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THE FUNDAMENTAL THEOREM AND MASCHKE'S THEOREM IN THE CATEGORY OF RELATIVE HOM-HOPF MODULES

BY

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Abstract. We introduce the concept of relative Hom-Hopf modules and investigate their structure in a monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. More particularly, the fundamental theorem for relative Hom-Hopf modules is proved under the assumption that the Hom-comodule algebra is cleft. Moreover, Maschke's theorem for relative Hom-Hopf modules is established when there is a multiplicative total Hom-integral.

1. Introduction. The theory of algebraic deformation has become an important branch of algebra. Multiplicative deformations were applied in quantum modules, as well as in analysis of complex systems and processes exhibiting complete or partial scaling invariance. Discretization of vector fields via twisted derivations leads to Hom-Lie structures in which the Jacobi identity is twisted by an endomorphism (see [22, 23]). The first example of q-deformations, in which the derivations are replaced by σ-derivations, concerned the Witt and Virasoro algebras [20]. Recently, Hom-Lie structures have been further studied by many scholars [1, 2, 21, 29, 34, 36]. Notions like Hom-Lie bialgebras, quasi-Hom-Lie algebras, Hom-Lie superalgebras, Hom-Lie color algebras, Hom-Lie admissible Hom-algebras, Hom-Nambu–Lie algebras and so on have been introduced. In [5], we study the construction of Hom-Lie bialgebras from Hom-Lie algebras and Hom-Lie coalgebras respectively. Quasi-triangular Hom-Lie bialgebras were investigated further in [6].

The ideals of tailoring associativity-like conditions by using endomorphisms has migrated to other algebraic structures. The concepts of Hom-algebras, Hom-coalgebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infinitesimal Hom-bialgebras, Hom-power associative algebras, quasi-triangular Hom-bialgebras, separable and Frobenius structures of monoidal Hom-

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algebras have been introduced and developed [3, 7, 8, 9, 10, 11, 16, 17, 25, 26, 27, 28, 33, 35]. Now, the associativity of Hom-algebras (A, α) is replaced by

$$\alpha(a)(bc) = (ab)\alpha(c),$$

where $\alpha \in \mathrm{Aut}(A)$ is an algebra map. Further, some actions and coactions on such objects with Hom-structures such as Hom-modules, Hom-comodules, Hom-Hopf modules and Hom-module algebras have been considered, and the structure theorem of Hom-Hopf modules in a monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (called the Hom-category) was investigated in [3]. Recently, the categories of relative Hom-Hopf modules, Hom-Long bimodules and Hom-Yetter–Drinfeld modules have been introduced and studied [8, 10, 18, 24].

Hopf modules are known to be important in the theory of Hopf algebras. As a generalization of Hopf modules, relative Hopf modules were introduced by Doi [13]. Furthermore, Doi investigated their structure in [14, 15], including the structure theorem for relative Hopf modules over cleft comodule algebras and Maschke's theorem for relative Hopf modules with total integrals.

In 2013, Chen, Wang and Zhang [7] introduced (total) integrals for Hom-Hopf algebras. In a similar way, we may introduce the concept of generalized Hom-integrals. The purpose of this paper is to investigate the structure theorem and Maschke's theorem for relative Hom-Hopf modules via generalized Hom-integrals. The paper is organized as follows. In Section 1, we recall some basic concepts in the Hom-category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. In Section 2, we introduce the concept of generalized Hom-integrals and study their application to questions of splitting in relative Hom-Hopf modules. In Section 3, we prove a structure theorem for relative Hom-Hopf modules under the assumption that the Hom-comodule algebra (A, β) is cleft. In Section 4, we obtain Maschke's theorem for relative Hom-Hopf modules, that is, every short exact sequence of right (H, A)-Hom-Hopf modules

$$0 \to (M, \mu) \xrightarrow{p} (N, \nu) \xrightarrow{q} (L, \iota) \to 0$$

which is split as a sequence of (A, β)-Hom modules, is also split as a sequence of (H, A)-Hom-Hopf modules when there is a multiplicative total Hom-integral $\phi : (H, \alpha) \to (Z(A), \beta)$.

2. Preliminaries. Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the category of k-modules. There is a new monoidal category $\mathcal{H}(\mathcal{M}_k)$. The objects of $\mathcal{H}(\mathcal{M}_k)$ are couples (M, μ), where $M \in \mathcal{M}_k$ and $\mu \in \mathrm{Aut}_k(M)$. The morphisms of $\mathcal{H}(\mathcal{M}_k)$ are morphisms $f : (M, \mu) \to (N, \nu)$ in $\mathcal{M}_k$ such that $\nu \circ f = f \circ \mu$. For any objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu) \quad \text{and} \quad (k, \mathrm{id}).$$

Roughly speaking, all Hom-structures are objects in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \mathrm{id}), \widetilde{a}, \widetilde{l}, \widetilde{r})$ introduced in [3], where the associator $\widetilde{a}$ is given by the fomula

$$\widetilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes \mathrm{id}) \otimes \varsigma^{-1}) = (\mu \otimes (\mathrm{id} \otimes \varsigma^{-1})) \circ a_{M,N,L} \tag{2.1}$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$, and the unitors $\widetilde{l}$ and $\widetilde{r}$ are

$$\widetilde{l}_M = \mu \circ l_M = l_M \circ (\mathrm{id} \otimes \mu), \quad \widetilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes \mathrm{id}).$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called the *Hom-category* associated to the monoidal category $\mathcal{M}_k$, where a k-submodule $N \subseteq M$ is called a *subobject* of (M, μ) if μ restricts to an automorphism of N, that is, $(N, \mu|_N) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. Since the category $\mathcal{M}_k$ has left duality, so does $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Now let M^* be the left dual of $M \in \mathcal{M}_k$, and $b_M : k \to M \otimes M^*$, $d_M : M^* \otimes M \to k$ be the coevaluation and evaluation maps. Then the left dual of $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ is $(M^*, (\mu^*)^{-1})$, and the coevaluation and evaluation maps are given by the formulas

$$\widetilde{b}_M = (\mu \otimes \mu^*)^{-1} \circ b_M, \quad \widetilde{d}_M = d_M \circ (\mu^* \otimes \mu).$$

In the following, we recall from [3] some information about Hom-structures.

Definition 2.1. A *Hom-algebra* is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \to A, \quad a \otimes b \mapsto m(a \otimes b) \equiv ab, \quad \alpha \in \mathrm{Aut}(A),$$

such that

$$2\alpha(a)(bc) = (ab)\alpha(c), \quad a1_A = 1_A a = \alpha(a), \tag{2.2}$$

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \alpha(1_A) = 1_A, \tag{2.3}$$

for all $a, b, c \in A$. In the following, we denote the Hom-algebra as above by (A, α).

In the language of Hopf algebras, m is called multiplication, α is called the twisting automorphism and 1_A is called the unit. Let (A, α) and (A', α') be two Hom-algebras. A *Hom-algebra map* $f : (A, \alpha) \to (A', \alpha')$ is a linear map such that $f \circ \alpha = \alpha' \circ f$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

Throughout, we use the terminology of [3] for convenience. The notion of monoidal Hom-associative algebras is different from that of unital Hom-associative algebras in [28, 29] in the following sense. The same twisted associativity condition holds in both cases. However, the unitality condition for unital Hom-associative algebras is the usual untwisted one: $a1_A = 1_A a = a$ for any $a \in A$, and the twisting map α need not be monoidal (that is, (2.3) is not required).

Definition 2.2. A *Hom-coalgebra* is an object (C, γ) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$, and

$\varepsilon : C \to k$ such that

$$\begin{aligned} \gamma^{-1}(c_1)\otimes\Delta(c_2) = \Delta(c_1)\otimes\gamma^{-1}(c_2), \quad & c_1\varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1)c_2, \\ \Delta(\gamma(c)) = \gamma(c_1)\otimes\gamma(c_2), \quad & \varepsilon(\gamma(c)) = \varepsilon(c), \end{aligned} \tag{2.4}$$

for all $c \in C$.

Note that the first part of (2.4) is equivalent to

$$c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2,$$

which is often used in the rest of the paper. Let (C, γ) and (C', γ') be two Hom-coalgebras. A *Hom-coalgebra map* $f : (C, \gamma) \to (C', \gamma')$ is a linear map such that $f \circ \gamma = \gamma' \circ f$, $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon \circ f = \varepsilon$.

The notion of monoidal Hom-coassociative coalgebra here is somewhat different from that of counital Hom-coassociative coalgebra in [28, 29]. Their coassociativity condition is twisted by some endomorphism, not necessarily by the inverse of the automorphism γ. The counitality condition is the usual untwisted one and counital Hom-coassociative coalgebras are not necessarily monoidal.

DEFINITION 2.3. A *Hom-bialgebra* $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H, α, m, η) is a Hom-algebra and $(H, \alpha, \Delta, \varepsilon)$ is a Hom-coalgebra such that Δ and ε are Hom-algebra maps, that is, for any $h, g \in H$,

$$\begin{aligned} \Delta(hg) = \Delta(h)\Delta(g), \quad & \Delta(1_H) = 1_H \otimes 1_H, \\ \varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad & \varepsilon(1_H) = 1_k. \end{aligned}$$

DEFINITION 2.4. A Hom-bialgebra (H, α) is a *Hom-Hopf algebra* if there exists a morphism (called the *antipode*) $S : H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e. $S \circ \alpha = \alpha \circ S$) such that

$$S * \mathrm{id} = \eta \circ \varepsilon = \mathrm{id} * S. \tag{2.5}$$

In fact, a Hom-Hopf algebra is a Hopf algebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Further, the antipode of Hom-Hopf algebras has almost all the properties of antipode of Hopf algebras such as

$$S(hg) = S(g)S(h), \quad S(1_H) = 1_H, \quad \Delta(S(h)) = S(h_2)\otimes S(h_1), \quad \varepsilon\circ S = \varepsilon.$$

That is, S is a Hom-anti-(co)algebra homomorphism. Since α is bijective and commutes with the antipode S, the inverse α^{-1} also commutes with S, that is, $S \circ \alpha^{-1} = \alpha^{-1} \circ S$. For a finite-dimensional Hom-Hopf algebra $(H, \alpha, m, \eta, \Delta, \varepsilon, S)$, the dual $(H^*, (\alpha^*)^{-1})$ is also a Hom-Hopf algebra with the following structures: for all $g, h \in H$ and $\phi, \varphi \in H^*$,

$$\langle\phi\varphi, h\rangle = \langle\phi, h_1\rangle\langle\varphi, h_2\rangle, \quad 1_{H^*} = \varepsilon,$$
$$\langle\Delta(\phi), g \otimes h\rangle = \langle\phi, gh\rangle, \quad \varepsilon_{H^*} = \eta,$$
$$(\alpha^*)^{-1}(\phi) = \phi \circ \alpha^{-1}, \quad S^*(\phi) = \phi \circ S^{-1}.$$

Now we recall actions and coactions over Hom-algebras and Hom-coalgebras respectively.

Definition 2.5. Let (A, α) be a Hom-algebra. A *right* (A, α)-*Hom-module* consists of (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : M \otimes A \to M$, $\psi(m \otimes a) = m \cdot a$, such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab), \quad m \cdot 1_A = \mu(m),$$
$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a),$$

for all $a, b \in A$ and $m \in M$.

Similarly, we can define a *left* (A, α)-*Hom-module.* The Hom-algebra (A, α) is both a left (A, α)-Hom-module and a right (A, α)-Hom-module via multiplication. Let $(M, \mu), (N, \nu)$ be two left (A, α)-Hom-modules. A map $f : M \to N$ is called a *left* (A, α)-*Hom-module morphism* if $f(a \cdot m) = a \cdot f(m)$ and $f \circ \mu = \nu \circ f$. The category of left (A, α)-Hom modules is denoted by $\widetilde{\mathcal{H}}({}_A\mathcal{M})$. If $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}({}_A\mathcal{M})$, then $(M \otimes N, \mu \otimes \nu) \in \widetilde{\mathcal{H}}({}_A\mathcal{M})$ via the left H-action

$$h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n. \tag{2.6}$$

Dually, we can define Hom-comodules.

Definition 2.6. Let $C = (C, \gamma)$ be a Hom-coalgebra. A *right* (C, γ)-*Hom-comodule* is an object (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that

$$\begin{aligned} &\mu^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = m_{(0)(0)} \otimes (m_{(0)(1)} \otimes \gamma^{-1}(m_{(1)})), \\ &\rho_M(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}), \quad m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m), \end{aligned} \tag{2.7}$$

for all $m \in M$.

Note that the first part of (2.7) is equivalent to

$$(\mathrm{id} \otimes \Delta) \circ \rho_M = \widetilde{a} \circ (\rho_M \otimes \mathrm{id}) \circ \rho_M. \tag{2.8}$$

(C, γ) is a Hom-comodule on itself via comultiplication. Let (M, μ) and (N, ν) be two right C-Hom-comodules. A map $g : M \to N$ is called a *right* (C, γ)-*Hom-comodule morphism* if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$. We denote by $\widetilde{\mathcal{H}}(\mathcal{M}^C)$ the category of right (C, γ)-Hom-comodules. If $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}^C)$, then $(M \otimes N, \mu \otimes \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}^C)$ with the Hom-comodule strucutre

$$\rho(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}. \tag{2.9}$$

Definition 2.7. A *right* (H,α)*-Hom-Hopf-module* (M,μ) is defined as being a right (H,α)-Hom-module and a right (H,α)-Hom-comodule as well, satisfying the following compatibility condition:

$$\rho_M(m\cdot h)=m_{(0)}\cdot h_1\otimes m_{(1)}h_2 \tag{2.10}$$

for $m\in M$ and $h\in H$.

A morphism between two right (H,α)-Hom-Hopf-modules is a k-linear map which is a morphism in the categories $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ and $\widetilde{\mathcal{H}}(\mathcal{M}^H)$ at the same time. We denote the category of right (H,α)-Hom-Hopf modules by $\widetilde{\mathcal{H}}(\mathcal{M}_H^H)$.

3. Generalized Hom-integrals. In this section, we introduce the concept of generalized Hom-integrals, and give different conditions for Hom-comodule algebras to have generalized Hom-integrals.

Definition 3.1. Let (H,α) be a Hom-Hopf algebra, and (A,β) a Hom-algebra. Then (A,β) is called *right* (H,α)*-Hom-comodule algebra* if there is a right (H,α)-Hom-comodule structure $\rho_A: A\to A\otimes H$ on (A,β) such that ρ_A is a morphism of Hom-algebras, that is,

$$\rho_A(ab)=\rho_A(a)\rho_A(b) \tag{3.1}$$

for any $a,b\in A$ and

$$\rho_A(1_A)=1_A\otimes 1_H,\qquad \rho_A\circ\beta=(\beta\otimes\alpha)\circ\rho_A.$$

Let $(H,m_H,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra and (A,m_A,ρ) a right H-comodule algebra. If $\alpha: H\to H$ is a Hopf algebra automorphism, then, by [3, Proposition 1.14], there is a Hom-Hopf algebra $H_\alpha=(H,m_\alpha=\alpha\circ m_H,\eta,\Delta_\alpha=\Delta\circ\alpha^{-1},\varepsilon,S,\alpha)$.

Let $\beta\in\mathrm{Aut}(A)$ be such that $\rho\circ\beta=(\beta\otimes\alpha)\circ\rho$. Then it is easy to show by direct computation that $A_\beta=(A,m_\beta=\beta\circ m_A,\rho_\beta=\rho\circ\beta^{-1},\beta)$ is a right (H_α,α)-Hom-comodule algebra. Moreover, the compatibility condition (3.1) for ρ_β and m_β follows from the compatibility $\rho(ab)=\rho(a)\rho(b)$ of the comodule algebra (A,m_A,ρ).

Definition 3.2. Let (A,β,ρ_A) be a right (H,α)-Hom-comodule algebra. A *right* (H,A)*-Hom-Hopf module* (M,μ) is an object in the categories $\widetilde{\mathcal{H}}(\mathcal{M}_A)$ and $\widetilde{\mathcal{H}}(\mathcal{M}^H)$ satisfying the compatibility condition

$$\rho_M(m\cdot a)=m_{(0)}\cdot a_{(0)}\otimes m_{(1)}a_{(1)} \tag{3.2}$$

for all $m\in M$ and $a\in A$. Morphisms of right (H,A)-Hom-Hopf modules are morphisms of right (A,β)-Hom-modules and morphisms of right (H,α)-Hom-comodules. The category of right (H,A)-Hom-Hopf modules is denoted by $\widetilde{\mathcal{H}}(M_A^H)$.

In fact, the right (H,α)-Hom-comodule algebra (A,β) is itself a right (H,A)-Hom-Hopf module via ρ_A and the multiplication $m_A : A\otimes A \to A$, since the compatibility condition of (H,A)-Hom-Hopf modules is just (3.1).

EXAMPLE 3.3. (1) We can induce a relative Hom-Hopf module from a relative Hopf module (M,ψ,ρ), which is similar to inducing a Hom-comodule algebra from a comodule algebra. We just need to twist the action ψ and the coaction ρ into $\psi_\mu = \mu\circ\psi$ and $\rho_\mu = \rho\circ\mu^{-1}$ respectively, where $\mu : M \to M$ is an automorphism such that $\mu\circ\psi = \psi\circ(\mu\otimes\beta)$ and $\rho\circ\mu = (\mu\otimes\alpha)\circ\rho$.

(2) For a right (H,α)-Hom-comodule algebra (A,β) and a right (A,β)-Hom-module (M,μ), there is a right (H,A)-Hom-Hopf module $(M\otimes H,\mu\otimes\alpha)$, where the right (A,β)-Hom-module structure is $\psi : (M\otimes H)\otimes A \to M\otimes H$, $(m\otimes h)\otimes a \mapsto (m\otimes h)\bullet a = m\cdot a_{(0)}\otimes ha_{(1)}$, and the right (H,α)-Hom-comodule structure is $\rho : M\otimes H \to (M\otimes H)\otimes H$, $m\otimes h \mapsto (\mu^{-1}(m)\otimes h_1)\otimes\alpha(h_2)$. We only show the compatibility condition: for any $m\in M$, $h\in H$ and $a\in A$,

$$\begin{aligned}
(m\otimes h)_{(0)}\bullet a_{(0)}\otimes(m\otimes h)_{(1)}a_{(1)} &= (\mu^{-1}(m)\otimes h_1)\bullet a_{(0)}\otimes\alpha(h_2)a_{(1)}\\
&= (\mu^{-1}(m)\cdot a_{(0)(0)}\otimes h_1a_{(0)(1)})\otimes\alpha(h_2)a_{(1)}\\
&\overset{(2.7)}{=} \big(\mu^{-1}(m)\cdot\beta^{-1}(a_{(0)})\otimes h_1a_{(1)1}\big)\otimes\alpha(h_2)\alpha(a_{(1)2})\\
&= (\mu^{-1}(m\cdot a_{(0)})\otimes h_1a_{(1)1})\otimes\alpha(h_2a_{(1)2})\\
&= \rho(m\cdot a_{(0)}\otimes ha_{(1)}) = \rho((m\otimes h)\bullet a).
\end{aligned}$$

DEFINITION 3.4. Let (A,β,ρ_A) be a right (H,α)-Hom-comodule algebra. A morphism $\phi : (H,\alpha)\to(A,\beta)$ is called a *generalized Hom-integral* for (A,β) if ϕ is a right (H,α)-Hom-comodule map, that is,

$$\phi\circ\alpha = \beta\circ\phi, \tag{3.3}$$

$$\rho_A\circ\phi = (\phi\otimes\mathrm{id})\circ\Delta_H. \tag{3.4}$$

The set of all generalized Hom-integrals for (A,β) is denoted $\widetilde{H}(\mathrm{Com}(H,A))$.

THEOREM 3.5. *Let (A,β) be a right (H,α)-Hom-comodule algebra. Then the following are equivalent:*

1. *all right (H,A)-Hom-Hopf modules are injective as (H,α)-Hom-comodules,*
2. *(A,β) is an injective (H,α)-Hom-comodule,*
3. *there is $\phi\in\widetilde{H}(\mathrm{Com}(H,A))$ with $\phi(1_H)=1_A$,*
4. *there is $\phi\in\widetilde{H}(\mathrm{Com}(H,A))$ with $\phi(1_H)$ invertible in A.*

Proof. (1)⇒(2) is clear, since $(A,\beta)\in\widetilde{\mathcal{H}}(M_A^H)$. For (2)⇒(3), consider the diagram of right (H,α)-Hom-comodules

$$\begin{array}{ccccc} 0 & \longrightarrow & k & \xrightarrow{\eta_H} & H \\ & & {\scriptstyle \eta_A}\downarrow & \swarrow {\scriptstyle \phi} & \\ & & A & & \end{array}$$

If (A,β) is an injective (H,α)-Hom-comodule, then there is an (H,α)-Hom-comodule map ϕ such that the diagram commutes.

Next we prove (3)⇒(1). For $(M,\mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_A^H)$, define a map $\rho_{M\otimes H}$ in $\widetilde{\mathcal{H}}(M_k)$ as follows:

$$\rho_{M\otimes H} : M \otimes H \xrightarrow{\mathrm{id}\otimes\Delta} M \otimes (H \otimes H) \xrightarrow{\widetilde{a}^{-1}} (M \otimes H) \otimes H. \tag{3.5}$$

So $\rho_{M\otimes H}(m\otimes h) = (\mu^{-1}(m)\otimes h_1)\otimes\alpha(h_2)$. It is easy to show that $\rho_{M\otimes H}$ is a right (H,α)-Hom-comodule structure on $(M\otimes H, \mu\otimes\alpha)$. Then the diagram

$$\begin{array}{ccc} M & \xrightarrow{\qquad\rho_M\qquad} & M\otimes H \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_{M\otimes H}} \\ M\otimes H & \xrightarrow{\rho_M\otimes\mathrm{id}} & (M\otimes H)\otimes H \end{array}$$

is commutative, which implies that ρ_M is an (H,α)-Hom-comodule map. It is enough to show that $\rho_M : M \to M\otimes H$ is split as an (H,α)-Hom-comodule map. That is, there is an (H,α)-Hom-comodule map $\lambda : M\otimes H \to M$ such that $\lambda\circ\rho_M = \mathrm{id}_M$, which implies that (M,μ) is injective as an (H,α)-Hom-comodule.

Construct $\lambda : M\otimes H \to M$ as the composite

$$\begin{aligned} M\otimes H &\xrightarrow{\rho_M\otimes\mathrm{id}} (M\otimes H)\otimes H \xrightarrow{\mathrm{id}\otimes S\otimes\mathrm{id}} (M\otimes H)\otimes H \xrightarrow{\widetilde{a}} \\ &M\otimes(H\otimes H) \xrightarrow{\mathrm{id}\otimes m_H} M\otimes H \xrightarrow{\mathrm{id}\otimes\phi} M\otimes A \xrightarrow{\psi_M} M. \end{aligned}$$

Then, for $m\in M$ and $h\in H$,

$$\lambda(m\otimes h) = \mu(m_{(0)})\cdot\phi(S(m_{(1)})\alpha^{-1}(h)).$$

For any $m\in M$,

$$\begin{aligned} \lambda\circ\rho_M(m) = \lambda(m_{(0)}\otimes m_{(1)}) &= \mu(m_{(0)(0)})\cdot\phi(S(m_{(0)(1)})\alpha^{-1}(m_{(1)})) \\ &= m_{(0)}\cdot\phi(S(m_{(1)1})m_{(1)2}) = m_{(0)}\cdot\phi(\varepsilon(m_{(1)})1_H) \\ &= \mu^{-1}(m)\cdot\phi(1_H) = \mu^{-1}(m)\cdot 1_A = m, \end{aligned}$$

so $\lambda\circ\rho_M = \mathrm{id}_M$. Moreover, λ is an (H,α)-Hom-comodule map since λ is in $\widetilde{\mathcal{H}}(M_k)$, and

$$\begin{aligned} \rho_M\circ\lambda(m\otimes h) &= \rho_M\big(\mu(m_{(0)})\cdot\phi(S(m_{(1)})\alpha^{-1}(h))\big) \\ &= \mu(m_{(0)})_{(0)}\cdot\phi\big(S(m_{(1)})\alpha^{-1}(h)\big)_{(0)}\otimes\mu(m_{(0)})_{(1)}\phi\big(S(m_{(1)})\alpha^{-1}(h)\big)_{(1)} \end{aligned}$$

$$
\begin{aligned}
&\overset{(3.4)}{=} \mu(m_{(0)(0)}) \cdot \phi(S(m_{(1)})_1\alpha^{-1}(h_1)) \otimes \alpha(m_{(0)(1)})\big(S(m_{(1)})_2\alpha^{-1}(h_2)\big)\\
&= m_{(0)} \cdot \phi(S(\alpha(m_{(1)2}))_1\alpha^{-1}(h_1)) \otimes \alpha(m_{(1)1})\big(S(\alpha(m_{(1)2}))_2\alpha^{-1}(h_2)\big)\\
&= m_{(0)} \cdot \phi(S(\alpha(m_{(1)22}))\alpha^{-1}(h_1)) \otimes \big(m_{(1)1}S(\alpha(m_{(1)21}))\big)h_2\\
&= m_{(0)} \cdot \phi(S(m_{(1)2})\alpha^{-1}(h_1)) \otimes \alpha(m_{(1)11}S(m_{(1)12}))h_2\\
&= m_{(0)} \cdot \phi(S(m_{(1)2})\alpha^{-1}(h_1)) \otimes \varepsilon(m_{(1)1})\alpha(h_2)\\
&= m_{(0)} \cdot \phi\big(S(\alpha^{-1}(m_{(1)}))\alpha^{-1}(h_1)\big) \otimes \alpha(h_2)\\
&= m_{(0)} \cdot \phi(\alpha^{-1}(S(m_{(1)})h_1)) \otimes \alpha(h_2)\\
&= (\lambda \otimes \mathrm{id})\big((\mu^{-1}(m) \otimes h_1) \otimes \alpha(h_2)\big)\\
&= (\lambda \otimes \mathrm{id}) \circ \rho_{M\otimes H}(m \otimes h)
\end{aligned}
$$

for any $m \in M$ and $h \in H$.

(3)⇒(4) is trivial. For (4)⇒(3), since $\phi(1_H)$ is invertible, replacing ϕ by the map $a \mapsto \phi(1_H)^{-1}\phi(h)$ we find that $\phi(1_H) = 1_A$ and $\phi \in \widetilde{H}(\mathrm{Com}(H, A))$. ■

4. The fundamental theorem for relative Hom-Hopf modules. In this section, we prove the fundamental theorem for relative Hom-Hopf modules under the assumption that the Hom-comodule algebra (A, β) is cleft.

LEMMA 4.1 ([3, Proposition 2.9]). *If (A, m_A, η_A, α) is a Hom-algebra, and $(C, \Delta_C, \varepsilon_C, \beta)$ is a Hom-coalgebra, then* $\mathrm{Hom}(C, A)$ *has a Hom-algebra structure under convolution $*$. For any $\phi, \varphi \in$* $\mathrm{Hom}(C, A)$*, the convolution product is defined by*

$$\phi * \varphi = m_A \circ (\phi \otimes \varphi) \circ \Delta_C.$$

The unit of $\mathrm{Hom}(C, A)$ *is $\eta_A \circ \varepsilon_C$, and the twisting automorphism γ is*

$$\gamma(\phi) = \alpha \circ \phi \circ \beta^{-1}.$$

In Lemma 4.1, the elements of $\mathrm{Hom}(C, A)$ are not necessarily in the category $\widetilde{\mathcal{H}}(M_k)$. But if they are, then $\gamma(\phi) = \phi$, and $\mathrm{Hom}(C, A)$ becomes an associative algebra. The set of all invertible (with respect to $*$) elements of $\mathrm{Hom}(C, A)$ is denoted by $\mathrm{Reg}(C, A)$.

DEFINITION 4.2. Let (H, α) be a Hom-Hopf-algebra and (A, β) a right (H, α)-Hom-comodule algebra. Then (A, β) is called *cleft* if there is $\phi \in \widetilde{H}(\mathrm{Com}(H, A)) \cap \mathrm{Reg}(H, A)$, or equivalently, ϕ satisfies (3.3), (3.4) and there is an element $\phi^{-1} \in \mathrm{Hom}(H, A)$ such that

$$\phi^{-1} * \phi = \phi * \phi^{-1} = \eta_A \circ \varepsilon_H. \tag{4.1}$$

Note that (A, β) satisfies condition (4) of Theorem 3.5 if it is cleft, so (A, β) is an injective (H, α)-Hom-comodule. In particular, the Hom-Hopf-algebra (H, α) is cleft.

For any right (H,α)-Hom-comodule (M,μ,ρ), we define the (H,α)-*coinvariant subspace* of (M,μ) to be the set

$$(M^{\operatorname{co} H}, \mu|_{M^{\operatorname{co} H}}) = \{m \in M \mid \rho(m) = \mu^{-1}(m) \otimes 1_H\}.$$

In the following, we always assume that (A,β) is a right (H,α)-Hom-comodule algebra, and the coinvariant of (A,β) is denoted by $(A^{\operatorname{co} H}, \beta|_{A^{\operatorname{co} H}}) = (B, \beta|_B)$. It is easy to see that $(B,\beta|_B)$ is a Hom-subalgebra of (A,β).

If (M,μ) is a right (H,α)-Hom-module, and (N,ν) a left (H,α)-Hom-module, the relative tensor product space $(M \otimes_H N, \mu \otimes \nu)$ in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is defined as

$$\{m \otimes n \in M \otimes_H N \mid m \cdot h \otimes \nu(n) = \mu(m) \otimes h \cdot n\}. \tag{4.2}$$

LEMMA 4.3. *Let (M,μ) be a right (A,β)-Hom-module. Then $(M \otimes_B A, \mu \otimes \beta)$ is a right (H,A)-Hom-Hopf module with the following action ψ and coaction ρ:*

$$\begin{aligned} \psi &: (M \otimes_B A) \otimes A \to M \otimes_B A, && (m \otimes_B a) \otimes a' \mapsto \mu(m) \otimes_B a\beta^{-1}(a'), \\ \rho &: M \otimes_B A \to (M \otimes_B A) \otimes H, && m \otimes_B a \mapsto (\mu^{-1}(m) \otimes_B a_{(0)}) \otimes \alpha(a_{(1)}). \end{aligned}$$

Proof. The proof is straightforward and is left to the reader. ∎

LEMMA 4.4. *If $\phi \in \widetilde{H}(\operatorname{Com}(H,A)) \cap \operatorname{Reg}(H,A)$, then the following diagram commutes:*

$$\begin{array}{ccccc} H & \xrightarrow{\phi^{-1}} & A & \xrightarrow{\rho_A} & A \otimes H \\ {\scriptstyle \Delta_H}\downarrow & & & & \uparrow{\scriptstyle \phi^{-1}\otimes S} \\ H \otimes H & & \xrightarrow{\quad\tau\quad} & & H \otimes H \end{array}$$

Thus, for any $h \in H$,

$$\rho_A \circ \phi^{-1}(h) = \phi^{-1}(h_2) \otimes S(h_1). \tag{4.3}$$

Proof. We just need to show $(\rho_A \circ \phi)^{-1} = \rho_A \circ \phi^{-1}$ and $((\phi \otimes \mathrm{id}) \circ \Delta_H)^{-1} = (\phi^{-1} \otimes S) \circ \tau \circ \Delta_H$. For any $h \in H$, since ρ_A is a morphism of Hom-algebras, we have

$$\begin{aligned} ((\rho_A \circ \phi) * (\rho_A \circ \phi^{-1}))(h) &= \rho_A \circ \phi(h_1) \rho_A \circ \phi^{-1}(h_2) = \rho_A(\phi(h_1)\phi^{-1}(h_2)) \\ &= \rho_A(\eta_A \circ \varepsilon_H(h)) = \eta_{A \otimes H} \varepsilon_H(h) \end{aligned}$$

and

$$\begin{aligned} &\big(((\phi \otimes \mathrm{id}) \circ \Delta_H) * ((\phi^{-1} \otimes S) \circ \tau \circ \Delta_H)\big)(h) \\ &\qquad = (\phi \otimes \mathrm{id}) \circ \Delta_H(h_1)(\phi^{-1} \otimes S) \circ \tau \circ \Delta_H(h_2) \end{aligned}$$

$$
\begin{aligned}
&= (\phi(h_{11}) \otimes h_{12})(\phi^{-1}(h_{22}) \otimes S(h_{21})) \\
&= \phi(h_{11})\phi^{-1}(h_{22}) \otimes h_{12}S(h_{21}) \\
&= \phi(h_{11})\phi^{-1}(\alpha^{-1}(h_2)) \otimes \alpha(h_{121})S(\alpha(h_{122})) \\
&= \phi(h_{11})\phi^{-1}(\alpha^{-1}(h_2)) \otimes \varepsilon(h_{12})1_H \\
&= \phi(\alpha^{-1}(h_1))\phi^{-1}(\alpha^{-1}(h_2)) \otimes 1_H \\
&= \eta_A \circ \varepsilon_H(\alpha^{-1}(h)) \otimes 1_H = \eta_A \circ \varepsilon_H(h) \otimes 1_H.
\end{aligned}
$$

Similarly, $(\rho_A \circ \phi^{-1}) * (\rho_A \circ \phi) = \eta_{A\otimes H}\varepsilon_H$ and

$$((\phi^{-1} \otimes S) \circ \tau \circ \Delta_H) * ((\phi \otimes \mathrm{id}) \circ \Delta_H) = \eta_A \circ \varepsilon_H \otimes 1_H.$$

Thus, $\rho_A \circ \phi^{-1} = (\rho_A \circ \phi)^{-1} \overset{(3.4)}{=} ((\phi \otimes \mathrm{id}) \circ \Delta_H)^{-1} = (\phi^{-1} \otimes S) \circ \tau \circ \Delta_H$. ∎

Now we proceed to the structure theorem for relative Hom-Hopf modules.

THEOREM 4.5. *Let (A,β) be a right (H,α)-Hom-comodule algebra. If (A,β) is cleft with a generalized Hom-integral $\phi : (H,\alpha) \to (A,\beta)$, then for every right (H,A)-Hom-Hopf module (M,μ),*

$$M^{\mathrm{co}\,H} \otimes_B A \cong M$$

as (H,A)-Hom-Hopf modules.

Proof. Let $p : (M,\mu) \to (M,\mu)$ be the composite

$$M \xrightarrow{\rho_M} M \otimes H \xrightarrow{\mathrm{id}\otimes\phi^{-1}} M \otimes A \xrightarrow{\psi_M} M.$$

Explicitly $p(m) = m_{(0)} \cdot \phi^{-1}(m_{(1)})$. We claim that $p(m) \in M^{\mathrm{co}\,H}$. Indeed,

$$
\begin{aligned}
\rho_M(p(m)) &= \rho(m_{(0)} \cdot \phi^{-1}(m_{(1)})) \\
&= m_{(0)(0)} \cdot \phi^{-1}(m_{(1)})_{(0)} \otimes m_{(0)(1)}\phi^{-1}(m_{(1)})_{(1)} \\
&\overset{(4.3)}{=} m_{(0)(0)} \cdot \phi^{-1}(m_{(1)2}) \otimes m_{(0)(1)}S(m_{(1)1}) \\
&= \mu^{-1}(m_{(0)}) \cdot \phi^{-1}(\alpha(m_{(1)22})) \otimes m_{(1)1}S(\alpha(m_{(1)21})) \\
&= \mu^{-1}(m_{(0)}) \cdot \phi^{-1}(m_{(1)2}) \otimes \alpha(m_{(1)11})S(\alpha(m_{(1)12})) \\
&= \mu^{-1}(m_{(0)}) \cdot \phi^{-1}(m_{(1)2}) \otimes \varepsilon(m_{(1)1})1_H \\
&= \mu^{-1}(m_{(0)}) \cdot \phi^{-1}(\alpha^{-1}(m_{(1)})) \otimes 1_H \\
&= \mu^{-1}(m_{(0)} \cdot \phi^{-1}(m_{(1)})) \otimes 1_H = \mu^{-1}(p(m)) \otimes 1_H.
\end{aligned}
$$

Thus $p(M) \subseteq M^{\mathrm{co}\,H}$, i.e. p is in fact a map $p : (M,\mu) \to (M^{\mathrm{co}\,H}, \mu|_{M^{\mathrm{co}\,H}})$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

Define two maps

$$\epsilon : M^{\mathrm{co}\,H} \otimes_B A \to M$$

by $\epsilon(m \otimes_B a) = m \cdot a$ for $m \in M^{\mathrm{co}\,H}$ and $a \in A$, and

$$\vartheta : M \xrightarrow{\rho_M} M \otimes H \xrightarrow{p\otimes\phi} M^{\mathrm{co}\,H} \otimes_B A$$

by $\vartheta(m) = p(m_{(0)}) \otimes_B \phi(m_{(1)})$ for all $m \in M$. We will show $\epsilon \circ \vartheta = \mathrm{id}_M$, $\vartheta \circ \epsilon = \mathrm{id}_{M^{\mathrm{co}\,H} \otimes_B A}$ and ϵ is a right (H, A)-Hom-Hopf module map.

For $m \in M$,

$$\begin{aligned}
\epsilon \circ \vartheta(m) &= \epsilon(p(m_{(0)}) \otimes_B \phi(m_{(1)})) = p(m_{(0)}) \cdot \phi(m_{(1)}) \\
&= (m_{(0)(0)} \cdot \phi^{-1}(m_{(0)(1)})) \cdot \phi(m_{(1)}) \\
&= \mu(m_{(0)(0)}) \cdot \big(\phi^{-1}(m_{(0)(1)})\beta^{-1}(\phi(m_{(1)}))\big) \\
&= m_{(0)} \cdot (\phi^{-1}(m_{(1)1})\phi(m_{(1)2})) = m_{(0)} \cdot \epsilon(m_{(1)})1_A \\
&= \mu^{-1}(m) \cdot 1_A = m.
\end{aligned}$$

Moreover, for any $m' \in M^{\mathrm{co}\,H}$ and $a \in A$,

$$\begin{aligned}
\vartheta \circ \epsilon(m' \otimes_B a) &= \vartheta(m' \cdot a) = p((m' \cdot a)_{(0)}) \otimes_B \phi((m' \cdot a)_{(1)}) \\
&= p(\mu^{-1}(m') \cdot a_{(0)}) \otimes_B \phi(1_H a_{(1)}) \\
&= (\mu^{-1}(m')_{(0)} \cdot a_{(0)(0)}) \cdot \phi^{-1}\big(\mu^{-1}(m')_{(1)} \cdot a_{(0)(1)}\big) \otimes_B \phi(\alpha(a_{(1)})) \\
&= (\mu^{-2}(m') \cdot a_{(0)(0)}) \cdot \phi^{-1}\big(\alpha^{-1}(1_H) \cdot a_{(0)(1)}\big) \otimes_B \phi(\alpha(a_{(1)})) \\
&= (\mu^{-2}(m') \cdot a_{(0)(0)}) \cdot \phi^{-1}(\alpha(a_{(0)(1)})) \otimes_B \phi(\alpha(a_{(1)})) \\
&= (\mu^{-2}(m') \cdot a_{(0)(0)}) \cdot \beta(\phi^{-1}(a_{(0)(1)})) \otimes_B \phi(\alpha(a_{(1)})) \\
&= \mu^{-1}(m') \cdot (a_{(0)(0)}\phi^{-1}(a_{(0)(1)})) \otimes_B \phi(\alpha(a_{(1)})).
\end{aligned}$$

It is easy to show that $a_{(0)(0)}(\phi^{-1}(a_{(0)(1)})) \in B$, which is similar to the proof of $p(m) \in M^{\mathrm{co}\,H}$ as above. Hence

$$\begin{aligned}
\vartheta \circ \epsilon(m' \otimes_B a) &= m' \otimes_B \big(a_{(0)(0)}\phi^{-1}(a_{(0)(1)})\big)\phi(a_{(1)}) \\
&= m' \otimes_B \big(\beta^{-1}(a_{(0)})\phi^{-1}(a_{(1)1})\big)\phi(\alpha(a_{(1)2})) \\
&= m' \otimes_B a_{(0)}\big(\phi^{-1}(a_{(1)1})\phi(a_{(1)2})\big) \\
&= m' \otimes_B a_{(0)}\varepsilon_H(a_{(1)})1_A \\
&= m' \otimes_B \beta^{-1}(a)1_A = m' \otimes_B a.
\end{aligned}$$

Finally, for any $m' \in M^{\mathrm{co}\,H}$ and $a, b \in A$, we have

$$\begin{aligned}
\epsilon((m' \otimes_B a) \cdot b) &= \epsilon\big(\mu(m') \otimes_B a\beta^{-1}(b)\big) = \mu(m') \cdot (a\beta^{-1}(b)) \\
&= (m' \cdot a) \cdot b = \epsilon(m' \otimes_B a) \cdot b
\end{aligned}$$

and

$$\begin{aligned}
\rho_M \circ \epsilon(m' \otimes_B a) &= \rho_M(m' \cdot a) = m'_{(0)} \cdot a_{(0)} \otimes m'_{(1)}a_{(1)} \\
&= \mu^{-1}(m') \cdot a_{(0)} \otimes 1_H a_{(1)} = \mu^{-1}(m') \cdot a_{(0)} \otimes \alpha(a_{(1)}) \\
&= (\epsilon \otimes \mathrm{id})\big((\mu^{-1}(m') \otimes_B a_{(0)}) \otimes \alpha(a_{(1)})\big) = (\epsilon \otimes \mathrm{id}) \circ \rho(m' \otimes_B a).
\end{aligned}$$

It is obvious that $\epsilon \circ (\mu \otimes \beta) = \mu \circ \epsilon$. So ϵ is a right (H, A)-Hom-Hopf module map and $M^{\mathrm{co}\,H} \otimes_B A \cong M$ as (H, A)-Hom-Hopf modules. ∎

As consequences of the above theorem, we have

REMARK 4.6. (1) Since (H, α) is a cleft (H, α)-Hom-comodule algebra, for every right (H, α)-Hom-Hopf module (M, μ), there is a Hom-Hopf module isomorphism

$$M^{\mathrm{co}\,H} \otimes H \cong M,$$

which is the fundamental theorem for Hom-Hopf modules given in [3], where the Hom-Hopf module structures on $M^{\mathrm{co}\,H} \otimes H$ are given by

$$(m \otimes h) \cdot g = \mu(m) \otimes h\alpha^{-1}(g),$$
$$\rho(m \otimes h) = (\mu^{-1}(m) \otimes h_1) \otimes \alpha(h_2),$$

for $m \in M^{\mathrm{co}\,H}$ and $h, g \in H$.

(2) Let (A, β) be a right (H, α)-Hom-comodule algebra and ϕ a generalized Hom-integral of (A, β). If ϕ is also a Hom-algebra morphism, then ϕ is invertible with $\phi^{-1} = \phi \circ S$. Thus for every right (H, A)-Hom-Hopf module (M, μ), we have

$$M^{\mathrm{co}\,H} \otimes_B A \cong M$$

as (H, A)-Hom-Hopf modules.

5. Maschke's theorem for (H, A)-Hom-Hopf modules. In this section we prove Maschke's theorem for (H, A)-Hom-Hopf modules.

DEFINITION 5.1. A generalized Hom-integral $\phi \in \widetilde{H}(\mathrm{Com}(H, A))$ is called a *total Hom-integral* if it preserves the unit. Equivalently, ϕ satisfies (3.3), (3.4) and $\phi(1_H) = 1_A$.

In the following, we assume that the given right (H, α)-Hom-comodule algebra (A, β) always has a total Hom-integral ϕ.

LEMMA 5.2. *Let $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}(M_A^H)$ and $f : (N, \nu) \to (M, \mu)$ in $\widetilde{\mathcal{H}}(M_k)$. Set*

$$f_\phi : N \xrightarrow{\rho_N} N \otimes H \xrightarrow{f \otimes \mathrm{id}} M \otimes H \xrightarrow{\lambda} M,$$

where λ is the map defined in Theorem 3.5. Explicitly,

$$f_\phi(n) = \mu(f(n_{(0)})_{(0)}) \cdot \phi\big(S(f(n_{(0)})_{(1)})\alpha^{-1}(n_{(1)})\big)$$

for any $n \in N$. Then f_ϕ is a right (H, α)-Hom-comodule map.

Furthermore, if f is a right (A, β)-Hom-module map and $\phi : (A, \beta) \to Z(A)$ is a multiplication map, then f_ϕ is a right (A, β)-Hom-module map.

Proof. Firstly, λ is a right (H,α)-Hom-comodule map by Theorem 3.5, that is, $\rho_M\circ\lambda=(\lambda\otimes\mathrm{id})\circ\rho_{M\otimes H}$, so we have

$$\begin{aligned}\rho_M\circ f_\phi &= \rho_M\circ\lambda\circ(f\otimes\mathrm{id})\circ\rho_N=(\lambda\otimes\mathrm{id})\circ\rho_{M\otimes H}\circ(f\otimes\mathrm{id})\circ\rho_N\\ &\overset{(3.5)}{=} (\lambda\otimes\mathrm{id})\circ\widetilde{a}^{-1}\circ(\mathrm{id}\otimes\Delta)\circ(f\otimes\mathrm{id})\circ\rho_N\\ &= (\lambda\otimes\mathrm{id})\circ\widetilde{a}^{-1}\circ(f\otimes(\mathrm{id}\otimes\mathrm{id}))\circ(\mathrm{id}\otimes\Delta)\circ\rho_N\\ &\overset{(2.8)}{=} (\lambda\otimes\mathrm{id})\circ\widetilde{a}^{-1}\circ(f\otimes(\mathrm{id}\otimes\mathrm{id}))\circ\widetilde{a}\circ(\rho_N\otimes\mathrm{id})\circ\rho_N\\ &= (\lambda\otimes\mathrm{id})\circ((f\otimes\mathrm{id})\otimes\mathrm{id})\circ\widetilde{a}^{-1}\circ\widetilde{a}\circ(\rho_N\otimes\mathrm{id})\circ\rho_N\\ &= (\lambda\otimes\mathrm{id})\circ((f\otimes\mathrm{id})\otimes\mathrm{id})\circ(\rho_N\otimes\mathrm{id})\circ\rho_N\\ &= (f_\phi\otimes\mathrm{id})\circ\rho_N.\end{aligned}$$

Moreover, $f_\phi\in\widetilde{\mathcal{H}}(M_k)$, so f_ϕ is a right (H,α)-Hom-comodule map.

Further, if f is a right (A,β)-Hom-module map and $\phi:(A,\beta)\to Z(A)$ is a multiplication map, then, for any $n\in N$, $a\in A$, we have

$$\begin{aligned}f_\phi(n\cdot a)&=\lambda\circ(f\otimes\mathrm{id})\circ\rho_N(n\cdot a)=\lambda(f(n_{(0)}\cdot a_{(0)})\otimes n_{(1)}a_{(1)})\\ &=\lambda(f(n_{(0)})\cdot a_{(0)}\otimes n_{(1)}a_{(1)})\\ &=\mu(f(n_{(0)})_{(0)}\cdot a_{(0)(0)})\cdot\phi\big(S(f(n_{(0)})_{(1)}a_{(0)(1)})\alpha^{-1}(n_{(1)}a_{(1)})\big)\\ &=(\mu(f(n_{(0)})_{(0)})\cdot a_{(0)})\cdot\phi\big(S(f(n_{(0)})_{(1)}a_{(1)1})(\alpha^{-1}(n_{(1)})a_{(1)2})\big)\\ &=(\mu(f(n_{(0)})_{(0)})\cdot a_{(0)})\cdot\phi\big((S(f(n_{(0)})_{(1)})S(a_{(1)1}))(a_{(1)2}\alpha^{-1}(n_{(1)}))\big)\\ &=(\mu(f(n_{(0)})_{(0)})\cdot a_{(0)})\cdot\phi\big((\alpha^{-1}(S(f(n_{(0)})_{(1)})S(a_{(1)1}))a_{(1)2})n_{(1)}\big)\\ &=(\mu(f(n_{(0)})_{(0)})\cdot a_{(0)})\cdot\phi\big((S(f(n_{(0)})_{(1)})\alpha^{-1}(S(a_{(1)1})a_{(1)2}))n_{(1)}\big)\\ &=(\mu(f(n_{(0)})_{(0)})\cdot a_{(0)})\cdot\phi\big((S(f(n_{(0)})_{(1)})\varepsilon(a_{(1)})1_H)n_{(1)}\big)\\ &=(\mu(f(n_{(0)})_{(0)})\cdot\beta^{-1}(a))\cdot\phi\big(\alpha(S(f(n_{(0)})_{(1)}))n_{(1)}\big)\\ &=\mu^2(f(n_{(0)})_{(0)})\cdot\big(\beta^{-1}(a)\phi(S(f(n_{(0)})_{(1)})\alpha^{-1}(n_{(1)}))\big)\\ &=\mu^2(f(n_{(0)})_{(0)})\cdot\big(\phi(S(f(n_{(0)})_{(1)})\alpha^{-1}(n_{(1)}))\beta^{-1}(a)\big)\\ &=\big(\mu(f(n_{(0)})_{(0)})\cdot\phi(S(f(n_{(0)})_{(1)})\alpha^{-1}(n_{(1)}))\big)\cdot a=f_\phi(n)\cdot a.\end{aligned}$$

Thus, f_ϕ is a right (A,β)-Hom-module map. ■

THEOREM 5.3 (Maschke's theorem). *If* $\mathrm{Im}(\phi)\subseteq Z(A)$ *and* ϕ *is a multiplication map, then every short exact sequence of right* (H,A)*-Hom-Hopf modules*

$$0\to(M,\mu)\xrightarrow{p}(N,\nu)\xrightarrow{q}(L,\iota)\to 0,$$

which is split as a sequence of (A,β)*-Hom-modules, is also split as a sequence of* (H,A)*-Hom-Hopf-modules.*

Proof. We claim that if $p:(M,\mu)\to(N,\nu)$ is a morphism in $\widetilde{\mathcal{H}}(M_A^H)$ such that there exists a right (A,β)-Hom-module map $f:(N,\nu)\to(M,\mu)$

with $f \circ p = \mathrm{id}_M$, then there exists a morphism $f_\phi : (N, \nu) \to (M, \mu)$ in $\widetilde{\mathcal{H}}(M_A^H)$ such that $f_\phi \circ p = \mathrm{id}_M$.

Indeed, by Lemma 5.2, f_ϕ is both a right (H, α)-Hom-module map and a right (A, β)-Hom-module map. Moreover,

$$\begin{aligned} f_\phi \circ p &= \lambda \circ (f \otimes \mathrm{id}) \circ \rho_N \circ p = \lambda \circ (f \otimes \mathrm{id}) \circ (p \otimes \mathrm{id}) \circ \rho_M \\ &= \lambda \circ (f \circ p \otimes \mathrm{id}) \circ \rho_M = \lambda \circ \rho_M = \mathrm{id}_M, \end{aligned}$$

where the last step follows by the proof of Theorem 3.5. ∎

Remark 5.4. Assume that (H, α) is commutative. Then, since (H, α) is a cleft (H, α)-Hom-comodule algebra, every short exact sequence of right (H, α)-Hom-Hopf modules

$$0 \to M \to N \to L \to 0$$

which splits as a sequence of right (H, α)-Hom-modules, also splits as a sequence of right (H, α)-Hom-Hopf modules.

Remark 5.5. Let us mention that Maschke's theorem and related problems for relative Hom-Hopf modules are also studied by S. Guo and his research group [18, 19]. In Maschke's theorem provided in [19] for Doi Hom-Hopf modules it is assumed that there exists a normalized (A, β)-integral $\theta : (H, \alpha) \otimes (H, \alpha) \to (A, \beta)$ satisfying three suitable conditions. This is different from the conditions for the total integral in our Theorem 4.3. Moreover, we prove Theorem 4.3 by different methods. In fact, there are some relations between normalized integrals and total integrals. A normalized (A, β)-integral is indeed a total integral, but for the converse additional conditions (similar to weak commutativity) are needed.

The structural construction of the proof for Maschke's theorem for relative Hom-Hopf modules in [18] is of interest. We prove Maschke's theorem in the same way, introduced by Doi. However, the morphism f_ϕ in [18, Lemma 5.5] is defined by $f_\phi(n) = \mu^{-1}(f(n_{(0)})_{(0)}) \cdot \phi(S(f(n_{(0)})_{(1)})\alpha(n_{(1)}))$, which is twisted in a different way from ours. Although with different twisting structures, the final results are the same. This yields some interesting questions about twisting automorphisms for Hom-structures in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Maybe, these Hom-structures are not uniquely determined by the twisting automorphisms but presented as a family. We guess there may be some relations among the family of Hom-objects twisted in different ways.

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