{\Delta}{2}} \}$ ,  $\Delta = \Delta(a, f)$ .

COROLLARY 2.9. *For meromorphic functions of finite lower order the following inclusion holds*

$$\Omega(f) \subset V(f).$$

The estimate in Theorem 2.8 is sharp. The appropriate example of a meromorphic function was given by Ryzhkov in [55].

In a way for meromorphic in the plane functions the structure of  $\Omega(f)$  resembles  $D(f)$  in case of finite lower order functions and is close to  $V(f)$  for functions of infinite lower order. For functions of finite lower order  $\Omega(f)$  is at most countable, which follows from (7). In case of a meromorphic function of infinite lower order Petrenko showed that the set of positive deviations may be of cardinality continuum. Nevertheless, it is of zero logarithmic capacity ([51]). Moreover, not only  $\Omega(f)$ , but even  $\Omega(f) \setminus V(f)$  may be of cardinality continuum.

For meromorphic in the unit disc functions the structure of  $\Omega(f)$  may strongly differ from  $D(f)$  also for finite order functions. For instance, for  $f(z) = \exp(\frac{1}{1-z})$ , the deviation  $\beta(\infty, f) = \infty$ . For the set of positive deviations Petrenko obtained the following theorem ([51]).

THEOREM 2.10. *For any value  $\rho$ ,  $0 \leq \rho \leq \infty$ , there exists a meromorphic in a disc function of order  $\rho$  with the set of positive deviations of cardinality continuum.*

For disc functions of big enough order, the set  $\Omega(f)$  is of zero logarithmic capacity. Petrenko in [51] proved this for meromorphic in the disc function of lower order  $\lambda > 6$ , Krytov in [37] was able to extend this statement to the functions of lower order  $\lambda > 2$ . On the other hand, there are examples of functions with  $\Omega(f)$  of positive logarithmic capacity. Petrenko in [51] presented an example of such a function with bounded characteristic, in [9] Chyzykhov constructed an example of a holomorphic in the disc function of any given lower order  $0 < \lambda \leq \frac{\sqrt{5}-1}{2}$  with this property. As far as Valiron exceptional values of meromorphic in the disc functions are concerned, we have the following theorem ([48]).

THEOREM 2.11. *Let  $f$  be a meromorphic function of unbounded characteristic in the unit disc. Then*

$$\lim_{r \rightarrow 1^-} \frac{N(r, a, f)}{T(r, f)} = 1,$$

for values  $a$  possibly outside a set of inner capacity zero.

It follows that if  $T(r, f) \rightarrow \infty$  ( $r \rightarrow 1^-$ ) then  $V(f)$  is a set of inner capacity zero.

For meromorphic in the plane functions of infinite lower order the quantity  $\beta(a, f)$  may be infinite, for instance,  $\beta(\infty, \exp(\exp z)) = \infty$ . In 1994 Bergweiler and Bock proved that for a meromorphic function of infinite lower order

$$\liminf_{r \rightarrow \infty} \frac{\mathcal{L}(r, \infty, f)}{rT'_-(r, f)} \leq \pi,$$

where  $T'_-(r, f)$  is the left derivative of the Nevanlinna's characteristic function ([4]). In connection with their theorem Eremenko in 1997 introduced the quantity

$$b(a, f) = \liminf_{r \rightarrow \infty} \frac{\mathcal{L}(r, a, f)}{A(r, f)},$$

where  $\pi A(r, f)$  is the spherical area, counting multiplicities of the covering, of the image on the Riemann sphere of the disc  $\{z : |z| \leq r\}$  under  $f$ .  $A(r, f)$  is called Ahlfors–Shimizu characteristic of a meromorphic function  $f$ . The connection between the result of Bergweiler and Bock and Eremenko’s definition follows from the fact that

$$rT'_-(r, f) = A(r, f) + O(1) \quad (r \rightarrow \infty).$$

Thus the theorem of Bergweiler and Bock implies that for all  $a \in \overline{\mathbb{C}}$

$$b(a, f) \leq \pi.$$

The value  $b(a, f)$  is called deviation in the sense of Eremenko. In 1997 Eremenko received an analogue of the estimate of the sum of defects for  $b(a, f)$  ([17]).

**THEOREM 2.12.** *For a meromorphic in the plane function such that the set*

$$\{a \in \overline{\mathbb{C}} : b(a, f) > 0\}$$

*contains more than one point the following inequality holds*

$$\sum_{a \in \overline{\mathbb{C}}} b(a, f) \leq 2\pi.$$

In 1998 Marchenko proved that for meromorphic functions of infinite lower order the estimate

$$b(a, f) \leq \pi \sqrt{\Delta(a, f)(2 - \Delta(a, f))}$$

holds for each  $a \in \overline{\mathbb{C}}$  ([40]). As a result in this case each value exceptional in the sense of Eremenko is also exceptional in the sense of Valiron.

In 2000 Marchenko and Nikolenko examined the properties of Eremenko deviation

$$b(a, f) := \lim_{r \rightarrow 1} \frac{\mathcal{L}(r, a, f)}{T'_-(r, f)}$$

for meromorphic in the unit disc functions. They obtained an exact upper estimate of this value and the analogue of Nevanlinna defect relationships (4) and (5) ([45]).

**THEOREM 2.13.** *Let  $f$  be a meromorphic in the disc function of lower order  $\lambda > 0$ . Then, for all  $a \in \overline{\mathbb{C}}$*

$$b(a, f) \leq \begin{cases} \pi \cos^{-1-\lambda} \frac{\pi}{2(\lambda + 1)} & \text{if } \lambda < \infty, \\ \pi & \text{if } \lambda = \infty. \end{cases}$$

Moreover,

$$\sum_{(a)} b(a, f) \leq \begin{cases} 2\pi \cos^{-1-\lambda} \frac{\pi}{2(\lambda + 1)} & \text{if } \lambda < \infty, \\ 2\pi & \text{if } \lambda = \infty. \end{cases}$$

Countability of the set of exceptional values in the sense of Eremenko for non-zero order functions meromorphic in the unit disc follows as a corollary of this result.

**3. Exceptional functions.** One of conjectures of Nevanlinna was that in the second main theorem it is possible to replace constants  $a_k$  with meromorphic functions  $a_k(z)$

under the condition  $T(r, a_k) = o(T(r, f))$  ( $r \rightarrow \infty$ ). We call such functions  $a_k$  *small with respect to  $f$*  and denote the set of all small functions of  $f$  as  $\mathcal{S}(f)$ . Then we put

$$\delta(a_k, f) := \delta(0, f - a_k).$$

Other values, such as  $m(r, a_k, f)$ ,  $N(r, a_k, f)$ ,  $\mathcal{L}(r, a_k, f)$ ,  $\beta(a_k, f)$ ,  $b(a_k, f)$  are defined in a similar manner. Nevanlinna himself proved so-called theorem on three small functions by methods which could not be extended to general case. Since 1920's the problem of generalizing Nevanlinna's second main theorem has been approached a number of times. In 1964 Chuang succeeded in obtaining such a generalisation for entire functions, for the first time introducing a Wronskian into the solution of the problem ([8]). In 1981 Osgood proved a result for meromorphic functions by different methods. His paper is also important as it contributed to growing interest in the analogies between Nevanlinna theory and diophantine approximation theory ([49]). In a paper from the same year Yang Le applied the spread relation to prove countability of the set of small defective functions of a meromorphic function ([63]). In 1986 the following extension of the second main theorem was shown by Frank and Weissenborn ([19]).

**THEOREM 3.1.** *Let  $f$  be a transcendental meromorphic function. Then for distinct rational functions  $q_1, \dots, q_k$  and every  $\varepsilon > 0$  we have*

$$m(r, f) + \sum_{\nu=1}^k m(r, q_\nu, f) \leq (2 + \varepsilon)T(r, f)$$

for  $r \rightarrow \infty$ , possibly except for  $r$  in a set of finite linear measure.

Also in 1986 Steinmetz proved a result in which small functions need not be rational ([58]).

**THEOREM 3.2.** *Let  $f$  be a non-constant meromorphic function in the plane and let  $\{a_\nu\}_{\nu=1}^k$  be a set of pairwise distinct meromorphic functions such that for  $1 \leq \nu \leq k$  we have  $T(r, a_\nu) = o(T(r, f))$  ( $r \rightarrow \infty$ ). Then for every  $\varepsilon > 0$ ,*

$$m(r, f) + \sum_{\nu=1}^k m(r, a_\nu, f) \leq (2 + \varepsilon)T(r, f)$$

for  $r \rightarrow \infty$ , possibly except for  $r$  in a set of finite linear measure.

The exact analogue of the second main theorem, including ramification factor, was finally obtained by Yamanoi in 2004 ([62]).

**THEOREM 3.3.** *Let  $f$  be a non-constant meromorphic function on  $\mathbb{C}$  and let  $a_1, \dots, a_k$  be distinct meromorphic functions on  $\mathbb{C}$ . Assume that  $T(r, a_\nu) = o(T(r, f))$  ( $r \rightarrow \infty$ ) for all  $\nu = 1, \dots, k$ . Then we have the second main theorem,*

$$(k - 2 - \varepsilon)T(r, f) \leq \sum_{\nu=1}^k \bar{N}(r, a_\nu, f)$$

for all  $\varepsilon > 0$ , for  $r \rightarrow \infty$ ,  $r \notin E$ ,  $\int_E d \log r < \infty$ , and the defect relation,

$$\sum_{a \in \mathcal{S}(f)} (\delta(a, f) + \theta(a, f)) \leq 2.$$

For exceptional functions in the sense of Petrenko the problem of obtaining the exact analogue of Theorem 2.6 has not been solved in generality. In 2007 the following theorem appeared ([10]).

**THEOREM 3.4.** *Let  $f(z)$  be a transcendental entire function of finite lower order  $\lambda$  and let  $\mathcal{Q}$  denote the set of all rational functions. The set  $\{q \in \mathcal{Q} : \beta(q, f) > 0\}$  is at most countable. Moreover, for distinct rational functions  $\{q_\nu\}$  we have*

$$\sum_{q \in \mathcal{Q}} \beta(q_\nu, f) \leq B(\lambda).$$

The theorem can also be formulated in a slightly more general way, where  $f$  is not necessarily entire, but  $N(r, f) = o(T(r, f))$  ( $r \rightarrow \infty$ ) ([11]).

In 2009 Kaluzhynova and Marchenko analysed the set of exceptional rational functions in the sense of Eremenko for entire functions of non-zero lower order and obtained the following estimate ([33]).

**THEOREM 3.5.** *Let  $f$  be an entire function of lower order  $\lambda > 0$ . Let also  $\mathcal{Q}$  be the set of all rational functions. Then the set  $\{q \in \mathcal{Q} : b(q, f) > 0\}$  is at most countable and*

$$\sum_{q \in \mathcal{Q}} b(q, f) \leq \begin{cases} \frac{\pi}{\sin \pi \lambda} & \text{if } 0 < \lambda \leq 1/2, \\ \pi & \text{if } 1/2 < \lambda \leq \infty. \end{cases}$$

It should be added that Theorem 3.5 holds in a slightly more general case when  $f$  is a meromorphic function with  $N(r, \infty, f) = o(T(r, f))$  ( $r \rightarrow \infty$ ) ([34]).

**4. Asymptotic values.** We call a value  $a \in \overline{\mathbb{C}}$  an *asymptotic value* of a meromorphic in the plane function  $f$  if there exists a continuous curve  $\Gamma \subset \mathbb{C}$ ,

$$\Gamma : z = z(t), \quad 0 \leq t < \infty,$$

joining 0 and  $\infty$ , such that

$$\lim_{z \rightarrow \infty, z \in \Gamma} f(z) = \lim_{t \rightarrow \infty} f(z(t)) = a.$$

A pair  $\{a, \Gamma\}$ , defined as above, is called an *asymptotic spot* of  $f$ . Two asymptotic spots  $\{a_1, \Gamma_1\}$  and  $\{a_2, \Gamma_2\}$  are considered equal if  $a_1 = a_2 = a$  and there exists a sequence of continuous curves  $\gamma_k$  with one end of each  $\gamma_k$  belonging to  $\Gamma_1$  and the other to  $\Gamma_2$ , and

$$\lim_{k \rightarrow \infty} \min_{z \in \gamma_k} |z| = \infty, \quad \lim_{z \rightarrow \infty, z \in \bigcup_k \gamma_k} f(z) = a.$$

The set of all asymptotic values of  $f$  is denoted by  $\text{As}(f)$  ([31]).

Although for every entire function  $\infty$  is an asymptotic value, the size of the whole set of asymptotic values depends on order. If an entire function is of infinite lower order the set of asymptotic spots may be infinite. It is true for  $f(z) = \exp(\exp z)$ . As early as 1918 Gross showed an example of an entire function of infinite order for which the set of asymptotic values coincides with the extended complex plane ([28]). In 1957 Heins proved that for every analytic set  $A$ , there is an entire function whose set of asymptotic values is  $A \cup \{\infty\}$  ([31]). Notice also that, by Wiman’s theorem, if  $f$  has got a finite asymptotic value then  $\varrho(f) \geq 1/2$ . The bound here is sharp, which the example of  $\sin \sqrt{z}/\sqrt{z}$  with an

asymptotic value 0 shows. If we consider entire functions of finite lower order, a classical theorem of Denjoy–Carleman–Ahlfors gives the sharp upper estimate of the number of asymptotic spots.

**THEOREM 4.1.** *If an entire function of finite lower order  $\lambda$  has got  $p$  different asymptotic spots  $\{a_j, \Gamma_j\}$  ( $a_j \in \mathbb{C}, 1 \leq j \leq p$ ), then*

$$\liminf_{r \rightarrow \infty} r^{-p/2} T(r, f) > 0.$$

**COROLLARY 4.2.** *An entire function of finite lower order  $\lambda$  cannot have more than  $[2\lambda]$  different asymptotic spots connected with finite values.*

Denjoy made this conjecture in 1907 and was able to prove it in a special case when the asymptotic curves are rays ([13]). Carleman addressed the general case, but his estimate was not sharp ([6]). Finally, in 1929 Ahlfors proved the theorem ([1]). For his works he was given the Fields medal in 1936.

In case of meromorphic functions we may have an infinite or even uncountable set of asymptotic values, which the following theorem of Eremenko shows ([15]).

**THEOREM 4.3.** *For every  $\varrho \geq 0$ , there is a meromorphic function of order  $\varrho$  whose set of asymptotic values is  $\mathbb{C}$ .*

Recently, Canton, Drasin and Granados in [5] proved a more general result improving earlier results of Heins and Eremenko.

**THEOREM 4.4.** *For every analytic (Suslin) set  $A$ , and every  $\varrho \geq 0$  there exists a meromorphic function of order  $\varrho$  whose set of asymptotic values is equal to  $A$ .*

Asymptotic behaviour of functions meromorphic in a disc is more complicated. An asymptotic curve may join zero either with a point  $w$  with  $|w| = 1$  or with an arc of the unit circle. If for  $a \in \overline{\mathbb{C}}$  there exists an asymptotic path  $\Gamma$  such that  $f(z) \rightarrow a$  when  $|z| \rightarrow 1$  then  $a$  is an asymptotic value of  $f$ . Thus it is not only possible that various asymptotic values exist if we approach an arbitrary point on the unit circle via different paths, but also we have infinite number of possible ends of asymptotic curves.

Mazurkiewicz in 1936 proved that for a holomorphic in the unit disc function  $f$ ,  $\text{As}(f)$  must be an analytic set ([47]). Also, by MacLane's result a holomorphic function must have at least one asymptotic value. The value, however, need not be  $\infty$  if  $f$  is unbounded ([39]). MacLane in the same paper also proved that a holomorphic in the unit disc function of order  $\varrho(f) < 1$  has got a finite number of asymptotic spots connected with each boundary point.

Although it is possible that at a point of the unit circle a holomorphic function may have more than one asymptotic value, such points are relatively rare, which the following result of Bagemihl shows ([3]).

**THEOREM 4.5.** *Let  $f$  be holomorphic in the unit disc. Then the set  $E$  of the points of the unit circle at which  $f$  has more than one asymptotic value is at most countable.*

What is more, a holomorphic in the unit disc function may have infinitely many asymptotic values at a point on the unit circle. Such a function was noted by MacLane in [39] with the set of asymptotic values at a fixed point equal with  $\mathbb{C}$ .

In 1961 MacLane showed that a function meromorphic in the unit disc need not have any asymptotic values at all ([38]).

**THEOREM 4.6.** *There exist functions meromorphic in the unit disc which have no asymptotic values, finite or infinite. Further, the Nevanlinna characteristic of such functions may have arbitrarily slow growth.*

Looking at the inverse problem, Kierst was the first to show that for any analytic set  $A$  we can find a meromorphic in the unit disc function  $f$  with  $\text{As}(f) = A$ . It is also true for holomorphic functions as long as  $\{\infty\} \subset A$  ([35]). However, in general not for every analytic set  $A$  it is possible to find a holomorphic function  $f$  with  $\text{As}(f) = A$  ([53, 54]).

An asymptotic value does not have to be an exceptional value. For example, an entire function  $f(z) = \sin z/z$  has got an asymptotic value  $a = 0$  (with two spots corresponding with asymptotic curves  $\Gamma_1 : z(t) = t$ ,  $t \in [0, \infty)$ ,  $\Gamma_2 : z(t) = -t$ ,  $t \in [0, \infty)$ ), while  $a = 0$  is not exceptional in any sense. The following theorem from 1914 due to Iversen is a result in the opposite direction ([32]).

**THEOREM 4.7.** *If  $a$  is a Picard exceptional value of a transcendental meromorphic function  $f$ , then  $a$  is also an asymptotic value of  $f$ , that is  $P(f) \subset \text{As}(f)$ .*

Functions without asymptotic values in the unit disc present an interesting value distribution property ([12, 7]).

**THEOREM 4.8.** *A function meromorphic in the unit disc which has no asymptotic values assumes every value infinitely often. Further, in every neighbourhood of each point of the unit circle the function assumes all except perhaps two values.*

It follows that such a function has got no Picard exceptional values.

Initially, a conjecture of Rolf Nevanlinna that each Nevanlinna deficient value is an asymptotic value seemed probable. An easy consequence of the first fundamental theorem is that if the number of  $a$ -points in the disc  $|z| \leq r$  is relatively smaller then the function compensates for this by being relatively closer to  $a$  on the circle  $|z| = r$ . It seemed natural to assume that an asymptotic curve might exist. For example, for  $f(z) = \exp z$  we have  $\text{As}(f) = D(f)$  and there are two asymptotic spots of  $f$ . However, non-asymptotic and deficient values in the sense of Nevanlinna may exist for functions of any order of growth  $\rho > 0$ . Teichmüller in 1939 showed that a function of infinite order can have a Nevanlinna deficient but not asymptotic value ([59]). Hayman showed this for functions of finite order ([29]) and Ter-Israelyan in 1971 for zero-order functions ([60]). Hayman in 1978 even gave an example of meromorphic function where  $\delta(\infty, f) = 1$  and  $\infty \notin \text{As}(f)$  ([30]). What is more, the set  $D(f) \setminus \text{As}(f)$  may be infinite, which was shown by Arakelyan in 1966 ([2]).

Unlike defective values in the sense of Nevanlinna, Petrenko deficient values are directly related with asymptotic values—so-called *strong asymptotic values*. In 1999 Marchenko introduced the notion of a strong asymptotic value of an entire function and obtained estimates of the number of strong asymptotic spots of entire functions of finite lower order ([41]).

DEFINITION 4.9. We say that  $a \in \overline{\mathbb{C}}$  is a *strong asymptotic value of an entire function*  $f$ , if there exists a continuous curve  $\Gamma : z = z(t)$ ,  $0 \leq t < \infty$ , joining 0 and  $\infty$  such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\log |f(z(t)) - a|^{-1}}{\log M(|z(t)|, f)} &= 1, & \text{if } a \neq \infty, \\ \lim_{t \rightarrow \infty} \frac{\log |f(z(t))|}{\log M(|z(t)|, f)} &= 1, & \text{if } a = \infty, \end{aligned}$$

where  $M(r, f) = \max_{|z|=r} |f(z)|$ .

It means that  $a \in \overline{\mathbb{C}}$  is a strong asymptotic value if the speed with which  $f$  approaches  $a$  on an asymptotic curve is equivalent with  $\log M(r, f)$ .

THEOREM 4.10. Let  $f$  be a non-constant entire function, of finite lower order  $\lambda$  with  $p$  different strong asymptotic spots  $\{\infty, \Gamma_j\}$ ,  $j = 1, 2, \dots, p$ . Then

$$p \leq \begin{cases} \max\left(1, \left[\frac{\pi\lambda}{\beta(\infty, f)}\right]\right) & \text{if } \lambda \leq 0.5, \\ \left[\frac{\pi\lambda}{\beta(\infty, f)}\right] & \text{if } \lambda > 0.5. \end{cases}$$

Here  $[x]$  is the integer part of  $x$ .

THEOREM 4.11. If  $f$  is an entire function of infinite lower order and has  $p$  different strong asymptotic spots  $\{\infty, \Gamma_j\}$ ,  $j = 1, 2, \dots, p$ , then

$$p \leq \left[\frac{\pi}{b(\infty, f)}\right].$$

COROLLARY 4.12. If  $f$  is an entire function satisfying the condition  $b(\infty, f) > 0$ , then the set of different strong asymptotic spots  $\{\infty, \Gamma_j\}$  of  $f(z)$  is finite.

We can easily see that the set of strong asymptotic spots  $\{\infty, \Gamma_j\}$  is infinite for the function  $\exp(\exp z)$ . Therefore the condition  $b(\infty, f) > 0$  is essential.

By Theorem 4.3 a meromorphic function, even of finite order need not have a finite set of asymptotic values. In 2004 Marchenko defined strong asymptotic values for meromorphic functions and proved an estimate of the number of strong asymptotic spots for functions of finite order ([42]).

DEFINITION 4.13. We say that  $a \in \overline{\mathbb{C}}$  is an  $\alpha_0$ -strong asymptotic value of a meromorphic function  $f$  if there exists a continuous curve  $\Gamma : z = z(t)$ ,  $0 \leq t < \infty$ , joining 0 with  $\infty$ , such that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{\log |f(z(t)) - a|^{-1}}{T(|z(t)|, f)} &= \alpha(a) \geq \alpha_0 > 0, & \text{if } a \neq \infty; \\ \liminf_{t \rightarrow \infty} \frac{\log |f(z(t))|}{T(|z(t)|, f)} &\geq \alpha_0 > 0, & \text{if } a = \infty. \end{aligned}$$

The speed of approach of  $f$  towards  $a$  on an asymptotic curve is here comparable with Nevanlinna characteristic.

As it is easy to see, if  $a$  is an  $\alpha_0$ -strong asymptotic value of  $f$ , then  $\beta(a, f) \geq \alpha_0$ . It means that  $a$  is also a defective value in the sense of Petrenko.

**THEOREM 4.14.** *Let  $f$  be a meromorphic function of finite lower order  $\lambda$  and  $\{a_\nu, \Gamma_\nu\}$  ( $\nu = 1, 2, \dots, k$ )—distinct  $\alpha_0$ -strong asymptotic spots of  $f$ . Then  $k \leq \lceil \frac{2B(\lambda)}{\alpha_0} \rceil$ .*

Looking back now at Theorem 2.5, we can see that it leads to the conclusion that the number of strong asymptotic values of a meromorphic function of finite lower order  $\lambda$  does not exceed  $\frac{816\pi(\lambda+1)^2}{\alpha_0}$ . By Theorem 2.6, the upper bound is  $\frac{2B(\lambda)}{\alpha_0}$ . Theorem 4.14, however is stronger. Let us also notice that the inequality  $\alpha(a) \geq \alpha_0 > 0$  in the definition of strong asymptotic values is necessary. As Gol'dberg in [22] showed, there is a meromorphic function of finite lower order with infinite set of asymptotic values, asymptotic curves given by equalities  $\{\arg z = \theta_n\}$  and fulfilling the inequality

$$\liminf_{r \rightarrow \infty} \frac{\log |f(re^{i\theta_n}) - a_n|^{-1}}{T(r, f)} > 0.$$

**5. Asymptotic functions. A conjecture of Denjoy.** Denjoy made the following conjecture concerning asymptotic functions:

*If  $f$  is an entire function of finite lower order  $\lambda$  and  $a_1, a_2, \dots, a_k$  are entire functions of order less than  $1/2$  such that  $f(z) - a_j(z) \rightarrow 0$  for  $z$  tending to infinity along the path  $\Gamma_j$ ,  $1 \leq j \leq k$ , then  $k \leq \lfloor 2\lambda \rfloor$ .*

This restriction of the number of asymptotic functions is true for asymptotic functions of order less than  $1/4$ , which was shown by Fenton in 1983 ([18]).

**THEOREM 5.1.** *If  $f$  is an entire function having  $k$  distinct asymptotic functions of order less than  $1/4$  then the condition*

$$\alpha = \liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{k/2}} > 0$$

*holds. Moreover, if  $\alpha < \infty$  then  $f$  has order  $k/2$ .*

In general case the conjecture is still open until now.

Denjoy's conjecture contains an assumption that the asymptotic functions should be of order less than  $1/2$ , which is essential. If, for example,  $f(z) = \frac{\sin \sqrt{z}}{\sqrt{z}}$  ( $f(0) = 1$ ) all the functions  $b_c(z) = c \sin \sqrt{z}/\sqrt{z}$ ,  $c \in \mathbb{C}$ , are its asymptotic functions. Indeed,

$$f(x) - b_c(x) = (1 - c) \frac{\sin \sqrt{x}}{\sqrt{x}} \rightarrow 0 \quad \text{for } x \rightarrow +\infty.$$

Instead of strong asymptotic values we may consider strong asymptotic functions and look for an estimate parallel with Denjoy's conjecture. The following estimates of the number of functional asymptotic spots have been shown in [11].

**THEOREM 5.2.** *Let  $f$  be a transcendental meromorphic function of finite lower order  $\lambda$ , such that  $N(r, f) = o(T(r, f))$  for  $r \rightarrow \infty$ , possibly outside an exceptional set of finite linear measure. The number  $m$  of distinct  $\alpha_0$ -strong rational asymptotic spots of  $f$  is finite and  $m \leq \lceil \frac{B(\lambda)}{\alpha_0} \rceil$ .*

**THEOREM 5.3.** *Let  $f$  be a meromorphic function of finite lower order  $\lambda$ . For polynomials of a degree not overcoming  $d$ , the number  $m$  of distinct  $\alpha_0$ -strong polynomial asymptotic spots of  $f$  is finite and  $m \leq \lceil \frac{(d+2)B(\lambda)}{\alpha_0} \rceil$ .*

In 2008 Marchenko introduced the notion of an  $A$ -strong asymptotic function of a meromorphic function and an estimate of the number of  $A$ -strong asymptotic rational functions. A meromorphic function  $a(z)$  is called an  $A$ -strong asymptotic function of a meromorphic function  $f(z)$  if there exists a curve  $\Gamma \subset \mathbb{C}$  given by the equation

$$z = z(t), \quad 0 \leq t < \infty, \quad z(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

such that

$$\liminf_{t \rightarrow \infty} \frac{\log |f(z(t)) - a(z(t))|^{-1}}{A(|z(t)|, f)} \geq \alpha_0 > 0.$$

Thus  $a(z)$  is an  $A$ -strong asymptotic function of a meromorphic function  $f(z)$  if the rate of convergence to zero of the difference  $f(z) - a(z)$  on the asymptotic curve is comparable with the growth of the Ahlfors–Shimizu characteristic  $A(r, f)$  ([43]).

**THEOREM 5.4.** *Let  $f(z)$  be an entire function of infinite lower order. Then the number of  $A$ -strong asymptotic rational functions is finite and less or equal to  $[\frac{\pi}{\alpha_0}]$ , where  $[x]$  is the integer part of  $x$ .*

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