CONSTRUCTIVE APPROXIMATION OF FUNCTIONS
BANACH CENTER PUBLICATIONS, VOLUME 107
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2015

TANGENTIAL MARKOV INEQUALITY IN L^p NORMS

AGNIESZKA KOWALSKA
Institute of Mathematics, Pedagogical University
Podchorążych 2, 30-084 Kraków, Poland
E-mail: kowalska@up.krakow.pl

Dedicated to Professor Wiesław Pleśniak on his 70th birthday

Abstract. In 1889 A. Markov proved that for every polynomial p in one variable the inequality $\|p'\|_{[-1,1]} \leq (\deg p)^2\|p\|_{[-1,1]}$ is true. Moreover, the exponent 2 in this inequality is the best possible one. A tangential Markov inequality is a generalization of the Markov inequality to tangential derivatives of certain sets in higher-dimensional Euclidean spaces. We give some motivational examples of sets that admit the tangential Markov inequality with the sharp exponent. The main theorems show that the results on certain arcs and surfaces, which have been proved earlier for the uniform norm, can be generalized to L^p norms.

1. Introduction. Generalizations of the classical Markov inequality have been the subject of research for 125 years. The development of the theory of the multivariate Markov inequality and the search for the best exponent in this inequality, called the Markov exponent, were very extensive in the second half of the twentieth century. Currently, this inequality is still being generalized in many different ways. Such research was carried on for curves and submanifolds in $\mathbb{R}^N$ (Bos, Brudnyi, Levenberg, Milman, Taylor, Totik, Baran, Pleśniak, Kosek, Coman, Poletski, Gendre); for Julia sets (Kosek); in Banach spaces (Sarantopoulos, Harris, Muñoz, Baran); in L^p norms (Tamarkin, Hille, Szegö, Goetgheluck, Baran, Sroka) in o-minimal structures (Pleśniak, Pierzchała). More information on the research carried out in connection with the various generalizations of the Markov inequality, particularly comprehensive bibliography on these topics, can be found in [P].

Searching for the sharp exponent is more difficult if we consider the Markov-type inequality in L^p norms. In this case, many more questions remain unanswered. The first inequality generalized to the case L^p norms was Bernstein's inequality for trigonometric polynomials [Z]. The classical Markov inequality in L^p norms was shown a few years later.

2010 *Mathematics Subject Classification*: Primary 41A17, 14P10; Secondary 41A25, 14P05.
Key words and phrases: tangential Markov inequality, semialgebraic sets.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc107-0-13

THEOREM 1.1 ([HST]). *For every* $p \in [1, \infty)$ *and a polynomial* P *such that* $\deg P \leq n$,

$$\|P'\|_p \leq C_p n^2 \|P\|_p, \tag{1}$$

where $\|P\|_p = \left(\int_{-1}^{1} |P(x)|^p \, dx\right)^{1/p}$.

Similarly, as in the uniform norm case, the exponent 2 of n^2 is the best possible for L^p norms. Recently, it was proved, that the constants C_p in the inequality (1) can be chosen such that $\lim_{p\to\infty} C_p = 1$ ([B]). Research related to the Markov exponent in L^p norms was also conducted for the multidimensional Markov property. It is known that for fat sets with locally Lipschitz boundary, the Markov exponent in L^p norms equals 2 ([Go2]). Moreover, Goetgheluck has shown the Markov inequality in L^p norms for UPC sets ([Go1]). It is unknown whether Goetgheluck's result is optimal, but the methods he used, have proved to be useful in many other cases. For sharp Markov exponent on the complex plane using area measures we refer to Theorem 10.1 in [TV]. In connection with the exploration of the sharp Markov exponent for sets with cusps in L^p norms, a number of important relationships were discovered, including the polynomial identity proved by Milówka ([M]) and their generalizations shown in [BMO]. Questions about the Markov type property in L^p norms can also be considered for the tangential Markov inequality.

Recall that a compact set $K \subset \mathbb{R}^N$ is said to admit the *tangential Markov inequality with an exponent* l if there exists a positive constant M depending only on K such that for all polynomials p,

$$\|D_T p\|_K \leq M (\deg p)^l \|p\|_K,$$

where $D_T p$ denotes any (unit) tangential derivative of p along K, $\|f\|_K = \sup_{x\in K} |f(x)|$ is the supremum norm of a function f on a set K. According to [BLMT1], a $\mathcal{C}^\infty$ submanifold K of $\mathbb{R}^N$ admits the tangential Markov inequality with the exponent one if and only if K is algebraic.

For a semialgebraic submanifold K, $\mathcal{C}^\infty$ class means that K cannot have singular points. In [BLMT2] we can find examples of curves which admit the tangential Markov inequality with the sharp exponent greater than 1 and examples of curves which do not satisfy this inequality with any finite exponent l.

EXAMPLE 1.2 ([BLMT2]). Consider the curves $\gamma_r : y = x^r$, $0 \leq x \leq 1$, where $r \in \mathbb{R}$ and $r \geq 1$. Then

- for $r = \frac{q}{p}$ with positive integers $q \geq p$ in lowest terms, γ_r admits the tangential Markov inequality with the exponent $l := 2p$ (moreover, this is the best possible),
- for r irrational, γ_r does not admit the tangential Markov inequality of any exponent.

A generalization of this example is the result by L. Gendre, who proved that every singular algebraic curve in $\mathbb{R}^N$ admits the local tangential Markov inequality at each of its points. Moreover, he showed that the Markov exponent at a point of a real algebraic curve A is less than or equal to twice the multiplicity of the smallest complex algebraic curve containing A (see [Ge]). In Example 1.2 the exponent is equal to twice the multiplicity and is the best possible. Now we show that there are curves for which the best exponent is less than twice the multiplicity.

Example 1.3. Consider a family of curves

$$K_q = \{(t^2, t^q) : t \in [-1, 1] \text{ and } q \geq 2 \text{ is an odd number}\}.$$

By Theorem 3.2 in [K], each such curve admits the tangential Markov inequality with the exponent 2. The multiplicity of the smallest complex algebraic curve containing K_q is also equal to 2. Now we show that 2 is the best exponent for K_q. Indeed, for these curves and for polynomials $P_n(x, y) = T_n(x-1)$, where T_n denotes the k-th Chebyshev polynomials, we have $|D_T P_n(0, 0)| = |\frac{\partial}{\partial x} P_n(0, 0)| = |T_n'(-1)| = n^2$ and $\deg P_n = \deg T_n = n$, which implies that for such curves K_q the exponent cannot be less than 2. Hence the sharp exponent is equal to 2, which is less than twice the multiplicity of the smallest complex algebraic curve containing K_q, i.e. 4.

One may expect that the tangential Markov inequality does not occur on sets which are not semialgebraic. However, this conjecture is false, the tangential Markov inequality with the sharp exponent 4 on some exponential curves has been shown in [BBLT].

A simpler proof of the tangential Markov inequality for semialgebraic curves can be found in [K]. The characterization of semialgebraic curves, proved by Baran and Pleśniak in [BP1], plays an important role in this proof.

In [BP2] Baran and Pleśniak solved the problem of the characterization of semialgebraic sets of a higher dimension in the class of subanalytic sets. In [K] this theorem was used to prove the tangential Markov inequality on semialgebraic surfaces in $\mathbb{R}^3$ with finitely many singular points.

2. Preliminaries.

We recall that a subset of $\mathbb{R}^N$ is *semialgebraic* if it is the union of finitely many subsets of the form

$$\{P = 0\} \cap \bigcap_{i=1}^{l} \{Q_i > 0\},$$

where $l \in \mathbb{N}$ and $P, Q_1, \dots, Q_l \in \mathbb{R}[X_1, \dots, X_N]$.

We call a function *semialgebraic* if it has a semialgebraic graph. For compact semialgebraic sets we have the Łojasiewicz inequality, which is very convenient in applications.

Theorem 2.1 ([S], Theorem 5.2.2). *Let $K \subset \mathbb{R}^N$ be a compact semialgebraic set and let $f : K \longrightarrow \mathbb{R}$ be a semialgebraic function continuous on K. Then there exist constants $C > 0$ and $L \in \mathbb{N}$ such that for every $u \in K$ we have*

$$|f(u)| \geq C \cdot \operatorname{dist}(u, Z)^L,$$

where $Z := \{x \in K : f(x) = 0\}$.

Let us notice that Theorem 2.1 is a special case of a more general Łojasiewicz inequality, which has been proved for any $\mathbb{R}$-analytic function f on an open set $U \subset \mathbb{R}^N$ and any compact subset K of U ([L], Theorem 17). In this section the Łojasiewicz inequality will be used to show that curves and surfaces which admit an analytic parametrization are not too thin.

Let K be a compact curve in $\mathbb{R}^N$ and let $I = [-1, 1]$. Following [BP1], K is said to admit an *analytic parametrization* if there exist $r \in \mathbb{N}$, $\gamma > 1$ and $\mathbb{R}$-analytic maps

$\varphi_j = (\varphi_{j1}, \ldots, \varphi_{jN}) : \gamma I \longrightarrow K$, $j = 1, \ldots, r$, such that each $\varphi_j|_I$ is a bijection onto $\varphi_j(I)$ and

$$\bigcup_{j=1}^{r} \varphi_j(I) = K.$$

By Puiseux's theorem one can prove that every semialgebraic curve in $\mathbb{R}^N$ admits an analytic parametrization.

Lemma 2.2. *Let K be a semialgebraic arc having the analytic parametrization*

$$\Phi(t) = (\varphi_1(t), \ldots, \varphi_N(t))$$

in the neighborhood of $I = [-1, 1]$ such that in the neighborhood of 0 (0 *is the only singular point*) $\varphi_i(t) = \alpha_{i0} + \sum_{n=k}^{\infty} \alpha_{in} t^n$ *and $\alpha_{1k} = 1$. Then for every constant $M \in (0, 1]$ there exists a positive constant c_0 such that for every $t_0 \in I$ and $n \in \mathbb{N}$,*

$$\int_{I \cap [t_0 - M/n, t_0 + M/n]} \Big(\sum_{i=1}^{N} |\varphi_i'(t)|^2 \Big)^{1/2} dt \ge c_0 n^{-k}. \tag{2}$$

Proof. Let

$$A_{nt_0} := I \cap \Big[t_0 - \frac{M}{n}, t_0 + \frac{M}{n} \Big].$$

By the assumption $\varphi_1 \not\equiv \text{const}$ and Φ is analytic on an open interval containing I, so there exists a positive constant c_1 such that

$$\Big(\sum_{i=1}^{N} |\varphi_i'(t)|^2 \Big)^{1/2} \ge c_1 |t|^{k-1}.$$

Moreover,

$$\int_{A_{nt_0}} |t|^{k-1}\, dt \ge \frac{M^k}{k} n^{-k}.$$

Hence, we get (2), where $c_0 = \frac{c_1 M^k}{k}$. ∎

Now let $\mathbb{B}^m(u, R) := \{x \in \mathbb{R}^m : \|x - u\| \le R\}$, $\mathbb{B}^m(R) := \mathbb{B}^m(0, R)$, $\mathbb{B}^m := \mathbb{B}^m(1)$ and K be a compact subset of $\mathbb{R}^N$. Following [BP2], K is said to have an *analytic parametrization of dimension m*, $1 \le m \le n$, if there exist $\rho > 1$, $r \in \mathbb{N}$ and $\mathbb{R}$-analytic maps $\phi_j = (\phi_{j1}, \ldots, \phi_{jn}) : \mathbb{B}^m(\rho) \longrightarrow K$, $j = 1, \ldots, r$, such that for each $j = 1, \ldots, r$ we have $\operatorname{rank} \phi_j = m$ and

$$K = \bigcup_{j=1}^{r} \phi_j(\mathbb{B}^m).$$

Let $\mathbb{M}$ be an m-dimensional real-analytic manifold of $\mathbb{R}^N$. It is known that every compact semialgebraic subset of $\mathbb{M}$ of pure dimension m admits an analytic parametrization in the sense of the above definition. It is a consequence of the Hironaka Rectilinearization Theorem.

Lemma 2.3. *Let $S \subset \mathbb{R}^3$ be a compact surface with an analytic parametrization*

$$\Phi(u_1, u_2) = \big(\varphi_1(u_1, u_2), \varphi_2(u_1, u_2), \varphi_3(u_1, u_2)\big), \quad (u_1, u_2) \in \mathbb{B}^2,$$

such that $\Phi(0,0)=0$ *is the only singular point. Then for each positive constant* M *there exists a constant* c_0 *such that for any* $u' \in \mathbb{B}^2$ *and all* $n \in \mathbb{N}$,

$$\iint_{\mathbb{B}^2\cap\mathbb{B}(u',M/n)} \left\| \frac{\partial}{\partial u_1}\Phi(u_1,u_2) \times \frac{\partial}{\partial u_2}\Phi(u_1,u_2) \right\| du_1\, du_2 \geq c_0 n^{-s}, \tag{3}$$

where $s := L+2$ *and* L *is the exponent from the Łojasiewicz inequality.*

Proof. We define

$$A_{nu'} := \mathbb{B}^2 \cap \mathbb{B}(u', M/n).$$

Then, by the Łojasiewicz inequality, there exist constants $C>0$ and $L\in\mathbb{N}$ such that for any $u'\in\mathbb{B}^2$ and all $n\in\mathbb{N}$,

$$\int_{A_{nu'}} \left\| \frac{\partial}{\partial u_1}\Phi(u_1,u_2) \times \frac{\partial}{\partial u_2}\Phi(u_1,u_2) \right\| du_1\, du_2 \geq \frac{1}{C}\int_{A_{nu'}} \|(u_1,u_2)\|^L \, du_1\, du_2. \tag{4}$$

Moreover,

$$\int_{A_{nu'}} \|(u_1,u_2)\|^L \, du_1\, du_2 \geq \frac{M^{L+2}}{L+2}\, n^{-L-2}. \tag{5}$$

Combining (4) and (5) we get (3) with $c_0 = \frac{M^{L+2}}{C(L+2)}$. ■

3. Tangential Markov inequality on curves in L^p norms. In this section we prove a tangential Markov inequality on semialgebraic arcs in L^p norms (the norms are normalized with the arc length). The proof of the main result will be preceded by a lemma, which gives an estimation of the uniform norm by L^p norms. This result is a generalization of Nikolskii type inequalities, which for $0<r\leq p\leq+\infty$ and all trigonometric polynomials T ($\not\equiv 0$) of degree at most n have the form

$$\|T\|_p \leq A_n(p,r)\|T\|_r,$$

where $A_n(p,r)$ is a positive constant which does not depend on T. More information about the research related to the inequalities of this type can be found for example in [MMR, pp. 495–498].

Lemma 3.1. *Let* K *be a semialgebraic arc with the analytic parametrization*

$$\Phi(t) = (\varphi_1(t),\ldots,\varphi_N(t))$$

in the neighborhood of $I=[-1,1]$ *such that in the neighborhood of* 0 (0 *is the only singular point*) $\varphi_i(t) = \alpha_{i0} + \sum_{n=k}^{\infty}\alpha_{in}t^n$ *and* $\alpha_{1k}=1$. *Then for every* $p\in(1,+\infty)$ *there exists a positive constant* M *such that for every polynomial* $P\in\mathbb{C}[x_1,\ldots,x_N]$ *with* $\deg P\leq n$, *we have*

$$\|P\|_K \leq M n^{k/p}\|P\|_p.$$

Proof. Let $t_0\in I$ be such that $\|P\|_K = |P(\varphi_1(t_0),\ldots,\varphi_N(t_0))|$. Then

$$\bigl|P(\varphi_1(t),\ldots,\varphi_N(t)) - P(\varphi_1(t_0),\ldots,\varphi_N(t_0))\bigr| \leq \sum_{k=1}^{\infty}\frac{1}{k!}\bigl|(P\circ\Phi)^{(k)}(t_0)\bigr|\,|t-t_0|^k.$$

From Theorem 2.1 in [BP1] there exist positive constants M_1, M_2 such that for every polynomial $P \in \mathbb{C}[x_1, \dots, x_N]$ we have

$$\bigl|P(\varphi_1(\xi), \dots, \varphi_N(\xi))\bigr| \le M_2 \|P\|_K \quad \text{if } \operatorname{dist}(\xi, I) \le \frac{M_1}{\deg P}\,. \tag{6}$$

Hence for $r \le \frac{M_1}{n}$ we obtain

$$\begin{aligned}\Bigl|\frac{1}{k!}(P\circ\Phi)^{(k)}(t_0)\Bigr| &= \Bigl|\frac{1}{2\pi i}\int_{|\xi - t_0| = r} \frac{P(\varphi_1(\xi), \dots, \varphi_N(\xi))}{(\xi - t_0)^{k+1}}\,d\xi\Bigr| \\ &\le \frac{1}{2\pi}\int_{|\xi - t_0| = r} \frac{|P(\varphi_1(\xi), \dots, \varphi_N(\xi))|}{|\xi - t_0|^{k+1}}\,d\xi \\ &\le M_2\,\frac{1}{2\pi}\cdot 2\pi r \cdot r^{-k-1}\|P\|_K = M_2 r^{-k}\|P\|_K.\end{aligned}$$

Putting $r = \frac{M_1}{n}$ we have

$$\bigl|P(\varphi_1(t), \dots, \varphi_N(t)) - P(\varphi_1(t_0), \dots, \varphi_N(t_0))\bigr| \le M_2 \sum_{k=1}^{\infty} n^k M_1^{-k}|t - t_0|^k \|P\|_K.$$

Therefore for $|t - t_0| < \frac{M_1}{n}$ we get

$$\begin{aligned}\bigl|P(\varphi_1(t), \dots, \varphi_N(t)) - P(\varphi_1(t_0), \dots, \varphi_N(t_0))\bigr| &\le M_2\,\frac{n|t - t_0|}{M_1 - n|t - t_0|}\,\|P\|_K \\ &= M_2\,\frac{n|t - t_0|}{M_1 - n|t - t_0|}\,|P(\varphi_1(t_0), \dots, \varphi_N(t_0))|.\end{aligned}$$

The assumption $|t - t_0| \le \frac{M_1}{(1 + 2M_2)n}$ yields $M_2 \frac{n|t - t_0|}{M_1 - n|t - t_0|} \le \frac{1}{2}$. Hence

$$\bigl|P(\varphi_1(t), \dots, \varphi_N(t)) - P(\varphi_1(t_0), \dots, \varphi_N(t_0))\bigr| \le \frac{1}{2}|P(\varphi_1(t_0), \dots, \varphi_N(t_0))|,$$

so

$$|P(\varphi_1(t), \dots, \varphi_N(t))| \ge \frac{1}{2}\,|P(\varphi_1(t_0), \dots, \varphi_N(t_0))| \quad \text{if } |t - t_0| \le \frac{M_3}{n}\,,$$

where $M_3 = \frac{M_1}{1 + 2M_2}$. From this and Lemma 2.2 we have

$$\begin{aligned}&\int_{I\cap[t_0 - M_3/n,\, t_0 + M_3/n]} |P(\varphi_1(t), \dots, \varphi_N(t))|^p \Bigl(\sum_{i=1}^{N} |\varphi_i'(t)|^2\Bigr)^{1/2} dt \\ &\quad\ge \Bigl[\frac{1}{2}\,|P(\varphi_1(t_0), \dots, \varphi_N(t_0))|\Bigr]^p \int_{I\cap[t_0 - M_3/n,\, t_0 + M_3/n]} \Bigl(\sum_{i=1}^{N} |\varphi_i'(t)|^2\Bigr)^{1/2} dt \\ &\quad\ge \Bigl[\frac{1}{2}\,\|P\|_K\Bigr]^p c_0 n^{-k}.\end{aligned}$$

Finally we get

$$\begin{aligned}\|P\|_p &= \Bigl(\frac{1}{|K|}\int_{-1}^{1} |P(\varphi_1(t), \dots, \varphi_N(t))|^p \Bigl(\sum_{i=1}^{N} |\varphi_i'(t)|^2\Bigr)^{1/2} dt\Bigr)^{1/p} \\ &\ge \Bigl(\frac{1}{|K|}\Bigr)^{1/p} \frac{1}{2}\,\|P\|_K c_0^{1/p} n^{-k/p},\end{aligned}$$

where $|K|$ is the arc length of K. Hence $\|P\|_K \le 2|K|^{1/p} c_0^{-1/p} n^{k/p} \|P\|_p$. ∎

We formulate the main result of this section.

THEOREM 3.2. *Let K be a semialgebraic arc with the analytic parametrization*

$$\Phi(t) = (\varphi_1(t), \dots, \varphi_N(t))$$

in the neighborhood of $I = [-1, 1]$ *such that in the neighborhood of* 0 (0 *is the only singular point*) $\varphi_i(t) = \alpha_{i0} + \sum_{n=k}^{\infty} \alpha_{in} t^n$ *and* $\alpha_{1k} = 1$. *Then there exists a positive constant* D_1 *such that for every polynomial* $P \in \mathbb{C}[x_1, \dots, x_N]$ *with* $\deg P \le n$, *we have*

$$\|D_{\mathcal{T}} P\|_p \le D_1 n^{k+k/p} \|P\|_p.$$

Proof. For every polynomial $P \in \mathbb{C}[x_1, \dots, x_N]$ we have

$$\|D_{\mathcal{T}} P\|_p = \bigg(\frac{1}{|K|} \int_{-1}^{1} |D_{\mathcal{T}} P(\varphi_1(t), \dots, \varphi_N(t))|^p \Big(\sum_{i=1}^{N} |\varphi_i'(t)|^2\Big)^{1/2} dt\bigg)^{1/p} \le \|D_{\mathcal{T}} P\|_K.$$

From the tangential Markov inequality on curves in the uniform norm ([K, Theorem 3.2]) there exists a positive constant D such that for every polynomial $P \in \mathbb{C}[x_1, \dots, x_N]$ with $\deg P \le n$, we get $\|D_{\mathcal{T}} P\|_K \le D n^k \|P\|_K$. Hence

$$\|D_{\mathcal{T}} P\|_p \le D n^k \|P\|_K.$$

Finally, Lemma 3.1 implies

$$\|D_{\mathcal{T}} P\|_p \le D_1 n^{k+k/p} \|P\|_p,$$

where $D_1 = D \cdot M$ and M is from Lemma 3.1. ■

4. Tangential Markov inequality on surfaces in L^p norms. Another generalization of the tangential Markov inequality in L^p norms can be proved on semialgebraic surfaces with finitely many singular points in $\mathbb{R}^3$. As before we start from providing estimates for the norms. In this section we consider $\|\cdot\|_p$ norms, which are normalized with the surface area.

LEMMA 4.1. *Let $S \subset \mathbb{R}^3$ be a compact surface with the analytic parametrization*

$$\Phi(u_1, u_2) = \big(\varphi_1(u_1, u_2), \varphi_2(u_1, u_2), \varphi_3(u_1, u_2)\big), \quad (u_1, u_2) \in \mathbb{B}^2,$$

such that $\Phi(0, 0) = 0$ *is the only singular point. Then for every* $p \in (1, +\infty)$ *there exists a positive constant* D_2 *such that for every polynomial* $P \in \mathbb{C}[x_1, \dots, x_N]$ *with* $\deg P \le n$,

$$\|P\|_S \le D_2 n^{s/p} \|P\|_p,$$

where s is defined in Lemma 2.3.

Proof. Let $(u_1^0, u_2^0) \in \mathbb{B}^2$ be such that $\|P\|_S = |(P \circ \Phi)(u_1^0, u_2^0)|$. Then

$$\begin{aligned} \big|(P \circ \Phi)(u_1, u_2) - (P \circ \Phi)(u_1^0, u_2^0)\big| &\le \sum_{l=1}^{\infty} \frac{1}{l!} \Big|\frac{\partial^l (P \circ \Phi)}{\partial^l u_2}(u_1^0, u_2^0)\Big| \, |u_2 - u_2^0|^l \\ &\quad + \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k! l!} \Big|\frac{\partial^{k+l} (P \circ \Phi)}{\partial^k u_1 \partial^l u_2}(u_1^0, u_2^0)\Big| \, |u_1 - u_1^0|^k \, |u_2 - u_2^0|^l. \end{aligned}$$

By Theorem 4.5 in [BP2] there exist positive constants C_3, δ_3 such that

$$\forall\, P \in \mathbb{C}[x,y,z] \quad |P(\Phi(\xi))| \le C_3 \|P\|_S, \quad \text{if } \operatorname{dist}(\xi, \mathbb{B}^2) \le \frac{\delta_3}{\deg P}\,. \tag{7}$$

Hence for $r_1, r_2 \le \frac{\delta_3}{2n}$ we have

$$\begin{aligned}\Big|\frac{1}{k!l!}\,\frac{\partial^{k+l}(P\circ\Phi)}{\partial^k u_1 \partial^l u_2}(u_1^0,u_2^0)\Big| \\ &= \bigg|\frac{1}{(2\pi i)^2}\int_{|\xi_1-u_1^0|=r_1}\int_{|\xi_2-u_2^0|=r_2} \frac{(P\circ\Phi)(\xi)}{(\xi_1-u_1^0)^{k+1}(\xi_2-u_2^0)^{l+1}}\,d\xi_1\,d\xi_2\bigg| \\ &\le C_3\,\frac{1}{(2\pi)^2}\cdot 2\pi r_1\cdot 2\pi r_2\cdot r_1^{-k-1} r_2^{-l-1}\|P\|_S = C_3 r_1^{-k} r_2^{-l}\|P\|_S.\end{aligned}$$

Then for $r_1 = r_2 = \frac{\delta_3}{2n}$ we obtain

$$\begin{aligned}\big|(P\circ\Phi)(u_1,u_2) - (P\circ\Phi)(u_1^0,u_2^0)\big| &\le C_3\sum_{l=1}^{\infty}\Big(\frac{2n}{\delta_3}\Big)^l |u_2-u_2^0|^l\|P\|_S \\ &\quad + C_3\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\Big(\frac{2n}{\delta_3}\Big)^{k+l}|u_1-u_1^0|^k|u_2-u_2^0|^l\|P\|_S.\end{aligned}$$

Hence for $|u_1-u_1^0| < \frac{\delta_3}{2n}$ and $|u_2-u_2^0| < \frac{\delta_3}{2n}$, since (u_1^0,u_2^0) is such that $\|P\|_S = |(P\circ\Phi)(u_1^0,u_2^0)|$, we get

$$\begin{aligned}&\big|(P\circ\Phi)(u_1,u_2) - (P\circ\Phi)(u_1^0,u_2^0)\big| \\ &\quad\le C_3\bigg(\frac{\frac{2n}{\delta_3}|u_2-u_2^0|}{1-\frac{2n}{\delta_3}|u_2-u_2^0|} + \frac{\frac{2n}{\delta_3}|u_1-u_1^0|}{(1-\frac{2n}{\delta_3}|u_2-u_2^0|)(1-\frac{2n}{\delta_3}|u_1-u_1^0|)}\bigg)|(P\circ\Phi)(u_1^0,u_2^0)|.\end{aligned}$$

If $|u_1-u_1^0| \le \frac{\delta_3}{6(1+4C_3)n}$ and $|u_2-u_2^0| \le \frac{\delta_3}{6(1+4C_3)n}$, we obtain

$$C_3\bigg(\frac{\frac{2n}{\delta_3}|u_2-u_2^0|}{1-\frac{2n}{\delta_3}|u_2-u_2^0|} + \frac{\frac{2n}{\delta_3}|u_1-u_1^0|}{(1-\frac{2n}{\delta_3}|u_2-u_2^0|)(1-\frac{2n}{\delta_3}|u_1-u_1^0|)}\bigg) \le \frac{1}{2}.$$

Hence

$$\big|(P\circ\Phi)(u_1,u_2) - (P\circ\Phi)(u_1^0,u_2^0)\big| \le \frac{1}{2}\,|(P\circ\Phi)(u_1^0,u_2^0)|,$$

which gives

$$|(P\circ\Phi)(u_1,u_2)| \ge \frac{1}{2}\,|(P\circ\Phi)(u_1^0,u_2^0)|, \quad \text{if } (u_1,u_2)\in\mathbb{B}((u_1^0,u_2^0), M_4/n), \tag{8}$$

where $M_4 = \frac{\delta_3}{6(1+4C_4)}$. Applying Lemma 2.3 we obtain

$$\begin{aligned}&\iint_{\mathbb{B}^2\cap\mathbb{B}((u_1^0,u_2^0),M_4/n)} |(P\circ\Phi)(u_1,u_2)|^p\,\Big\|\frac{\partial}{\partial u_1}\Phi(u_1,u_2)\times\frac{\partial}{\partial u_2}\Phi(u_1,u_2)\Big\|\,du_1\,du_2 \\ &\ge \Big[\frac{1}{2}|(P\circ\Phi)(u_1^0,u_2^0)|\Big]^p \iint_{\mathbb{B}^2\cap\mathbb{B}((u_1^0,u_2^0),M_4/n)}\Big\|\frac{\partial}{\partial u_1}\Phi(u_1,u_2)\times\frac{\partial}{\partial u_2}\Phi(u_1,u_2)\Big\|\,du_1\,du_2 \\ &\qquad\qquad\ge \Big[\frac{1}{2}\|P\|_S\Big]^p c_0 n^{-s}.\end{aligned}$$

Finally we obtain

$$\|P\|_p = \Big(\frac{1}{|S|}\iint_{\mathbb{B}^2} |(P\circ\Phi)(u_1,u_2)|^p \Big\|\frac{\partial}{\partial u_1}\Phi(u_1,u_2)\times\frac{\partial}{\partial u_2}\Phi(u_1,u_2)\Big\|\,du_1\,du_2\Big)^{1/p} \geq \Big(\frac{1}{|S|}\Big)^{1/p}\frac{1}{2}\,\|P\|_S c_0^{1/p} n^{-s/p},$$

where $|S|$ is the surface area of S. Therefore $\|P\|_S \leq 2|S|^{1/p}c_0^{-1/p}n^{s/p}\|P\|_p$. ■

THEOREM 4.2. *Let $S\subset\mathbb{R}^3$ be a compact surface with analytic parametrization*

$$\Phi(u_1,u_2) = \big(\varphi_1(u_1,u_2),\varphi_2(u_1,u_2),\varphi_3(u_1,u_2)\big),\quad (u_1,u_2)\in\mathbb{B}^2,$$

such that $\Phi(0,0)=0$ is the only singular point. Then there exist positive constants M_1 and k such that for every polynomial $P\in\mathbb{C}[x,y,z]$ such that $\deg P\leq n$*,*

$$\|D_{\mathcal{T}}P\|_p \leq M_1 n^{k+s/p}\|P\|_p,$$

where s is given in Lemma 2.3.

Proof. For every $P\in\mathbb{C}[x,y,z]$ we have

$$\|D_{\mathcal{T}}P\|_p = \Big(\frac{1}{|S|}\iint_{\mathbb{B}^2} |D_{\mathcal{T}}P(\Phi(u_1,u_2))|^p \Big\|\frac{\partial}{\partial u_1}\Phi(u_1,u_2)\times\frac{\partial}{\partial u_2}\Phi(u_1,u_2)\Big\|\,du_1\,du_2\Big)^{1/p} \leq \|D_{\mathcal{T}}P\|_S.$$

By the tangential Markov inequality on surfaces in uniform norm ([K], Theorem 4.1), there exist constants $D>0$ and $k\in\mathbb{N}$ such that for each polynomial $P\in\mathbb{C}[x_1,x_2,x_3]$, $\|D_{\mathcal{T}}P\|_S\leq Dn^k\|P\|_S$. Therefore

$$\|D_{\mathcal{T}}P\|_p \leq Dn^k\|P\|_S.$$

Finally, using Lemma 4.1, we have

$$\|D_{\mathcal{T}}P\|_p \leq M_1 n^{k+s/p}\|P\|_p$$

with $M_1 = D_2\cdot D$, where D_2 is from Lemma 4.1. ■

Notice that we presented proofs in $\mathbb{R}^3$, but it is a minor modification to state and prove Theorem 4.2 for surfaces in $\mathbb{R}^N$ for $N\geq 3$. Namely we can consider a compact surface $S\subset\mathbb{R}^N$ with an analytic parametrization $\Phi(u_1,u_2) = (\varphi_1(u_1,u_2),\ldots,\varphi_N(u_1,u_2))$, $(u_1,u_2)\in\mathbb{B}^2$, such that $\Phi(0,0)=0$ is the only singular point and L^p norms associated with the surface measure in $\mathbb{R}^N$, i.e.

$$\|f\|_p = \Big(\frac{1}{|S|}\iint_{\mathbb{B}^2} |f(\Phi(u_1,u_2))|^p \times\Big(\sum_{1\leq i_1<i_2\leq N}\begin{vmatrix}\frac{\partial}{\partial u_1}\varphi_{i_1}(u_1,u_2) & \frac{\partial}{\partial u_2}\varphi_{i_1}(u_1,u_2)\\ \frac{\partial}{\partial u_1}\varphi_{i_2}(u_1,u_2) & \frac{\partial}{\partial u_2}\varphi_{i_2}(u_1,u_2)\end{vmatrix}^2\Big)^{1/2} du_1\,du_2\Big)^{1/p}.$$

Then by the same argument as before we get the generalization of Lemmas 2.3 and 4.1, in particular,

$$\int_{A_{nu'}}\Big(\sum_{1\leq i_1<i_2\leq N}\begin{vmatrix}\frac{\partial}{\partial u_1}\varphi_{i_1}(u_1,u_2) & \frac{\partial}{\partial u_2}\varphi_{i_1}(u_1,u_2)\\ \frac{\partial}{\partial u_1}\varphi_{i_2}(u_1,u_2) & \frac{\partial}{\partial u_2}\varphi_{i_2}(u_1,u_2)\end{vmatrix}^2\Big)^{1/2} du_1\,du_2 \geq c_1 n^{-l} \tag{9}$$

with some positive constants c_1 and l. Moreover,

$$\|P\|_S \leq D_3 n^{l/p} \|P\|_p$$

for every $p \in (1, +\infty)$ and $P \in \mathbb{C}[x_1, \ldots, x_N]$ with $\deg P \leq n$, where l is from (9) and D_3 is a positive constant, independent of polynomial P. Finally, the tangential Markov inequalities in L^p norms hold for every compact and semialgebraic sets K in $\mathbb{R}^N$ with an analytic parametrization of order m having finitely many singular points.

Acknowledgments. The author wishes to express her gratitude to the anonymous referee for all his/her comments and especially for the suggestion of the simpler proof of Lemma 2.2.

References

[B] M. Baran, *New approach to Markov inequality in L^p norms*, in: Approximation Theory, Monogr. Textbooks Pure Appl. Math. 212, Marcel Dekker, New York 1998, 75–85.

[BMO] M. Baran, B. Milówka, P. Ozorka, *Markov's property for k-th derivative*, Ann. Polon. Math. 106 (2012), 31–40.

[BP1] M. Baran, W. Pleśniak, *Bernstein and van der Corput–Schaake type inequalities on semialgebraic curves*, Studia Math. 125 (1997), 83–96.

[BP2] M. Baran, W. Pleśniak, *Characterization of compact subsets of algebraic varieties in terms of Bernstein type inequalities*, Studia Math. 141 (2000), 221–233.

[BBLT] L. P. Bos, A. Brudnyi, N. Levenberg, V. Totik, *Tangential Markov inequalities on transcendental curves*, Constr. Approx. 19 (2003), 339–354.

[BLMT1] L. P. Bos, N. Levenberg, P. Milman, B. A. Taylor, *Tangential Markov inequalities characterize algebraic submanifolds of* $\mathbb{R}^N$, Indiana Univ. Math. J. 44 (1995), 115–138.

[BLMT2] L. Bos, N. Levenberg, P. Milman, B. A. Taylor, *Tangential Markov inequalities on real algebraic varieties*, Indiana Univ. Math. J. 47 (1998), 1257–1272.

[Ge] L. Gendre, *Inégalités de Markov tangentielles locales sur les algébriques singulières de* $\mathbb{R}^n$, Ann. Polon. Math. 86 (2005), 59–77.

[Go1] P. Goetgheluck, *Inégalité de Markov dans les ensembles efillés*, J. Approx. Theory 30 (1980), 149–154.

[Go2] P. Goetgheluck, *Markov's inequality on locally Lipschitzian compact subsets of* $\mathbb{R}^n$ *in L^p-spaces*, J. Approx. Theory 49 (1987), 303–310.

[Go3] P. Goetgheluck, *On the Markov inequality in L^p-spaces*, J. Approx. Theory 62 (1990), 197–205.

[HST] E. Hille, G. Szegö, J. Tamarkin, *On some generalisation of a theorem of A. Markoff*, Duke Math. J. 3 (1937), 729–739.

[K] A. Kowalska, *Tangential Markov inequalities on semialgebraic curves and some semialgebraic surfaces*, Ann. Polon. Math. 106 (2012), 215–222.

[L] S. Łojasiewicz, *Sur le problème de la division*, Studia Math. 18 (1959), 87–136.

[MMR] G. V. Milovanović, D. S. Mitrinović, T. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, River Edge, NJ 1994.

[M] B. Milówka, *Markov's inequality and a generalized Pleśniak condition*, East J. Approx. 11 (2005), 291–300.

[P] W. Pleśniak, *Recent progress in multivariate Markov inequality*, in: Approximation Theory, Monogr. Textbooks Pure Appl. Math., Dekker, New York 1998, 449–464.

[S] J. Stasica, *On roots of analytic equations*, Wydawnictwo Uniwersytetu Jagiellońskiego, Kraków 2003.

[TV] V. Totik, T. Varga, *Chebyshev and fast decreasing polynomials*, Proc. London Math. Soc. (3) 110 (2015), 1057–1098.

[Z] A. Zygmund, *A remark on conjugate series*, Proc. London Math. Soc. (2) 34 (1932), 392–400.