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Equilateral sets in Banach spaces of the form $C(K)$

by

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Abstract. We show that for "most" compact nonmetrizable spaces, the unit ball of the Banach space $C(K)$ contains an uncountable 2-equilateral set. We also give examples of compact nonmetrizable spaces K such that the minimum cardinality of a maximal equilateral set in $C(K)$ is countable.

1. Introduction. A subset S of a metric space (M,d) is said to be *equilateral* if there is a constant $\lambda > 0$ such that $d(x,y) = \lambda$ for all distinct $x, y \in S$; we also call such a set a λ-*equilateral* set. An equilateral set $S \subseteq M$ is said to be *maximal* if there is no equilateral set $B \subseteq M$ with $A \subsetneq B$.

Equilateral sets have been studied mainly in finite-dimensional spaces (see [18], [20], and [19] for a survey). More recently there are also results in infinite dimensions [16], [8] and also results on maximal equilateral sets [21].

In this paper we study equilateral sets in Banach spaces of the form $C(K)$, where K is a compact space. In the first section we introduce the combinatorial concept of a linked family of pairs in a set Γ; using this concept we characterize those compact spaces K such that the unit ball of $C(K)$ contains a $(1+\varepsilon)$-separated (equivalently, 2-equilateral) set of a given cardinality (Theorem 2.6). Then we show that in "most" cases a compact nonmetrizable space K admits an uncountable linked family of closed pairs and hence its unit ball contains an uncountable 2-equilateral set (Theorem 2.9). We note in this connection that the unit sphere of every infinite-dimensional Banach space contains an infinite $(1+\varepsilon)$-separated set [5].

In the second section we focus on maximal equilateral sets in the space $C(K)$. Following [21], given a normed space E, we denote by $m(E)$ the minimum cardinality of a maximal equilateral set in E. The main results here are the following. For every infinite locally compact space K we have

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$m(C_0(K)) \geq \omega$ (Theorem 3.5) (thus in particular $m(C(K)) = \omega$ for any infinite compact metric space K). For every infinite product $K = \prod_{\gamma\in\Gamma} K_\gamma$ of nontrivial compact metric spaces, $m(C(K)) = |\Gamma|$ (Theorem 3.7). Then we give a variety of examples of compact nonmetrizable spaces K (including scattered compact spaces and the Stone–Čech compactification $\beta\Gamma$ of any infinite discrete set Γ) such that $m(C(K)) = \omega$ (Theorems 3.8, 3.10 and Corollaries 3.9, 3.11).

If E is any (real) Banach space then B_E denotes its closed unit ball. If K is any compact [locally compact] Hausdorff space, then $C(K)$ [$C_0(K)$] is the Banach space of all continuous real functions on K [that vanish at infinity], endowed with the supremum norm $\|\cdot\|_\infty$.

When K is a compact Hausdorff space and $\mu \in P(K)$ (= the space of regular Borel probability measures on K), then μ is called *countably determined* if there is a sequence (K_n) of compact subsets of K, such that for every open $U \subseteq K$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $K_n \subseteq U$ and $\mu(U\setminus K_n) \leq \varepsilon$. If the sequence (K_n) consists of closed G_δ sets then μ is called *strongly countably determined* (or *uniformly regular*). For these concepts we refer the reader to [15].

2. Linked families and equilateral sets in $C(K)$ spaces. In this section we introduce the concept of a linked family of pairs in a set Γ and then use it to investigate the existence of equilateral sets in $C(K)$, where K is any compact (nonmetrizable) space.

DEFINITION 2.1. Let $\mathcal{F} = \{(A_\alpha, B_\alpha) : \alpha \in \mathcal{A}\}$ be a family of pairs of subsets of a nonempty set Γ. We say that $\mathcal{F}$ is *linked* (or *intersecting*) if

(i) $A_\alpha \cap B_\alpha = \emptyset$ for $\alpha \in \mathcal{A}$,
(ii) $A_\alpha \cup B_\alpha \neq \emptyset$ for $\alpha \in \mathcal{A}$, and
(iii) for distinct $\alpha, \beta \in \mathcal{A}$, either $A_\alpha \cap B_\beta \neq \emptyset$ or $A_\beta \cap B_\alpha \neq \emptyset$.

If we replace condition (ii) with the stronger one: $A_\alpha \neq \emptyset \neq B_\alpha$ for $\alpha \in \mathcal{A}$, we shall say that $\mathcal{F}$ is a *linked family of nonempty pairs*.

We note the following easily verified facts:

(a) If $\alpha \neq \beta \in \mathcal{A}$ then $A_\alpha \neq A_\beta$ and $B_\alpha \neq B_\beta$, and hence
(b) there is at most one $\alpha \in \mathcal{A}$ such that $A_\alpha = \emptyset$ and at most one $\beta \in \mathcal{A}$ such that $B_\beta = \emptyset$.
(c) If $\mathcal{F}$ is a linked family of nonempty pairs in the set Γ, then the family $\mathcal{F} \cup \{(\Gamma, \emptyset), (\emptyset, \Gamma)\}$ is a linked family of pairs in Γ.

EXAMPLE 2.2. Let $\{A_\alpha : \alpha \in \mathcal{A}\}$ be a family of distinct subsets of a set Γ. Then the family $\{(A_\alpha, \Gamma \setminus A_\alpha) : \alpha \in \mathcal{A}\}$ is linked. Assuming furthermore that $\emptyset \neq A_\alpha \neq \Gamma$ for $\alpha \in \mathcal{A}$, we find that $\mathcal{F}$ is a linked family of nonempty pairs. It follows in particular that $\{(A, \Gamma \setminus A) : A \subseteq \Gamma\}$ (resp.

$\{(A, \Gamma \setminus A) : \emptyset \neq A \subsetneq \Gamma\}$) is linked (resp. a linked family of nonempty pairs). For further examples of linked families see Remarks 3.12(3).

Now we are going to examine the interrelation between the concepts of linked families and equilateral sets in Banach spaces of the form $C(K)$, where K is a compact Hausdorff space.

LEMMA 2.3. *Let K be a compact Hausdorff space and $S \subseteq [0,1]^K \cap C(K)$. Set*

$$A_f = f^{-1}(\{0\}) \quad \textit{and} \quad B_f = f^{-1}(\{1\})$$

for $f \in S$. Then the following are equivalent:

(i) *The family $\mathcal{F} = \{(A_f, B_f) : f \in S\}$ of (closed) pairs in K is linked.*
(ii) *The set S is 1-equilateral in $C(K)$.*

Proof. (i)⇒(ii). Let $f, g \in S$ with $f \neq g$; clearly $0 < \|f - g\|_\infty \leq 1$. Since either $A_f \cap B_g \neq \emptyset$ or $A_g \cap B_f \neq \emptyset$, there is a $t_0 \in K$ such that $|f(t_0) - g(t_0)| = 1$, hence $\|f - g\|_\infty = 1$.

(ii)⇒(i). If $f \neq g \in S$, then $\|f - g\|_\infty = 1$; so by the compactness of K, there is a $t_0 \in K$ such that $|f(t_0) - g(t_0)| = \|f - g\|_\infty = 1$. Since $0 \leq f(t_0), g(t_0) \leq 1$ we get $\{f(t_0), g(t_0)\} = \{0, 1\}$. Therefore, either $t_0 \in A_f \cap B_g$ or $t_0 \in A_g \cap B_f$, and $\mathcal{F}$ is as required. ∎

NOTE. Since there is at most one $f_0 \in S$ with $A_{f_0} = \emptyset$ ($\Leftrightarrow \inf(f_0) > 0$) and at most one $g_0 \in S$ with $B_{g_0} = \emptyset$ ($\Leftrightarrow \|g_0\|_\infty < 1$), we see that $\{(A_f, B_f) : f \in S \setminus \{f_0, g_0\}\}$ is a linked family of nonempty closed pairs in K, and $S \setminus \{g_0\}$ is a subset of the positive part $S^+_{C(K)}$ of the unit sphere $S_{C(K)}$.

LEMMA 2.4. *Let $\mathcal{F} = \{(A_\alpha, B_\alpha) : \alpha \in \mathcal{A}\}$ be a linked family of closed pairs in the compact space K. Then we can associate with $\mathcal{F}$ a 1-equilateral subset S of $C(K)$ with $|S| = |\mathcal{A}|$ and $S \subseteq [0,1]^K \cap C(K)$.*

Proof. Let $\alpha \in \mathcal{A}$; we distinguish the following cases:

(I) Assume that $A_\alpha \neq \emptyset \neq B_\alpha$. We consider an Urysohn function $f_\alpha : K \to [0,1]$ such that $f_\alpha(x) = 0$ for $x \in A_\alpha$ and $f_\alpha(x) = 1$ for $x \in B_\alpha$; clearly $\inf(f_\alpha) = 0 < \|f_\alpha\|_\infty = 1$.

(II) Assume that $A_\alpha = \emptyset$, thus $B_\alpha \neq \emptyset$. If $B_\alpha \neq K$, pick $t_0 \in K \setminus B_\alpha$ and consider an Urysohn function $f_\alpha : K \to [0,1]$ such that $f_\alpha | B_\alpha = 1$ and $f_\alpha(t_0) = 0$. In case $B_\alpha = K$, we let $f_\alpha = 1$ on K.

(III) Assume that $B_\alpha = \emptyset$, thus $A_\alpha \neq \emptyset$. This case is similar to (II). So we consider an Urysohn function $f_\alpha : K \to [0,1]$ such that $f_\alpha | A_\alpha = 0$ and $f_\alpha(t_0) = 1$ for some $t_0 \in K \setminus A_\alpha$ if $A_\alpha \neq K$, and define f_α to be the constant zero function in case $A_\alpha = K$.

Now set $A'_\alpha = f_\alpha^{-1}(\{0\})$ and $B'_\alpha = f_\alpha^{-1}(\{1\})$ for $\alpha \in \mathcal{A}$. Since $A'_\alpha \cap B'_\alpha = \emptyset$, $A_\alpha \subseteq A'_\alpha$ and $B_\alpha \subseteq B'_\alpha$ for $\alpha \in \mathcal{A}$, we see that $\{(A'_\alpha, B'_\alpha) : \alpha \in \mathcal{A}\}$ is a linked family of closed pairs in K, hence by Lemma 2.3, $S = \{f_\alpha : \alpha \in \mathcal{A}\}$ is a 1-equilateral subset of $[0,1]^K \cap C(K)$. ∎

REMARKS 2.5. (1) If in the proof of Lemma 2.4 we consider (as we may) continuous functions $f_\alpha : K \to [-1,1]$ such that $f_\alpha|A_\alpha = 1$ and $f_\alpha|B_\alpha = -1$, then $\{f_\alpha : \alpha \in \mathcal{A}\}$ is a 2-equilateral subset of the unit ball of $C(K)$.

(2) Let $\mathcal{F} = \{(A_\alpha, B_\alpha) : \alpha \in \mathcal{A}\}$ be a family of disjoint pairs in a set Γ. Set $\bar{\mathcal{F}} = \{(\bar{A}_\alpha, \bar{B}_\alpha) : \alpha \in \mathcal{A}\}$ where $\bar{A}_\alpha = \mathrm{cl}_{\beta\Gamma} A_\alpha$, $\bar{B}_\alpha = \mathrm{cl}_{\beta\Gamma} B_\alpha$ and $\beta\Gamma$ is the Stone–Čech compactification of the discrete set Γ. Then it is easy to see that $\mathcal{F}$ is a linked family of (nonempty) pairs in Γ iff $\bar{\mathcal{F}}$ is a linked family of (nonempty) pairs in $\beta\Gamma$.

(3) Let K be a compact space and $S \subseteq [0,1]^K \cap C(K)$ be a 1-equilateral set. We consider the linked family $\mathcal{F} = \{(A_f, B_f) : f \in S\}$ given by Lemma 2.3. Then it is not difficult to verify that $\mathcal{F}$ is a maximal linked family of closed pairs in K iff S is a maximal (with respect to inclusion) 1-equilateral subset of $[0,1]^K \cap C(K)$, endowed with the norm metric (S is not necessarily a maximal equilateral set in $C(K)$ (see Remark 3.12(4)).

THEOREM 2.6. *Let K be a compact Hausdorff space and α be an infinite cardinal. The following are equivalent:*

(i) *The unit ball $B_{C(K)}$ contains a λ-equilateral set, with $\lambda > 1$, of size α.*
(ii) *The unit sphere $S_{C(K)}$ (resp. its positive part $S^+_{C(K)}$) admits a 2-equilateral (resp. 1-equilateral) set of size α.*
(iii) *$B_{C(K)}$ contains a $(1+\varepsilon)$-separated set, for some $\varepsilon > 0$, of size α.*
(iv) *There exists a linked family of closed (nonempty) pairs in K of size α.*

Proof. (ii)⇒(i). Let S be a 1-equilateral subset of $S^+_{C(K)}$ with $|S| = \alpha$. Then by Lemma 2.3, $\mathcal{F} = \{(A_f, B_f) : f \in S\}$ is a linked family of closed pairs in K with $|\mathcal{F}| = \alpha$. Therefore, by Lemma 2.4 and Remark 2.5(1), $\mathcal{F}$ defines a 2-equilateral set contained in $B_{C(K)}$.

(i)⇒(iii) is obvious.

(iii)⇒(iv). Let $D \subseteq B_{C(K)}$ be a $(1+\varepsilon)$-separated set ($\varepsilon > 0$) with $|D| = \alpha$. We may assume that $\|f\|_\infty = 1$ for $f \in D$ (see [14, Lemma 6, p. 8], and also [11, Lemma 2.2]). We define

$$A_f = f^{-1}([-1, -\varepsilon/2]) \quad \text{and} \quad B_f = f^{-1}([\varepsilon/2, 1])$$

for $f \in D$; clearly $A_f \cup B_f \neq \emptyset$. Let $f, g \in D$ with $f \neq g$, so there is a $t_0 \in K$ such that $\|f - g\|_\infty = |f(t_0) - g(t_0)| \geq 1 + \varepsilon$. Assume without loss of generality that $f(t_0) < g(t_0)$. We will show that $f(t_0) \leq -\varepsilon/2$ and $g(t_0) \geq \varepsilon/2$, that is, $A_f \cap B_g \neq \emptyset$.

Indeed, suppose otherwise; then either $f(t_0) > -\varepsilon/2$ or $g(t_0) < \varepsilon/2$. Assuming the former, we get $-\varepsilon/2 < f(t_0) < g(t_0) \le 1$, hence $g(t_0)-f(t_0) < 1+\varepsilon/2$, a contradiction. In a similar way we get a contradiction assuming that $g(t_0) < \varepsilon/2$.

It follows that $\mathcal{F} = \{(A_f, B_f) : f \in D\}$ is a linked family of closed pairs in K of size α.

(iv)$\Rightarrow$(ii) is a direct consequence of Lemma 2.4. ∎

Let K be an infinite compact space; as is well known, the Banach space c_0 can be isometrically embedded in $C(K)$, hence the assertions of Theorem 2.6 hold true for $\alpha = \omega$. The following questions were open when this article became public:

QUESTIONS 2.7. Let K be a compact Hausdorff nonmetrizable space.

(1) (a) Does there exist an uncountable $(1+\varepsilon)$-separated $D \subseteq B_{C(K)}$?
  (b) Does there exist an uncountable $D \subseteq B_{C(K)}$ with the property that $f \ne g \in D \Rightarrow \|f-g\|_\infty > 1$? (Note that, by transfinite induction it can be shown that there is an uncountable $D \subseteq S_{C(K)}$ such that $f \ne g \in D \Rightarrow \|f-g\|_\infty \ge 1$.)

(2) Does $C(K)$ contain an uncountable equilateral set?

We can show that in "most" cases the answers to the above questions are positive. For this purpose we recall the following definitions. Let X be a Hausdorff and completely regular topological space.

X is said to be *hereditarily Lindelöf* (HL) if every subspace Y of X is Lindelöf. It is well known that X is HL iff there is no uncountable *right separated* family in X, that is, a family $\{t_\alpha : \alpha < \omega_1\} \subseteq X$ such that $t_\alpha \notin \mathrm{cl}_X\{t_\beta : \alpha < \beta < \omega_1\}$ for $\alpha < \omega_1$.

X is said to be *hereditarily separable* (HS) if every subspace Y of X is separable. It is also well known that X is HS iff there is no uncountable *left separated* family in X, that is, a family $\{t_\alpha : \alpha < \omega_1\} \subseteq X$ such that $t_\alpha \notin \mathrm{cl}_X\{t_\beta : \beta < \alpha\}$ for $1 \le \alpha < \omega_1$ (see [9, p. 151]).

We are going to use the following standard

FACT 2.8. *A compact space K is HL if and only if it is perfectly normal* (*i.e. each closed subset of K is G_δ*).

THEOREM 2.9. *Let K be a compact space. If K satisfies one of the following conditions, then K admits an uncountable linked family of closed pairs* (*and hence by Theorem* 2.6 *the unit ball of $C(K)$ contains an uncountable* 2*-equilateral set*).

(i) *There exists a closed subset Ω of K admitting uncountably many relatively clopen sets* (*in particular Ω is nonmetrizable and totally disconnected*).

(ii) *K is not hereditarily Lindelöf.*
(iii) *K is not hereditarily separable.*
(iv) $|K| > 2^\omega$.
(v) *K admits a Radon probability measure which is not strongly countably determined.*

Proof. (i) Let $\mathcal{B}$ be any uncountable family of clopen sets in $\Omega \subseteq K$. Then clearly $\mathcal{F} = \{(V, \Omega \setminus V) : V \in \mathcal{B}\}$ is an uncountable linked family of closed pairs in K. (It is clear that condition (i) can be stated as follows: there is a closed subset $\Omega \subseteq K$ such that the unit ball of $C(\Omega)$ has uncountably many extreme points.)

(ii) Let $\{t_\alpha : \alpha < \omega_1\} \subseteq K$ be an uncountable right separated family. Set $A_\alpha = \{t_\alpha\}$ and $B_\alpha = \mathrm{cl}_K\{t_\beta : \alpha < \beta < \omega_1\}$ for $\alpha < \omega_1$. Then it is easy to see that $\{(A_\alpha, B_\alpha) : \alpha < \omega_1\}$ is a linked family of closed (nonempty) pairs in K.

(iii) Since K is non-HS, there exists an uncountable left separated family in K and the proof is similar to that of the previous case.

(iv) This follows from (ii), since if $|K| > 2^\omega$ then K is not HL. Indeed, any compact HL space is first countable (each one-point subset of K is G_δ by Fact 2.8). By a classical result of Arkhangel'skii each compact first countable space has cardinality $\leq 2^\omega$.

(v) We may assume by (ii) that K is HL. Therefore (by Fact 2.8) our assumption is equivalent to the existence of a measure μ on K which is not countably determined. We shall construct by transfinite induction a family of closed pairs (A_α, B_α), $\alpha < \omega_1$, as follows: Given $\alpha < \omega_1$ and pairs for $\beta < \alpha$, there are an open set $V_\alpha \subseteq K$ and $\varepsilon(\alpha) > 0$ (as μ is not countably determined) such that

$$\beta < \alpha \text{ and } A_\beta \subseteq V_\alpha \ \Rightarrow\ \mu(V_\alpha) - \mu(A_\beta) \geq \varepsilon(\alpha). \tag{1}$$

Then set $B_\alpha = K \setminus V_\alpha$ and consider a closed set $A_\alpha \subseteq V_\alpha$ with

$$\mu(V_\alpha \setminus A_\alpha) = \mu(V_\alpha) - \mu(A_\alpha) < \varepsilon(\alpha)/3. \tag{2}$$

It is clear that we can choose $\varepsilon > 0$ and an uncountable set $I \subseteq \omega_1$ such that $\varepsilon(\alpha) \geq \varepsilon$ for $\alpha \in I$ and furthermore

$$|\mu(A_\beta) - \mu(A_\alpha)| < \varepsilon/3, \qquad \forall \alpha, \beta \in I. \tag{3}$$

We claim that the family (A_α, B_α), $\alpha \in I$, is linked. So let $\beta < \alpha \in I$. Then $A_\beta \not\subseteq V_\alpha$ (otherwise by (1) and (2) we would get $\mu(A_\alpha) - \mu(A_\beta) > 2\varepsilon(\alpha)/3 \geq 2\varepsilon/3$, which contradicts (3)). Hence $(K \setminus V_\alpha) \cap A_\beta = B_\alpha \cap A_\beta \neq \emptyset$. ∎

Remarks 2.10. (1) Both of the above questions were included in a preliminary version of this article and have recently been answered. Specifically, P. Koszmider [12] with the aid of MA+¬ CH gave a positive answer to Question 2.7(1)(a) and on the other hand constructed a consistent example with

$C(K)$ containing no uncountable equilateral set. T. Kania and T. Kochanek [11] answered Question 2.7(1)(b) in the affirmative in ZFC.

It seems to remain open if there is a consistent example of a compact space K such that $C(K)$ contains an uncountable equilateral set but its unit ball contains no uncountable 2-equilateral set.

(2) We initially proved assertion (v) of Theorem 2.9 with the stronger assumption that K admits a measure of uncountable type (in that case the resulting linked family has the stronger property that $\mu(A_\beta \cap A_\alpha) > 0$ for $\alpha, \beta < \omega_1$). The present form of (v) and its proof are due to the referee. It is worth mentioning that, consistently, a compact space which is HL and HS may carry a measure of uncountable type [4].

The above theorem has some interesting consequences. If K is any compact space, then recall that both spaces $P(K)$ and $B_{C(K)^*}$ are compact with the weak* topology.

Corollary 2.11. *Let K be any compact nonmetrizable space. Denote by Ω any of the compact spaces $K \times K$, $P(K)$ and $B_{C(K)^*}$. Then the unit ball of $C(\Omega)$ contains an uncountable 2-equilateral set.*

Proof. The compact space Ω is not HL. Indeed, if $\Omega = K \times K$, then since K is not metrizable, its diagonal $\Delta = \{(x, x) : x \in K\}$ is closed but not G_δ (by a classical result, if the diagonal of a compact space K is a G_δ subset of $K \times K$ then K is metrizable). So by Fact 2.8 the space $K \times K$ is not HL.

Let $\Omega = P(K)$. We consider the continuous map $\Phi : K \times K \to P(K)$, $\Phi(x, y) = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$ (δ_x is the Dirac measure at $x \in K$). Then $\Delta = \Phi^{-1}(\{\delta_x : x \in K\})$. If $P(K)$ were HL, then by Fact 2.8, K would be a closed G_δ subset of $P(K)$, therefore Δ would be a G_δ subset of $K \times K$, a contradiction.

If $\Omega = B_{C(K)^*}$ then since $P(K)$ is a weak* closed subset of Ω, we conclude that Ω is not HL. ■

Corollary 2.12. *Let X be a nonseparable Banach space. Denote by Ω its closed dual unit ball B_{X^*} with the weak* topology. Then the unit ball of $C(\Omega)$ contains an uncountable 2-equilateral set.*

Proof. Using transfinite induction and the Hahn–Banach Theorem we may construct for each $\varepsilon > 0$ two long sequences $\{x_\alpha : \alpha < \omega_1\} \subseteq B_X$ and $\{f_\alpha : \alpha < \omega_1\} \subseteq (1 + \varepsilon)B_{X^*}$ satisfying $f_\beta(x_\alpha) = 0$ for $\beta > \alpha$ and $f_\alpha(x_\alpha) = 1$ for $\alpha < \omega_1$ (see [9, Fact 4.27]). It is easy to see that the sequence $\{f_\alpha : \alpha < \omega_1\}$ is right separated in the compact space $(1 + \varepsilon)B_{X^*}$. So $(1 + \varepsilon)B_{X^*}$ is not HL and neither is $\Omega = B_{X^*}$. ■

A subspace Y of a topological space X is said to be *weakly separated* if there are open sets U_y, $y \in Y$, in X such that $y \in U_y$ for all $y \in Y$ and whenever $y_1 \neq y_2 \in Y$ we have either $y_1 \notin U_{y_2}$ or $y_2 \notin U_{y_1}$ ([2], [13]). The following can easily be verified:

(i) If $Y = \{t_\alpha : \alpha < \omega_1\}$ is any right (resp. left) separated family in the topological space X, then Y is an uncountable weakly separated subspace of X; we say in this case that Y is an *uncountable right* (resp. *left*) *separated subspace* of X.

(ii) Let Y be any weakly separated subspace of X by the family of open sets U_y, $y \in Y$. Then the family $\{(\{y\}, X \setminus U_y) : y \in Y\}$ is a linked family of closed pairs in X.

As we shall see, linked families of closed pairs in a space X are a special kind of weakly separated subspaces in $\exp X$, the hyperspace of closed nonempty subsets of X endowed with the Vietoris topology. If $G_1, \ldots, G_n$ are subsets of X we denote

$$\langle G_1, \ldots, G_n\rangle = \Big\{F \in \exp X : F \subseteq \bigcup_{k=1}^{n} G_k \text{ and } F \cap G_k \neq \emptyset \ \forall k = 1, \ldots, n\Big\}.$$

The Vietoris topology on $\exp X$ has as base the sets of the form $\langle G_1, \ldots, G_n\rangle$, where $G_1, \ldots, G_n$ are open subsets of X.

We shall say that a weakly separated subspace Y of $\exp X$ is *separated by open subsets* of X if the sets U_y, $y \in Y$, of the definition above are of the form $U_y = \langle V_y\rangle$, $y \in Y$, where V_y are open subsets of X. This is equivalent to both: $y \subseteq V_y$ for $y \in Y$ and if $y_1 \neq y_2 \in Y$, then either $y_1 \nsubseteq V_{y_2}$ or $y_2 \nsubseteq V_{y_1}$.

The (easy) proof of the following proposition is left to the reader.

Proposition 2.13. *Let X be a topological space. Then X admits a linked family of closed pairs of cardinality κ if and only if $\exp X$ admits a weakly separated subspace by open subsets of X of cardinality κ.*

Remarks 2.14. (1) As was shown by Todorcevic assuming Martin's Axiom and the negation of the continuum hypothesis, if K is compact and nonmetrizable then $C(K)$ admits an uncountable (bounded) biorthogonal system [22, Th. 11]. So by using [16, Theorem 3], the space $C(K)$ can be given an equivalent norm that admits an uncountable equilateral set.

(2) It is consistent with ZFC to assume that there exists a compact nonmetrizable space K having no uncountable weakly separated subspace (see [2]). The space K constructed there, among its many interesting properties, is totally disconnected and hence admits an uncountable linked family of closed (and open) pairs.

(3) Let K be a compact nonmetrizable space. Then the hyperspace $\exp K$ of K is not HL. Actually its closed subspace $[K]^{\leq 2} = \{A \subseteq K : |A| \leq 2\}$ is not HL. (The proof is similar to the proof that $(P(K), w^*)$ is not HL; we consider the continuous map $\Phi : K \times K \to \exp K$, $\Phi(x, y) = \{x, y\}$, and

note that $\Phi(K \times K) = [K]^{\leq 2}$.) It follows that there is an uncountable right separated subspace $Y = \{F_\alpha : \alpha < \omega_1\}$ of $\exp K$, which by the result of P. Koszmider [12] is not necessarily separated by open subsets of K.

3. Maximal equilateral sets in $C(K)$ spaces. Our goal here is the study of maximal equilateral sets of minimum cardinality, mainly in Banach spaces of the form $C(K)$.

Definition 3.1. Let (M, d) be a metric space. We define, for $x \in M$,

$$m(M, x) = \min\{|A| : x \in A \text{ and } A \text{ is a maximal equilateral set in } M\}.$$

We also define

$$m(M) = \min\{|A| : A \text{ is a maximal equilateral set in } M\}.$$

It is clear that $m(M) = \min\{m(M, x) : x \in M\}$.

Lemma 3.2. *Let $(X, \|\cdot\|)$ be a normed space. Then*

$$m(X) = m(S_X \cup \{0\}, 0).$$

Proof. Let $A \subseteq X$ be any maximal equilateral set in X. Assume that A is λ-equilateral. Let $x_0 \in A$; then the set $B = \{\lambda^{-1}(x - x_0) : x \in A\}$ is a 1-equilateral subset of $S_X \cup \{0\}$ containing the point 0. Note that $|B| = |A|$ and that B is a maximal equilateral set (in X and hence) in $S_X \cup \{0\}$.

In the converse direction, consider any maximal equilateral subset B of the metric space $S_X \cup \{0\}$ with $0 \in B$. Then clearly B is 1-equilateral. We claim that B is a maximal equilateral subset of X; indeed, if $x \in X$ with $x \notin B$ is such that $B \cup \{x\}$ is equilateral then $1 = \|x - 0\| = \|x\|$, so $x \in S_X$, which contradicts the maximality of B in $S_X \cup \{0\}$. ∎

Lemma 3.3. *Let $(X, \|\cdot\|)$ be a normed space. Then $m(B_X) \leq m(X)$ $(= m(S_X \cup \{0\}, 0))$.*

Proof. By the (method of proof of the) previous lemma any maximal equilateral set in X gives rise to a maximal equilateral set in X of the same cardinality, contained in $S_X \cup \{0\} \subseteq B_X$, so we are done. ∎

Remarks 3.4. (1) Swanepoel and Villa [21] have shown the following result, generalizing an example of Petty [18]:

If X is any Banach space with $\dim X \geq 2$ having a norm which is Gâteaux differentiable at some point, and $Y = (X \oplus \mathbb{R})_1$, then $m(Y) = 4$.

(Their proof is based on the following simple result: Let X be any normed space with $\dim X \geq 2$ and also let $x, u \in S_X$ be such that $\|u - x\| = \|u + x\| = 2$. Then the unit ball of the subspace $Z = \langle u, x\rangle$ of X is the parallelogram with vertices $\pm u, \pm x$.) One can easily check that the result of Swanepoel and Villa can be generalized (by the method of its proof) as follows:

If $\dim X \geq 2$ and the norm of X is either strictly convex or Gâteaux differentiable at some point, then $m(Y) = 4$ and $m(B_X) = 2$ $(= m(S_X))$.

(2) For the Hilbert space $X = \ell_2$ we clearly have $m(X) = \omega$, and since the norm of X is strictly convex, the preceding remark yields $m(B_X) = 2$. So the inequality in Lemma 3.3 may be strict.

(3) Let Γ be any set with $|\Gamma| \geq 2$ and let $\|\cdot\|$ be an equivalent strictly convex norm on the Banach space $\ell_1(\Gamma)$ (see [3]). Now set $X = (\ell_1(\Gamma), \|\cdot\|)$. Then (1) shows that $m(X \oplus \mathbb{R})_1 = 4$.

The following theorem can also be proved by using a result of Swanepoel and Villa [21] (if $d \in \mathbb{N}$ then $m(\ell_\infty^d) = d + 1$). But we are going to give a direct proof.

THEOREM 3.5. *Let X be any infinite locally compact Hausdorff space. Then $m(C_0(X)) \geq \omega$.*

Proof. We shall show that each finite equilateral subset of $C_0(X)$ can be extended. So let $S = \{f_1, \dots, f_n\}$, $n \geq 2$, be any 1-equilateral set in $C_0(X)$. Then there is a finite set $A \subseteq X$ such that $S|A = \{f_k|A : k \leq n\}$ is 1-equilateral. Set $g = \min_{k \leq n} f_k$. Then there is an open set $\emptyset \neq V \subseteq X \setminus A$ and $k \leq n$ such that $g = f_k$ on V. Indeed, set $Y = X \setminus A$ and $Y_k = \{x \in Y : g(x) = f_k(x)\}$ for $k \leq n$. Then $Y = \bigcup_{k \leq n} Y_k$, so there is $k \leq n$ such that $V = \operatorname{int}_Y(Y_k) \neq \emptyset$; since Y is an open subset of X, we see that V is open in X and clearly satisfies our requirements.

Let $h \in C_0(X)$ be a norm-one nonnegative function vanishing on $X \setminus V$. Then it is easy to verify that the set $S \cup \{f_k + h\}$ is 1-equilateral. ■

Let Γ be an infinite set endowed with discrete topology. Then $c_0(\Gamma)$ is the space of all functions $f : \Gamma \to \mathbb{R}$ that vanish at infinity. We shall show that $m(c_0(\Gamma))$ is as large as possible.

PROPOSITION 3.6. *Let Γ be an infinite set. Then $m(c_0(\Gamma)) = |\Gamma|$.*

Proof. We claim that each equilateral subset S in $c_0(\Gamma)$ with $|S| < |\Gamma|$ can be extended. If Γ is countable, then S is finite and can be extended by the previous theorem. So assume Γ is uncountable. It is also clear by Lemma 3.2 that we may assume S is contained in $B_{c_0(\Gamma)}$ and is 1-equilateral. Set $\Delta = \bigcup\{\operatorname{supp} x : x \in S\}$; since $|S| < |\Gamma| \geq \omega_1$ and each element of $c_0(\Gamma)$ has at most countable support, we get $|\Delta| < |\Gamma|$. Let $\gamma_0 \in \Gamma \setminus \Delta$, then it is easy to see that the set $S \cup \{e_{\gamma_0}\}$ (e_{γ_0} is the γ_0-member of the usual basis of $c_0(\Gamma)$) is 1-equilateral. Now we can proceed by transfinite induction, using the above claim to show that $m(c_0(\Gamma)) = |\Gamma|$. We omit the details of this (easy) proof. ■

Let K be any infinite compact metric space. Then the Banach space $C(K)$ is separable and by Theorem 3.5 we get $m(C(K)) = \omega$. This can be generalized as follows:

THEOREM 3.7. *Let $\{X_\gamma : \gamma \in \Gamma\}$ be an infinite family of compact metric spaces with $|X_\gamma| \geq 2$ for all $\gamma \in \Gamma$. Set $X = \prod_{\gamma\in\Gamma} X_\gamma$. Then $m(C(X)) = |\Gamma|$.*

Proof. We remind the reader that each continuous function $f : X \to \mathbb{R}$ depends on countably many coordinates; that is, there is a countable set $A \subseteq \Gamma$ such that $f(x) = f(y)$ for every pair $x = (x_\gamma)$, $y = (y_\gamma)$ of points of X satisfying $x_\gamma = y_\gamma$ for $\gamma \in A$ (see [6, pp. 157–159 and 194–195]).

Assume that $|\Gamma| \geq \omega_1$ (if $|\Gamma| = \omega$, then X is compact metric and the result holds true). So let S be any equilateral set in $C(X)$ with $|S| < |\Gamma|$. Since by Theorem 3.5, $m(C(K)) \geq \omega$ for any infinite compact space K, we may also assume that S is infinite. We are going to prove that S can be extended (cf. the proof of Proposition 3.6). Since each continuous function on X depends on countably many coordinates and $|S| < |\Gamma|$, it follows that there is an $A \subseteq \Gamma$ with $|A| = |S| < |\Gamma|$ such that each member of S depends on A. Pick $\gamma \in \Gamma \setminus A$ and take distinct $t_0, t_1 \in X_\gamma$ and a continuous function $h : X_\gamma \to [0,1]$ such that $h(t_0) = 0$ and $h(t_1) = 1$. Pick a nonzero function $g_0 \in S$ and let $f = g_0 \cdot (h \circ \pi_\gamma)$, where π_γ is the projection on coordinate γ. Now it is easy to check that $S \cup \{f\}$ is equilateral. ∎

NOTE. The same proof yields $m(C(X)) \geq |\Gamma|$ when $X = \prod_{\gamma\in\Gamma} X_\gamma$ with each X_γ compact Hausdorff with at least two points and Γ infinite.

THEOREM 3.8. *Let K be a compact space containing a countable infinite set D of isolated points such that $\overline{D}$ is G_δ. Then $m(C(K)) = \omega$.*

Proof. Let $D = \{x_n : n \geq 1\}$ be a sequence of distinct isolated points of K. We consider the sequence $(t_n)_{n\geq 1}$ of reals, where

$$t_{2n-1} = 1 - \frac{1}{n+1} \quad \text{and} \quad t_{2n} = \frac{1}{n+1}.$$

Since $\overline{D}$ is G_δ, there is a continuous function $f_0 : K \to [0,1/2]$ such that $\overline{D} = f_0^{-1}(\{0\})$. For $n \geq 1$ take a continuous function f_n such that $f_n(x_n) = 1$, $f_n(x_k) = 0$ for $k < n$ and $f_n(x) = t_n$ elsewhere (cf. also Remark 3.12(3)). Then $\{f_0, f_1, \dots\}$ is a 1-equilateral set in $C(K)$. We shall prove that it is maximal. Indeed, if $g \in C(K)$ and $\|g - f_n\| = 1$ for every $n \geq 1$ then:

(i) $g(x) \in [0,1]$ for $x \in K$.
(ii) $g = 0$ on $\overline{D}$. Otherwise there would be a minimal $N \in \mathbb{N}$ such that $g(x_N) > 0$; then we would have $\|g - f_N\| < 1$.

It then follows from (i), (ii) and the properties of f_0 that $\|g - f_0\| < 1$, which proves our claim. ∎

COROLLARY 3.9.

(i) *Let K be any compactification of the discrete set $\mathbb{N}$. Then $m(C(K)) = \omega$.*

(ii) *Let* $K = \beta\Gamma$ (= *the Stone–Čech compactification of the infinite discrete set* Γ). *Then* $m(C(K)) = \omega$.

Proof. (i) is an immediate consequence of Theorem 3.8, since $\overline{\mathbb{N}} = K$.

(ii) Let D be an infinite subset of $K = \beta\Gamma$. Then $\overline{D}$ is a clopen subset of K. (Recall that $C(\beta\Gamma)$ is isometric to $\ell_\infty(\Gamma)$.) ■

Using similar ideas we obtain the following result (cf. also Remark 3.12(3)).

Theorem 3.10. *Let* K *be a compact space containing a countable infinite set* D *of isolated points such that* $\overline{D}$ *is not homeomorphic to* $\beta\mathbb{N}$. *Then* $m(C(K)) = \omega$.

Proof. By our assumption we can write $D = \{x_n : n \geq 1\}$, so that for $D_0 = \{x_{2n} : n \geq 1\}$ and $D_1 = \{x_{2n-1} : n \geq 1\}$ we have $\overline{D_0} \cap \overline{D_1} \neq \emptyset$. Let $\sigma = (\sigma_1, \sigma_2, \ldots) \in 2^\omega$ (= the Cantor set $\{0,1\}^{\mathbb{N}}$) be defined by $\sigma_{2n} = 0$ and $\sigma_{2n-1} = 1$. We shall also use the sequence (t_n) from the proof of Theorem 3.8.

We define a sequence $(f_n)_{n\geq 1}$ of continuous functions on K as follows:

$$f_n(x_n) = \sigma_n + 1 \bmod 2,\ f_n(x_k) = \sigma_k \text{ for } k < n \text{ and } f_n(x) = t_n \text{ elsewhere.}$$

It is clear that the set $\{f_1, f_2, \ldots\}$ is 1-equilateral in $C(K)$. We claim that it is maximal. Consider any function $g : K \to \mathbb{R}$ such that $\|g - f_n\| = 1$ for $n \geq 1$. It follows that:

(i) $g(x) \in [0,1]$ for $x \in K$, and (hence)
(ii) $|g(x) - f_n(x)| < 1$ for $x \in K \setminus D$.

So we get $\|g - f_n\|_\infty = \sup_{x\in D} |g(x) - f_n(x)| = 1$ for $n \geq 1$. Now we check by induction that $g(x_n) = \sigma_n$ for $n \geq 1$, and this implies that g is not continuous. ■

Corollary 3.11. *Let* K *be an infinite compact scattered space. Then* $m(C(K)) = \omega$.

Proof. Since K is scattered, the set D of isolated points of K is dense in K. Clearly $\overline{D} = K$ is not homeomorphic to $\beta\mathbb{N}$, so Theorem 3.10 implies the conclusion. ■

Remarks 3.12. (1) Let X, Y be compact spaces, $\pi : X \to Y$ a continuous surjective map which is nonirreducible (i.e., there is $\Omega \subsetneq X$ compact such that $\pi(\Omega) = Y$) and $T : C(Y) \to C(X)$ the linear isometry given by $T(f) = f \circ \pi$ for $f \in C(Y)$. Let S be a 1-equilateral subset of $S^+_{C(Y)}$. Then it is fairly easy to prove that there is $g \in S^+_{C(X)}$ such that $T(S) \cup \{g\}$ is equilateral.

Given this result, it can be shown by transfinite induction that if a compact nonmetrizable space K is roughly the "limit" of a long system of "smaller" compact spaces connected by nonirreducible maps, then

$$m(S^+_{C(K)} \cup \{0\}, 0) \geq \omega_1.$$

This is the case for instance when:

(a) K is an Eberlein, or more generally a Corson compact space, because then K admits a long sequence of compatible retractions (see [17], [3]), and when

(b) K is a compact group, since then K is a projective limit of compact metrizable groups (see [10]).

(2) Note that, given any infinite set Γ, the Banach spaces $c_0(\Gamma)$ and $c(\widetilde{\Gamma})$ are isomorphic, where $\widetilde{\Gamma}$ is the one-point compactification of the discrete set Γ. But if Γ is uncountable, then by Proposition 3.6, Corollary 3.11 and the above remark we have

$$m(C(\widetilde{\Gamma})) = \omega < m(S^+_{C(\widetilde{\Gamma})} \cup \{0\}, 0) = m(c_0(\Gamma)) = |\Gamma|.$$

(3) Let $T = \bigcup_{n=0}^{\infty} \{0,1\}^n$ be the dyadic tree. Consider any antichain $A = \{s_n : n \geq 1\}$ of T. For $n \in \mathbb{N}$ let

$$A_n = \{k \leq |s_n| : s_n(k) = 1\} \quad \text{and} \quad B_n = \{k \leq |s_n| : s_n(k) = 0\},$$

where $|s|$ is the length of $s \in T$. Then it is easy to see that $\mathcal{F}(A) = \{(A_n, B_n) : n \in \mathbb{N}\}$ is a linked family of (finite) pairs in $\mathbb{N}$. Note that each $\sigma = (\sigma_1, \sigma_2, \dots) \in 2^\omega$ gives rise to an antichain of T by letting

$$A(\sigma) = \{(\varphi(\sigma_1)), (\sigma_1, \varphi(\sigma_2)), \dots, (\sigma_1, \dots, \sigma_{n-1}, \varphi(\sigma_n)), \dots\},$$

where $\varphi(0) = 1$ and $\varphi(1) = 0$, and hence to a linked family $\mathcal{F}(A(\sigma))$. Moreover, $\mathcal{F}(A(\sigma)) \cup \{(A_\omega, B_\omega)\}$, where $A_\omega = \{n \in \mathbb{N} : \sigma_n = 1\}$ and $B_\omega = \{n \in \mathbb{N} : \sigma_n = 0\}$, is a maximal linked family of pairs in $\mathbb{N}$.

Indeed, let (t_n) be an arbitrary sequence in $(0,1)$; for $n \geq 1$ define a function $f_n : \mathbb{N} \to \mathbb{R}$ so that

$$f_n(n) = \varphi(\sigma_n), \qquad f_n(k) = \sigma_k,\ k < n, \qquad f_n(k) = t_n,\ k > n,$$

and also let $f_\omega(k) = \sigma_k$ for $k \in \mathbb{N}$.

Then it can be shown that the set $\{f_n : n \geq 1\} \cup \{f_\omega\}$ is a maximal 1-equilateral subset of ℓ_∞ (cf. the proofs of Theorems 3.8 and 3.10) and that the linked family of pairs in $\mathbb{N}$ corresponding to it according to Lemma 2.3 (cf. also Remarks 2.5(2) and (3)) is the family $\mathcal{F}(A(\sigma)) \cup \{(A_\omega, B_\omega)\}$.

(4) The following example is related to Remark 2.5(3). Let K be a compact nonempty space. We denote by Ω the disjoint union of the compact spaces K and $\widetilde{\mathbb{N}}$, where $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the one-point compactification of the dicrete space $\mathbb{N}$. We define a 1-equilateral set $S = \{f_n : n \leq \omega\} \subseteq [0,1]^\Omega \cap C(\Omega)$ as follows:

$$f_n(k) = 0,\, k < n, \quad f_n(n) = 1, \quad f_n(x) = 1/2,\, x \in \Omega \setminus \{1, \dots, n\},$$

for $n \in \mathbb{N}$, and $f_\omega(x) = 0$, $x \in \widetilde{\mathbb{N}}$, while $f_\omega(x) = 1/2$, $x \in K$.

It is easy to verify that the linked family $\mathcal{F}$ corresponding to this equilateral set according to Lemma 2.3 is maximal and also that S can be extended to an equilateral set in $C(\Omega)$. On the other hand, by Theorem 3.10 we know that $m(C(\Omega)) = \omega$.

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